Malaya Journal of Matematik, Vol. 8, No. 4, 1417-1426, 2020

https://doi.org/10.26637/MJM0804/0014

On intuitionistic L-fuzzy primary and P-primary submodules

P. K. Sharma[1]* and Kanchan[2]

Abstract

In the present manuscript, we introduce and study the notion of primary submodules as well as P-primary submodules of a module in the intuitionistic L-fuzzy environment. Apart from investigating basic properties of these submodules, we explore some foundational results analogous to corresponding submodules. A suitable characterization of intuitionistic L-fuzzy primary (P-primary) submodules in terms of primary (P-primary) submodules are presented.

Keywords

Intuitionistic L-fuzzy module(ILFM); Intuitionistic L-fuzzy (prime, primary) ideal; Intuitionistic L-fuzzy (primary, P-primary) submodule.

AMS Subject Classification

03F55, 16D10, 13C99.

[1] *Post-Graduate Department of Mathematics, DAV College, Jalandhar, Punjab, India.*

[2] *Research Scholar, IKGPT University, Jalandhar, Punjab, India.*

***Corresponding author**: [1] pksharma@davjalandhar.com; [2] kanchan4usoh@gmail.com

Article History: Received **21** July **2020**; Accepted **02** September **2020**

Contents

1. Introduction

One of the famous problems in ideal (module) theory is the decomposition of an ideal (module) in terms of intersection of finite number of primary ideals (submodules). It gives the algebraic footing for the decomposition of an algebraic variety in terms of irreducible components. From another perspective, it is an extension of factoring an integer as a product of prime′s powers. A prime ideal in a ring R is in some sense a generalization of a prime number. Also, primary ideal is some sort of generalization of prime ideal. An ideal $I(\neq R)$ in a ring R is called primary if

$$ab \in I \Rightarrow \text{either } a \in I \text{ or } b^m \in I \text{ for some } m \in \mathbb{N}.$$

Viz., I is primary ideal $\Leftrightarrow R/I \neq 0$ and every non-zero divisors in R/I is nilpotent. In a similar manner, a primary submodule is a generalization of prime submodule in module theory. A proper submodule N of an R-module M is called a primary submodule if

$$xy \in N \Rightarrow \text{either } x \in N \text{ or } y^m M \subseteq N \text{ for some } m \in \mathbb{N}.$$

Viz., N is primary submodule of M, if whenever $xy \in N \Rightarrow$ either $x \in N$ or y^n belong to the annihilator of M/N, $\sqrt{(N:M)}$. Also when N is a primary submodule of M then $\sqrt{(N:M)}$ is a prime ideal of R. If $P=\sqrt{(N:M)}$, then N is called a P-primary submodule of M (see [9])

After the foundation of the theory fuzzy sets by Zadeh [30]. Mathematician started fuzzifying the algebraic concepts. Rosenfeld [23] was the first one to introduce the notion of fuzzy subgroup of a group. The concept of fuzzy subrings and ideals were introduced and studied by Liu in [13]. The notion like fuzzy (prime, primary, semi-prime, nil radical etc.) ideals were studied by Swamy at al. in [15] and Malik et al. in [28]. The concept of fuzzy modules was introduced by Negoita and Ralescu in [20]. The notion of fuzzy prime submodule and fuzzy primary submodules was studied by Mashinchi and Zaidi in [16], [29] and Makamba and Murali in [14], which was further extended by Mohammed M. Ali

Radman Al-Shamiri in [18]. A detailed study of different algebraic structures in fuzzy setting can be found in [19].

One of the prominent generalizations of fuzzy sets theory is the theory of intuitionistic fuzzy sets introduced by Atanassov [1], [2], [3] and [4]. Biswas introduced the notion of intuitionistic fuzzy subgroup of a group in [7]. The concept of intuitionistic fuzzy subrings and ideals was introduced and studied by Hur and other in [10]. The notion like intuitionistic fuzzy (prime, primary, semi-prime, nil etc.) ideals were studied in [5], [12], [17], [21] and [26]. The notion of intuitionistic fuzzy submodule of a module was introduced by Davvaz et al. in [8] which was further studied by Basnet, Isaac and John, Rahman and Saikia, Sharma etc. in [6], [11], [22], [24] and [25]. The notation of intuitionistic fuzzy prime submodules was introduced by Sharma et al. in [27].

