Malaya Journal of Matematik, Vol. 8, No. 4, 1413-1416, 2020
https://doi.org/10.26637/MJM0804/0013

Planarity of a unit graph part -III $|Max(R)| \geq 3$ case

Jaydeep Parejiya[1]*, Patat Sarman [2] and Pravin Vadhel[3]

Abstract
The rings considered in this article are commutative with identity $1 \neq 0$. Recall that the unit graph of a ring R is a simple undirected graph whose vertex set is the set of all elements of the ring R and two distinct vertices x,y are adjacent in this graph if and only if $x+y \in U(R)$ where $U(R)$ is the set of all unit elements of ring R. We denote this graph by $UG(R)$. In this article we classified rings R with $|Max(R)| \geq 3$ such that $UG(R)$ is planar.

Keywords
Planar graph, (Ku_1^*) and (Ku_2^*).

2010 Mathematics Subject Classification
13A15, 05C25.

[1] Department of Mathematics, Government Polytechnic, Rajkot-360005, India.
[2] Department of Mathematics, Sir B.P.T.I, Bhavnagar, Bhavnagar-364002, India.
[3] Department of Mathematics, V.V.P. Engineering College, Rajkot-360005, India.
*Corresponding author: [1] parejiyajay@gmail.com; [2]patat.sarman03@gmail.com; [3]vadhelpravin@gmail.com
Article History: Received **19** May **2020**; Accepted **22** August **2020**

Contents

1. Introduction

We first recall the following definitions and results from graph theory. A graph G=(V,E) is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n [4, Definition 1.1.11]. A graph G=(V,E) is said to be bipartite if the vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and other in Y. The pair (X,Y) is called a bipartition of G. A bipartite graph G with bipartition (X,Y) is denoted by G(X,Y). A bipartite graph G(X,Y) is said to be complete if each vertex of X is adjacent to all the vertices of Y. If G(X,Y) is a complete bipartite graph with $|X| = m$ and $|Y| = n$, then it is denoted by $K_{m,n}$ [4, Definition 1.1.12]. Let G=(V,E) be a graph.By a clique of G, we mean a complete subgraph of G [4, Definition 1.2.2]. We say that the clique number of G equals n if n is the largest positive integer such that K_n is a subgraph of G [4, p.185]. The clique number of a graph G is denoted by the notation $\omega(G)$. If G contains K_n as a subgraph for all $n \geq 1$, then we set $\omega(G) = \infty$.

A graph G is said to be planar if it can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G [4, Definition 8.1.1]. Two adjacent edges of a graph G are said to be in series if their common vertex is of degree two [5, p.9]. Two graphs are said to be homeomorphic if one graph can be obtained from the other graph by the creation of edges in series (i.e by insertion of vertices of degree two) or by the merger of edges in series[5, p.100]. Recall from [5, p.93] that K_5 is referred to as Kuratowski's first graph and $K_{3,3}$ is referred to as Kuratowski's second graph. A celebrated theorem of Kuratowski says that a necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [5, Theorem 5.9].

In view of Kuratowski's Theorem, [5, Theorem 5.9] we introduce the following definitions. We say that a graph G=(V,E) satisfies Ku_1 if G does not contain K_5 as a subgraph and we say that graph G=(V,E) satisfies Ku_2 if G does not contain $K_{3,3}$ as a subgraph. We say that a graph G = (V,E) satisfies Ku_1^* if G satisfies Ku_1 and moreover, G does not contain any subgraph homeomorphic to K_5. We say that a graph G = (V,E) satisfies Ku_2^* if G satisfies Ku_2 and moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

If a graph G is planar, then it follows from Kuratowski's theorem [5, Theorem 5.9] that G satisfies both Ku_1^* and Ku_2^*. Hence G satisfies both Ku_1 and Ku_2. It is interesting to note that a graph G may be nonplanar even if it satisfies both

Ku_1 and Ku_2. For example of this type refer [5, Figure 5.9(a), p.101] and the graph G in this example does not satisfies Ku_2^*. We do not know an example of a graph G such that G satisfies Ku_1 but G does not satisfy Ku_1^*.

The rings considered in this article are commutative with identity and are nonzero. A ring R which has a unique maximal ideal is referred to as a quasilocal ring. A ring R which has only a finite number of maximal ideals is referred to as a semiquasilocal ring. A Noetherian quasilocal (respctively, semiquasilocal) ring is referred to as a local (respectively, semilocal) ring. We denote the set of all maximal ideals of a ring R by $Max(R)$. We used $J(R)$ to denote Jacobson radical of ring R.