The purpose of this paper is to introduced and investigate primary submodule and P-primary submodule in the intuitionistic fuzzy environment and lay down the foundation for the primary decomposition theorem in the intuitionistic fuzzy setting.

2. Preliminaries

During this article, R stand for a commutative ring with identity, M stand for a unitary R-module with zero element θ and L denote a complete lattice with smallest element 0 and largest element 1.

An element $1 \neq \alpha \in L$, is termed as prime in L if for any $a;b \in L$ such that $a \wedge b \leq \alpha$ implies either $a \leq \alpha$ or $b \leq \alpha$.

Via an intuitionistic L-fuzzy subset (ILFS) A of X we mean a mapping $A = (f_A, g_A) : X \to L \times L$. We denote by $ILFS(X)$ the set of all ILFSs of X. For $A, B \in ILFS(X)$ we say $A \subseteq B$ iff $f_A(x) \leq f_B(x)$ and $g_A(x) \geq g_B(x)$ $\forall x \in X$. Let $P \in ILFS(X)$ and $a, b \in L$. Then the crisp set $P_{(a,b)} = \{r \in X : f_P(r) \geq a$ and $g_P(r) \leq b\}$ is called the (a,b)-cut subset of A. By an intuitionistic L-fuzzy point $(ILFP)$ $r_{(a,b)}$ of $X, r \in X$ and $a, b \in L \backslash \{0\}$ with $a \vee b \leq 1$, we mean $r_{(a,b)} \in ILFS(X)$ defined by

$$r_{(a,b)}(s) = \begin{cases} (a,b), & \text{if } s = r \\ (0,1), & \text{if otherwise.} \end{cases}$$

If $r_{(a,b)}$ is an ILFP of X and $r_{(a,b)} \subseteq A \in ILFS(X)$, we write $r_{(a,b)} \in A$.

Definition 2.1 ([17]). *If $A \in ILFS(R)$, then A is termed as intuitionistic L-fuzzy ideal $(ILFI)$ of R if $\forall r, s \in R$, following holds*

(i) $f_A(r-s) \geq f_A(r) \wedge f_A(s)$;

(ii) $f_A(rs) \geq f_A(r) \vee f_A(s)$;

(iii) $g_A(r-s) \leq g_A(r) \vee g_A(s)$;

(iv)$g_A(rs) \leq g_A(r) \wedge g_A(s)$.

Definition 2.2 ([27]). *If $A \in ILFS(M)$, then A is termed as intuitionistic L-fuzzy module $(ILFM)$ of M if $\forall m, n \in M, r \in R$, following holds*

(i) $f_A(m-n) \geq f_A(m) \wedge f_A(n)$;

(ii) $f_A(rm) \geq f_A(m)$;

(iii) $f_A(\theta) = 1$;

(iv) $g_A(m-n) \leq g_A(m) \vee g_A(n)$;

(v)$g_A(rm) \leq g_A(m)$;

(vi)$g_A(\theta) = 0$.

We refer by $IF_L(M)$, the set of ILFMs of M and refer $IF_L(R)$, the set of ILFIs of R. Note that when $R = M$, then $A \in IF_L(M)$ iff $f_A(\theta) = 1, g_A(\theta) = 0$ and $A \in IF_L(R)$.

Definition 2.3. *If $C \in ILFS(R)$ and $B \in ILFS(M)$, then the product $C \circ B$ and CB are defined as follows: For all $x \in M$,*

$$f_{C \circ B}(x) = \begin{cases} Sup[f_C(r) \wedge f_B(m)] & \text{if } x = rm, r \in R, m \in M \\ 0, & \text{otherwise} \end{cases}$$

$$g_{C \circ B}(x) = \begin{cases} Inf[g_C(r) \vee g_B(m)] & \text{if } x = rm, r \in R, m \in M \\ 1, & \text{otherwise .} \end{cases}$$

$$f_{CB}(x) = \begin{cases} Sup[Inf_{i=1}^{n}\{f_C(r_i) \wedge f_B(m_i)\}] & \text{if } x = \Sigma_{i=1}^{n} r_i m_i \\ 0, & \text{otherwise} \end{cases}$$

$$g_{CB}(x) = \begin{cases} Inf[Sup_{i=1}^{n}\{g_C(r_i) \vee g_B(m_i)\}] & \text{if } x = \Sigma_{i=1}^{n} r_i m_i \\ 1, & \text{otherwise .} \end{cases}$$

Clearly, $C \circ B \subseteq CB$.