2. On the classification of rings R with $|Max(R)| \geq 3$ in order that $UG(R)$ is planar

Let R be a semiquasilocal ring with $|Max(R)| = n \geq 3$. We next try to classify such rings R in order that $UG(R)$ is planar. If $UG(R)$ is planar, then we know from Kuratowski's theorem [5,Theorem 5.9] that $UG(R)$ satisfies (Ku_2). Hence, we obtain from [10, Proposition 2.3] and [10, Remark 2.4] that there exist finite local rings $(R_i, \mathfrak{m}_i)$ for each $i \in \{1,2,\ldots,n\}$ such that $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$ as rings.

Lemma 2.1. *Let $n \geq 3$ and let $R = R_1 \times R_2 \times R_3 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a quasilocal ring for each $i \in \{1,2,\ldots,n\}$. If there exist at least two values of $i \in \{1,2,\ldots,n\}$ such that R_i is not a field, then $UG(R)$ does not satisfy (Ku_2).*

Proof. We are assuming that there are at least two values of $i \in \{1,2,\ldots,n\}$ such that R_i is not a field. Without loss of generality, we can assume that R_1 and R_2 are not fields. As $|\mathfrak{m}_i| \geq 2$ for each $i \in \{1,2\}$, we obtain from Lemma [10, Lemma 3.22] that $UG(R_1 \times R_2)$ does not satisfy (Ku_2). Hence, it follows from [10, Lemma 2.2] that $UG(R)$ does not satisfy (Ku_2). □

Remark 2.2. *Let $n \geq 3$ and let F_i be a field for each $i \in \{1,2,3,\ldots,n\}$. Let $R = F_1 \times F_2 \times F_3 \times \cdots \times F_n$. If $UG(R)$ satisfies (Ku_2), then the following hold.*

(i) $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_5\}$ for each $i \in \{1,2,3,\ldots,n\}$.

(ii) There exists at most one $i \in \{1,2,3,\ldots,n\}$ such that $|F_i| \geq 4$.

Proof. (i) Let $i \in \{1,2,3,\ldots,n\}$. We are assuming that $UG(R)$ satisfies (Ku_2). Then we obtain from [10, Lemma 2.2] that $UG(F_i)$ satisfies (Ku_2). Hence, it follows from [9, Lemmas 2.2 and 2.3] that $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_5\}$.
(ii) Suppose that $|F_i| \geq 4$ for at least two values of $i \in \{1,2,3,\ldots,n\}$. Without loss of generality, we can assume that $|F_1| \geq 4$ and $|F_2| \geq 4$. Note that $|U(F_1)| \geq 3$ and $|U(F_2 \times F_3 \times \cdots \times F_n)| \geq 3$. In such a case, we obtain from [10, Lemma 3.2] that $UG(R)$ does not satisfy (Ku_2). This is in contradiction to the assumption that $UG(R)$ satisfies (Ku_2). Therefore, there exists at most one $i \in \{1,2,3,\ldots,n\}$ such that $|F_i| \geq 4$. □

Lemma 2.3. *Let $n \geq 3$ and let $R = F_1 \times F_2 \times F_3 \times \cdots \times F_n$, where F_i is a field for each $i \in \{1,2,3,\ldots,n\}$. If $UG(R)$ satisfies (Ku_2^*), then $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4\}$ for each $i \in \{1,2,3,\ldots,n\}$.*

Proof. Assume that $UG(R)$ satisfies (Ku_2^*). Then $UG(R)$ satisfies (Ku_2). Hence, we obtain from Remark 2.2 (i) that $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_5\}$ for each $i \in \{1,2,3,\ldots,n\}$. We want to show that $F_i \neq \mathbb{Z}_5$ for each $i \in \{1,2,3,\ldots,n\}$. Suppose that $F_i = \mathbb{Z}_5$ for some $i \in \{1,2,3,\ldots,n\}$. Without loss of generality, we can assume that $F_1 = \mathbb{Z}_5$. In such a case, we know from Remark 2.2 (ii) that $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3\}$ for each $i \in \{2,\ldots,n\}$. Since $|U(\mathbb{Z}_5)| = 4$, it follows from [10, Lemma 3.2] that $|U(F_2 \times F_3 \times \cdots \times F_n)| \leq 2$. Hence, there exists at most one $i \in \{2,3,\ldots,n\}$ such that $F_i = \mathbb{Z}_3$. We consider the following cases.

Case(1) $F_i = \mathbb{Z}_2$ for each $i \in \{2,3,\ldots,n\}$

In this case, $R \cong \mathbb{Z}_5 \times T$ as rings, where $T = F_2 \times F_3 \times \cdots \times F_n$ is such that $char(T) = 2$. We know from [10, Proposition 3.12] that $UG(R)$ does not satisfy (Ku_2^*). This is in contradiction to the assumption that $UG(R)$ satisfies (Ku_2^*).

Case(2) $F_i = \mathbb{Z}_3$ for a unique $i \in \{2,3,\ldots,n\}$

Without loss of generality, we can assume that $F_2 = \mathbb{Z}_3$. Note that $F_i = \mathbb{Z}_2$ for each $i \in \{3,\ldots,n\}$. Let us denote the ring $F_3 \times \cdots \times F_n$ by T_1. Then $R \cong \mathbb{Z}_5 \times \mathbb{Z}_3 \times T_1$ as rings. Observe that $char(T_1) = 2$. Consider the mapping $f : V(UG(\mathbb{Z}_5 \times T_1)) = \mathbb{Z}_5 \times T_1 \to V(UG(\mathbb{Z}_5 \times \mathbb{Z}_3 \times T_1)) = \mathbb{Z}_5 \times \mathbb{Z}_3 \times T_1$ by $f(x,y) = (x,1,y)$ for any $(x,y) \in \mathbb{Z}_5 \times T_1$. It is clear that f is one-one and for any $(x_1,y_1),(x_2,y_2) \in \mathbb{Z}_5 \times T_1$ are adjacent in $UG(\mathbb{Z}_5 \times T_1)$ if and only if $f(x_1,y_1)$ and $f(x_2,y_2)$ are adjacent in $UG(\mathbb{Z}_5 \times \mathbb{Z}_3 \times T_1)$. Therefore, $UG(\mathbb{Z}_5 \times T_1)$ is isomorphic to a subgraph of $UG(\mathbb{Z}_5 \times \mathbb{Z}_3 \times T_1)$. We know from [10, Proposition 3.12] that $UG(\mathbb{Z}_5 \times T_1)$ does not satisfy (Ku_2^*). Hence, $UG(\mathbb{Z}_5 \times \mathbb{Z}_3 \times T_1)$ does not satisfy (Ku_2^*). This is in contradiction to the assumption that $UG(R)$ satisfies (Ku_2^*).

Thus if $UG(R)$ satisfies (Ku_2^*), then $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4\}$ for each $i \in \{1,2,3,\ldots,n\}$. □

Lemma 2.4. *Let $n \geq 3$ and let $R = F_1 \times F_2 \times F_3 \times \cdots \times F_n$, where F_i is a field for each $i \in \{1,2,3,\ldots,n\}$. Suppose that $F_i = \mathbb{F}_4$ for some $i \in \{1,2,3,\ldots,n\}$. If $UG(R)$ satisfies (Ku_2^*), then $F_j = \mathbb{Z}_2$ for all $j \in \{1,2,3,\ldots,n\} \backslash \{i\}$.*

Proof. We are assuming that $F_i = \mathbb{F}_4$ for some $i \in \{1,2,3,\ldots,n\}$ and $UG(R)$ satisfies (Ku_2^*). Without loss of generality, we can assume that $F_1 = \mathbb{F}_4$. Since $|U(\mathbb{F}_4)| = 3$ and $UG(R)$ satisfies (Ku_2), it follows from [10, Lemma 3.2] that $|U(F_2 \times F_3 \times \cdots \times F_n)| \leq 2$. We claim that $F_j = \mathbb{Z}_2$ for each $j \in \{2,3,\ldots,n\}$. Suppose that $F_j = \mathbb{Z}_3$ for some $j \in \{2,3,\ldots,n\}$. Without loss of generality, we can assume that $F_2 = \mathbb{Z}_3$. Since $|U(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 4$, it follows that $F_j = \mathbb{Z}_2$ for each $j \in \{3,\ldots,n\}$. Let us denote the ring $F_3 \times \cdots \times F_n$ by T_1. Observe that $char(T_1) = 2$ and $R \cong \mathbb{F}_4 \times \mathbb{Z}_3 \times T_1$ as rings. We know from [10, Proposition

3.18] that $UG(\mathbb{F}_4 \times \mathbb{Z}_3 \times T_1)$ does not satisfy (Ku_2^*). This is in contradiction to the assumption that $UG(R)$ satisfies (Ku_2^*).