The next lemma can be found in [5], [17]. It provides the pivotal relation between ILFIs and ILFMs.

Lemma 2.4. *Let $A, B \in IF_L(M), C \in IF_L(R)$ and let L be a complete lattice satisfying the infinite distributive law. Then:*

(i) $CB \subseteq A$ iff $C \circ B \subseteq A$.

(ii) If $r_{(s,t)} \in ILFP(R), x_{(p,q)} \in ILFP(M)$. Then $r_{(s,t)} \circ x_{(p,q)} = (rx)_{(s \wedge p, t \vee q)}$.

iii If $f_C(0) = 1, g_C(0) = 0$ then $CA \in IF_L(M)$.

(iv) Let $r_{(s,t)} \in ILFP(R)$. Then for all $x \in M$,

$$f_{r_{(s,t)} \circ B}(x) = \begin{cases} Sup[s \wedge f_B(m)] & \text{if } x = rm, r \in R, m \in M \\ 0, & \text{otherwise} \end{cases}$$

$$g_{r_{(s,t)} \circ B}(x) = \begin{cases} Inf[t \vee g_B(m)] & \text{if } x = rm, r \in R, m \in M \\ 1, & \text{otherwise .} \end{cases}$$

The next theorem provide relationship between ILFMs and submodules of M.

Theorem 2.5. *Let $A \in ILFS(M)$. Then A is ILFM iff $A_{(\alpha,\beta)}$ is R-submodule of $M, \forall \alpha, \beta \in L$ with $\alpha \vee \beta \leq 1$.*

Proof. Simple proof □

Definition 2.6. *([5], [21]) For non-constant $C \in ILFI(R), C$ is termed as intuitionistic L-fuzzy prime (respectively, primary) ideal of R if for any $x_{(p,q)}, y_{(r,s)} \in ILFP(R)$ such that $x_{(p,q)}y_{(r,s)} \in C$ inferred that $x_{(p,q)} \in C$ or $y_{(r,s)} \in C$ (or respectively, $x_{(p,q)} \in C$ or $y^n_{(r,s)} \in C$, for some $n \in \mathbb{N}$).*

The set of ILF-prime ideals of R is written as $IF_L Spec(R)$.

3. Intuitionistic L-Fuzzy Primary Submodules(ideals)

In this segment, we will explore the characterization of intuitionistic L-fuzzy primary submodule (ILFPSM) of M.

Definition 3.1. *For $A, B \in IF_L(M), A$ is termed as intuitionistic L-fuzzy submodule(ILFSM) of B iff $A \subseteq B$. In case $B = \chi_M$, then, A is called an ILFSM of M.*

Definition 3.2. *Let A be an ILFSM of B, A is called an ILFPSM of B, if $r_{(s,t)} \in ILFP(R), x_{(p,q)} \in ILFP(M)$ ($r \in R, x \in M, s,t,p,q \in L$), $r_{(s,t)}x_{(p,q)} \in A \Rightarrow x_{(p,q)} \in A$ or $r^n_{(s,t)}B \subseteq A$, for some $n \in \mathbb{N}$.*

In particular, taking $B = \chi_M$, if for $r_{(s,t)} \in ILFP(R), x_{(p,q)} \in ILFP(M)$ we have $r_{(s,t)}x_{(p,q)} \in A$ inferred as $x_{(p,q)} \in A$ or $r^n_{(s,t)}\chi_M \subseteq A$, for some $n \in \mathbb{N}$, then A is called an ILFPSM of M.

The subsequent result authenticate the coincidence between ILFPSM and intuitionistic L-fuzzy primary ideal (ILFPI).

Theorem 3.3. *If $M = R$, then $A \in ILFS(M)$, is an ILFPSM of M iff $A \in IF_L(R)$ is an ILFPI.*

Proof. Let A be an ILFPSM of M. As $A \in IF_L(M)$ and R is commutative ring, $A \in IF_L(R)$.
For $a_{(p,q)}, b_{(s,t)} \in ILFP(R), a_{(p,q)}b_{(s,t)} \in A$ inferred that $a_{(p,q)} \in A$ or $b^n_{(s,t)}\chi_M \subseteq A$, for some $n \in \mathbb{N}$.

If $a_{(p,q)} \in A$, then A is an ILFPI.

If $b^n_{(s,t)}\chi_M \subseteq A$, then $f_{b^n_{(s,t)}}\chi_M(b^n m) \leq f_A(b^n m)$ and $g_{b^n_{(s,t)}}\chi_M(b^n m) \geq g_A(b^n m), \forall m \in M$ and for some $n \in \mathbb{N}$. As R has identity so $b^n = b^n 1$ and $f_{b^n_{(s,t)}}\chi_M(b^n 1) = s \leq f_A(b^n)$ and $g_{b^n_{(s,t)}}\chi_M(b^n 1) = t \geq g_A(b^n)$ inferred that $s = f_{b^n_{(s,t)}}(b^n) \leq f_A(b^n)$ and $t = g_{b^n_{(s,t)}}(b^n) \geq g_A(b^n)$, hence $b^n_{(s,t)} \in A$.

Conversely, let A be an ILFPI of R. Then $A \subset \chi_R$ and $A \in IF_L(M)$. Now, suppose $r_{(s,t)}x_{(p,q)} \in A$ for any $r_{(s,t)} \in ILFP(R), x_{(p,q)} \in ILFP(M)$.

If $x_{(p,q)} \in A$, then A is an ILFPSM of M.

If $x_{(p,q)} \notin A$ then $r^n_{(s,t)} \in A \Rightarrow f_{r^n_{(s,t)}}\chi_M(r^n m) = s \leq f_A(r^n) \leq f_A(r^n m)$ and $g_{r^n_{(s,t)}}\chi_M(r^n m) = t \geq g_A(r^n) \geq g_A(r^n m)$ by the definition of ILFPI of R. Thus, $r^n_{(s,t)}\chi_M \subseteq A$. □

The next theorem, connects ILFPSM to primary submodules, which will help in proving Theorem (3.6).

Theorem 3.4. *Let A be an ILFPSM of B. If $A_{(\alpha,\beta)} \neq B_{(\alpha,\beta)}$, $\alpha, \beta \in L$, then $A_{(\alpha,\beta)}$ is a primary submodule of $B_{(\alpha,\beta)}$.*

Proof. Suppose $A_{(\alpha,\beta)} \neq B_{(\alpha,\beta)}$ and $rx \in A_{(\alpha,\beta)}$ for some $r \in R, x \in M$. If $rx \in A_{(\alpha,\beta)}$, then $f_A(rx) \geq \alpha$ and $g_A(rx) \leq \beta \Rightarrow (rx)_{(\alpha,\beta)} = r_{(\alpha,\beta)}x_{(\alpha,\beta)} \in A$, since A is an ILFPSM of B, either $x_{(\alpha,\beta)} \in A$ or $r^n_{(\alpha,\beta)}B \subseteq A$, for some $n \in \mathbb{N}$.

Case(i) If $x_{(\alpha,\beta)} \in A$ then $f_A(x) \geq \alpha$ and $g_A(x) \leq \beta \Rightarrow x \in A_{(\alpha,\beta)}$.

Case(ii) Let $r^n_{(\alpha,\beta)}B \subseteq A$, then for any $w \in r^n B_{(\alpha,\beta)}, w = r^n z$, for some $z \in B_{(\alpha,\beta)}$. So, $f_B(z) \geq \alpha$ and $g_B(z) \leq \beta$. Now, $\alpha = \alpha \wedge f_B(z) \leq Sup\{\alpha \wedge f_B(x) : w = r^n x\} = f_{r^n_{(\alpha,\beta)}B}(w) \leq f_A(w)$. Similarly, we have $\beta = \beta \vee g_B(z) \geq Inf\{\beta \vee g_B(x) : w = r^n x\} = g_{r^n_{(\alpha,\beta)}B}(w) \geq g_A(w)$.
Thus, $w \in A_{(\alpha,\beta)}$. Thereby $r^n B_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a primary submodule of $B_{(\alpha,\beta)}$. □

Corollary 3.5. *Let A be an ILFPSM of M. Then A_* is a primary submodule of M.*

Proof. Follows from Theorem (3.4) because $A_{(\alpha,\beta)} = A_*$, for $\alpha = f_A(\theta)$ and $\beta = g_A(\theta)$. □

Remark 3.6. *The reverse of Theorem (3.4) may not be true, see the subsequent example;*