Thus if $UG(R)$ satisfies (Ku_2^*), then $F_j = \mathbb{Z}_2$ for each $j \in \{1,2,3,\ldots,n\} \backslash \{i\}$. □

Proposition 2.5. *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times S$, where S is a nonzero ring. Then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. We consider the following cases.
Case (1) $2 \notin U(S)$

Let $V_1 = \{(0,1,0),(0,2,1),(1,2,1)\}$ and let $V_2 = \{(1,2,0),(2,2,0),(1,1,1)\}$. Note that V_1 and V_2 are independent sets of $UG(R)$, $(0,2,1)$ is adjacent to both $(1,2,0)$ and $(2,2,0)$ in $UG(R)$, and $(1,2,1)$ is adjacent to $(1,2,0)$ in $UG(R)$. Let H be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(1,0,1),(2,0,1),(1,0,0),(0,2,0),(2,2,1),(0,0,0)\}$. It is not hard to verify that $(0,1,0)-(1,0,1)-((1,2,0)$, $(0,1,0)-(2,0,1)-(2,2,0)$, $(0,2,1)-(1,0,0)-(1,1,1)$, $(1,2,1)-(0,2,0)-(2,2,1)-(2,2,0)$, and $(1,2,1)-(0,0,0)-(1,1,1)$ are paths in $UG(R)$. Consider the subgraph H_1 of H shown in Figure 1. Observe that H_1 is homeomorphic to $K_{3,3}$. Therefore, we obtain that $UG(R)$ does not satisfy (Ku_2^*).

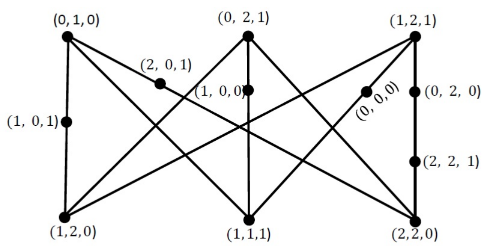

Figure 1. H_1

Case(2) $2 \in U(S)$

Let $V_1 = \{(0,1,2),(0,2,0),(0,2,2)\}$ and let $V_2 = \{(1,2,2),(2,0,2),(1,0,2)\}$. Let H be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(1,0,0)\}$. Note that (0, 2, 0) and $(0,2,2)$ are adjacent to each element of V_2 in H, $(0,1,2)$ is adjacent to both $(2,0,2)$ and $(1,0,2)$ in H, $(0,1,2)-(1,0,0)-(1,2,2)$ is a path in H. Let us denote the edges of H, $(1,0,0)-(0,2,2)$ and $(1,2,2)-(1,0,2)$ by e_1 and e_2. Let $H_2 = H - \{e_1, e_2\}$. The subgraph H_2 of $UG(R)$ is shown in Figure 2.

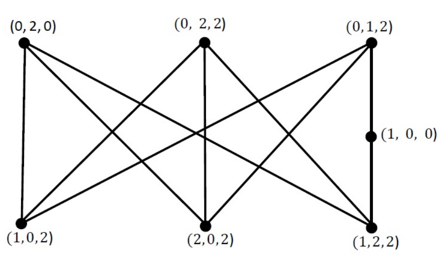

Figure 2. H_2

Observe that H_2 is homeomorphic to $K_{3,3}$. Hence, we obtain that $UG(R)$ does not satisfy (Ku_2^*). □

Proposition 2.6. *Let $R_1 = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors, $n \geq 1$) and let $R = R_1 \times \mathbb{Z}_6$. Then $UG(R)$ is planar.*