Example 3.7. *Let $L = [0,1], M = R = Z$. Define $A, B \in ILFS(M)$ as:*

$$(f_A(x), g_A(x)) = \begin{cases} (1,0), & \text{if } x = 0 \\ (0.5, 0.3), & \text{if } x \in 4Z - \{0\} \\ (0,1), & \text{if otherwise} \end{cases}$$

and

$$(f_B(x), g_B(x)) = \begin{cases} (1,0), & \text{if } x \in 4Z \\ (0.5, 0.3), & \text{if } x \in 2Z - 4Z \\ (0,1), & \text{if otherwise.} \end{cases}$$

Clearly, $A, B \in IF_L(M)$ with $A \subseteq B$. By some manipulation we can see that $\forall \alpha, \beta \in (0,1]$, $A_{(\alpha,\beta)}$ is a primary submodule *of $B_{(\alpha,\beta)}$. But it can be easily checked that A is not ILFPSM of M. For if we take $x=5, r=4$ then $(5)_{(1/3,1/2)}(4)_{(2/3,1/3)} = (20)_{(1/3,1/2)} \subseteq A$, but $(4)_{(2/3,1/3)} \notin A$ and $(5^n)_{(1/3,1/2)}B \nsubseteq A, \forall n \in \mathbb{N}$.*

The following theorem characterized, ILFPSM completely.

Theorem 3.8. *(a) Let N be a primary submodule of M and α a prime element in L. If A is an ILFS of M defined by*

$$f_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_A(x) = \begin{cases} 0, & \text{if } y \in N \\ \alpha', & \text{otherwise.} \end{cases}$$

for all $x \in M$, where α' is complement of α in L. Then A is an ILFPSM of M.
(b)Conversely, any ILFPSM can be acquired as in (a).

Proof. **(a)** Since $N \neq M$, is a primary submodule of M, we have that A is non-constant ILFSM of M. We show that A is an ILFPSM of M.

Suppose $r_{(s,t)} \in ILFP(R), x_{(p,q)} \in ILFP(M)$ are such that $r_{(s,t)}x_{(p,q)} \in A$ and $x_{(p,q)} \notin A$. If $x_{(p,q)} \notin A$ then $f_A(x) = \alpha$ and $g_A(x) = \alpha'$, hence $x \notin N$.

If $r_{(s,t)}x_{(p,q)} \in A$, then $f_{(rx)_{(s\wedge p, t\vee q)}}(rx) \leq f_A(rx)$ and $g_{(rx)_{(s\wedge p, t\vee q)}}(rx) \geq g_A(rx) \Rightarrow s\wedge p \leq f_A(rx)$ and $t \vee q \geq g_A(rx)$.

If $f_A(rx) = 1$ and $g_A(rx) = 0$, so $rx \in N$. As $x \notin N$ and N is a primary submodule of M, we have $r^nM \subseteq N$, for some $n \in \mathbb{N}$. Hence $f_A(r^nm) = 1$ and $g_A(r^nm) = 0$, for all $m \in M$. Thus $f_{r^n_{(s,t)}\chi_M}(r^nm) = s \leq f_A(r^nm)$ and $g_{r^n_{(s,t)}\chi_M}(r^nm) = t \geq g_A(r^nm)$.

If $f_A(r^nx) = \alpha$ and $g_A(r^nx) = \alpha'$, then $s\wedge p \leq \alpha$ and $t \vee q \geq \alpha'$. As α is prime element of L, we have $s \wedge p \leq \alpha$ and $p \nleq \alpha$ implies $s \leq \alpha$ and $t\vee q \geq \alpha'$ implies $t' \vee q' \geq \alpha$ and $q' \nleq \alpha$ implies $t' \leq \alpha$ i.e., $t \geq \alpha'$.

Thus $f_{r^n_{(s,t)}\chi_M}(w) = s \leq \alpha \leq f_A(w)$ and $g_{r^n_{(s,t)}\chi_M}(w) = t \geq \alpha' \geq g_A(w)$, for all $w \in M$. Therefore, $r^n_{(s,t)}\chi_M \subseteq A$. Hence A is an ILFPSM of M.

(b) Let A be an ILFPSM of M. We show that A can be represented in the form

$$f_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_A(x) = \begin{cases} 0, & \text{if } y \in N \\ \alpha', & \text{otherwise.} \end{cases}$$

$\forall x \in M$, here α' is complement of the prime element α in L.