Proof. Observe that $|R| = 2^n \times 6$, $|U(R)| = |U(R_1)||U(\mathbb{Z}_6| = 2$, and $2 \notin U(R)$. Therefore, we obtain from [2, Proposition 3.4 (i)] that $deg_{UG(R)} r = 2$ for any $r \in R$. Note that any element of R_1 is of the form $(x_1, \ldots, x_n)$, where $x_i \in \{0,1\}$ for each $i \in \{1,\ldots,n\}$. Let $(x_1,\ldots,x_n)$ be any element of R_1. Let H be the component of $UG(R)$ containing $(x_1,\ldots,x_n,0)$. It is not hard to verify that H is the cycle Γ of length 6 given by $\Gamma : (x_1,\ldots,x_n,0)-(1+x_1,\ldots,1+x_n,1)-(x_1,\ldots,x_n,4)-(1+x_1,\ldots,1+x_n,3)-(x_1,\ldots,x_n,2)-(1+x_1,\ldots,1+x_n,5)-(x_1,\ldots,x_n,0)$. Similarly, it can be shown that for any $i \in \mathbb{Z}_6$, the component of $UG(R)$ containing $(x_1,\ldots,x_n,i)$ is a cycle of length 6. It is clear that the number of components of $UG(R)$ equals $\frac{|R|}{6} = \frac{2^n \times 6}{6} = 2^n$. Since any component of $UG(R)$ is planar, we obtain that $UG(R)$ is planar. □

Theorem 2.7. *Let $n \geq 3$ and let $R = F_1 \times F_2 \times F_3 \times \cdots \times F_n$, where F_i is a field for each $i \in \{1,2,3,\ldots,n\}$. Then the following statements are equivalent:*

(i) $UG(R)$ is planar.

(ii) $UG(R)$ satisfies both (Ku_1^*) and (Ku_2^*).

(iii) $UG(R)$ satisfies (Ku_2^*).

(iv) There exists at most one $i \in \{1,2,3,\ldots,n\}$ such that $F_i \neq \mathbb{Z}_2$. If $i \in \{1,2,3,\ldots,n\}$ is such that $F_i \neq \mathbb{Z}_2$, then $F_i \in \{\mathbb{Z}_3, \mathbb{F}_4\}$.

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [5, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ We are assuming that $UG(R)$ satisfies (Ku_2^*). We know from Lemma 2.3 that $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4\}$ for each $i \in \{1,2,3,\ldots,n\}$. If $F_i = \mathbb{F}_4$ for some $i \in \{1,2,3,\ldots,n\}$, then we know from Lemma 2.4 that $F_j = \mathbb{Z}_2$ for all $j \in \{1,2,3,\ldots,n\} \backslash \{i\}$. If $F_i = \mathbb{Z}_3$ for some $i \in \{1,2,3,\ldots,n\}$, then it follows from Lemma 2.5 that $F_j = \mathbb{Z}_2$ for all $j \in \{1,2,3,\ldots,n\} \backslash \{i\}$.

$(iv) \Rightarrow (i)$ If $F_i = \mathbb{Z}_2$ for all $i \in \{1,2,3,\ldots,n\}$, then we know from [10, Proposition 3.5] that $UG(R)$ is planar. If $F_i = \mathbb{F}_4$ for some $i \in \{1,2,3,\ldots,n\}$, then by hypothesis, $F_j = \mathbb{Z}_2$ for each $j \in \{1,2,3,\ldots,n\} \backslash \{i\}$. In such a case, it follows from [10, Proposition 3.9] that $UG(R)$ is planar. If $F_i = \mathbb{Z}_3$ for some $i \in \{1,2,3,\ldots,n\}$, then by hypothesis, $F_j = \mathbb{Z}_2$ for each $j \in \{1,2,3,\ldots,n\} \backslash \{i\}$. In this case, it follows from Proposition 2.6 that $UG(R)$ is planar. □

Let R be a semiquasilocal ring with $|Max(R)| = n \geq 3$. If $UG(R)$ satisfies (Ku_2), then we know from [10, Proposition 2.3 and Remark 2.4] that there exist finite local rings $(R_i, \mathfrak{m}_i)$ for each $i \in \{1,2,3,\ldots,n\}$ such that $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$ as rings. Let us denote $R_1 \times R_2 \times R_3 \times \cdots \times R_n$ by T. If $UG(T)$ satisfies (Ku_2), then we know from Lemma 2.1 that R_i is not a field for at most one $i \in \{1,2,3,\ldots,n\}$. We assume that R_i is not a field for some $i \in \{1,2,3,\ldots,n\}$

and try to classify such rings $T = R_1 \times R_2 \times R_3 \times \cdots \times \times R_n$ in order that $UG(T)$ is planar.

Remark 2.8. *Let $t \geq 2$ and let $R = R_1 \times F_2 \times \cdots \times F_t$, where R_1 is a quasilocal ring which is not a field and F_i is a field for each $i \in \{2, \ldots, t\}$. If $|F_i| \geq 3$ for some $i \in \{2, \ldots, t\}$, then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. Without loss of generality, we can assume that $|F_2| \geq 3$. Then either $t = 2$ or $t \geq 3$. If $t = 2$, then it follows from [10, Proposition 3.14] that $UG(R)$ does not satisfy (Ku_2^*). Suppose that $t \geq 3$. Let $T = F_3 \times \cdots \times F_n$. Note that $R \cong R_1 \times F_2 \times T$ as rings. In this case, we obtain from [10, Proposition 3.15] that $UG(R)$ does not satisfy (Ku_2^*). □

Theorem 2.9. *Let $n \geq 3$ and let $R = R_1 \times F_2 \times F_3 \times \cdots \times F_n$, where R_1 is a quasilocal ring which is not a field and F_i is a field for each $i \in \{2, 3, \ldots, n\}$. The following statements are equivalent:*

(i) $UG(R)$ is planar.

(ii) $UG(R)$ satisfies both (Ku_1^*) and (Ku_2^*).

(iii) $UG(R)$ satisfies (Ku_2^*).

(iv) R_1 is isomorphic to one of the rings from the collection $\{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ and $F_i = \mathbb{Z}_2$ for each $i \in \{2, 3, \ldots, n\}$.

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [5, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ We are assuming that $UG(R)$ satisfies (Ku_2^*).and so, $UG(R)$ satisfies (Ku_2). It follows from [10, Lemma 2.2] that $UG(R_1)$ satisfies (Ku_2). Therefore, we obtain from $(iii) \Rightarrow (iv)$ of [9, Lemma 2.4] that R_1 is isomorphic to one of the rings from the collection $\{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Moreover, we know from Remark 2.8 that $F_i = \mathbb{Z}_2$ for each $i \in \{2, 3, \ldots, n\}$.

$(iv) \Rightarrow (i)$ We are assuming that R_1 is isomorphic to one of the rings from the collection $\{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ and $F_i = \mathbb{Z}_2$ for each $i \in \{2, 3, \ldots, n\}$. Now, it follows from [10, Proposition 3.7] that $UG(R)$ is planar. □

References

[1] S. Akbari, B. Miraftar and R. Nikandish, A note on comaximal ideal graph of commutative rings, arxiv, 2013.

[2] N. Ashrafi, H.R. Mainmani, M.R. Pournaki and S. Yassemi, Unit graph associated with rings, *Comm. Algebra,* 38(2010), 2851–2871.

[3] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley publishing Company, 1969.

[4] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Springer-Verlag, New York, 2000.

[5] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice Hall of India Private Limited, New Delhi, 1994.

[6] M.I. Jinnah and S.C. Mathew, When is the comaximal graph split?, *Comm. Algebra* 40(7)(2012), 2400–2404.

[7] H.R. Maimani, M. Salimi, A. Sattari, and S. Yassemi, Comaximal graph of commutative rings, *J. Algebra,* 319(2008), 1801–1808.

[8] S.M. Moconja and Z.Z. Petrovic, On the structure of comaximal graphs of commutative rings with identity, *Bull. Aust. Math. Soc.,* 83(2011), 11–21.

[9] J. Parejiya, S. Patat and P. Vadhel, Planarity of unit graph Planarity Part $-I$ Local Case, *Malaya Journal of Matematik*, 8(3)(2020), 1155–1157.

[10] J. Parejiya, P. Vadhel and S. Patat, Planarity of unit graph Planarity Part $-I$ Local Case, *Malaya Journal of Matematik* , 8(3)(2020), 1162–1170.

[11] K. Samei, On the comaximal graph of a commutative ring, *Canad. Math. Bull.,* 57(2)(2014), 413–423.

[12] P.K. Sharma and S.M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra,* 176(1995), 124–127.

[13] S. Visweswaran and Jaydeep Parejiya, When is the complement of the comaximal graph of a commutative ring planar?, *ISRN Algebra,* 2014 (2014), 8 pages.

[14] M.Ye and T.Wu, Comaximal ideal Graphs of commutative rings, *J. Algebra Appl.,* 6(2012), DOI : 10.1142/S0219498812501149.

⋆⋆⋆⋆⋆⋆⋆⋆⋆

ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

⋆⋆⋆⋆⋆⋆⋆⋆⋆