Claim (1) $A_* = \{x \in M : f_A(x) = f_A(\theta) \text{ and } g_A(x) = g_A(\theta)\}$ is a primary submodule of M.

Since A is a non-constant ILFPSM of M, so $A_* \neq M$. For all $r \in R, m \in M$, if $rm \in A_*$ implies $f_A(rm) = f_A(\theta)$ and $g_A(rm) = g_A(\theta)$ so that
$(rm)_{(f_A(\theta),g_A(\theta))} = r_{(f_A(\theta),g_A(\theta))}m_{(f_A(\theta),g_A(\theta))} \in A$, then
$m_{(f_A(\theta),g_A(\theta))} \in A$ or $r^n_{(f_A(\theta),g_A(\theta))}\chi_M \subseteq A$, for some $n \in \mathbb{N}$.

Case(i) If $m_{(f_A(\theta),g_A(\theta))} \in A$, then $f_A(\theta) \leq f_A(m)$ and $g_A(\theta) \geq g_A(m)$ but $f_A(\theta) \geq f_A(m)$ and $g_A(\theta) \leq g_A(m)$ [by definition of *ILFSM*]. Hence $f_A(m) = f_A(\theta)$ and $g_A(m) = g_A(\theta)$ so $m \in A_*$.

Case(ii) If $r^n_{(f_A(\theta),g_A(\theta))}\chi_M \subseteq A$, then $f_A(\theta) \leq f_A(r^nm)$ and $g_A(\theta) \geq g_A(r^nm)$, thus $r^nm \in A_*$ for all $m \in M$, for some $n \in \mathbb{N}$. Now, $\theta \in N$ and $f_A(\theta) = 1, g_A(\theta) = 0$. For all $x \in A_*, f_A(\theta) = f_A(x) = 1$ and $g_A(\theta) = g_A(x) = 0$. Now, $A_* = N$.

Claim (2) A has two values.

As A_* is a primary submodule of M, $A_* \neq M$. Then $\exists\, z \in M\backslash A_*$. We will show that $f_A(y) = f_A(z) < f_A(\theta)$ and $g_A(y) = g_A(z) > g_A(\theta)$, $\forall y \in M$ such that $y \notin A_*$. Now $z \notin A_* \Rightarrow f_A(z) < 1 = f_A(\theta)$ and $g_A(z) > 0 = g_A(\theta)$ so $z_{(1,0)} \notin A$ and $z_{(f_A(z),g_A(z))} = z_{(1,0)}1_{(f_A(z),g_A(z))} \in A$. Thus $1^n_{(f_A(z),g_A(z))}\chi_M \subseteq A$, since $w = 1^n.w$, for all $w \in M$, we have $f_A(z) \leq f_A(w)$ and $g_A(z) \geq g_A(w)$.

Let $w = y$. Then, $f_A(z) \leq f_A(y)$ and $g_A(z) \geq g_A(y)$. In a same manner, $f_A(y) \leq f_A(z)$ and $g_A(y) \geq g_A(z)$. Hence $f_A(z) = f_A(y)$ and $g_A(z) = g_A(y)$.

Claim (3) Let $f_A(z) = \alpha$ and $g_A(z) = \alpha'$, where α is prime element in L and α' be its complement in L. First, let $s \wedge p \leq \alpha$ and $t \vee q \geq \alpha'$ i.e., $t' \wedge q' \leq \alpha$ and let $p \nleq \alpha$ and $q' \nleq \alpha$.

Suppose $x \in M\backslash A_*$, then $x_{(p,q)} \notin A$. Hence $1_{(s,t)}x_{(p,q)} = x_{(s\wedge p, t\vee q)} \in A \Rightarrow 1_{(s,t)}\chi_M \subseteq A$, and for all $w \in M$, $f_{1_{(s,t)}\chi_M}(w) \leq f_A(w)$ and $g_{1_{(s,t)}\chi_M}(w) \geq g_A(w)$. Let $w = x$. Then,
$s = \mu_{1_{(s,t)}\chi_M}(w) \leq f_A(x) = \alpha$ and $t = g_{1_{(s,t)}\chi_M}(w) \geq g_A(x) = \alpha'$. Thus $s \leq \alpha$ and $t' \leq \alpha$. Thus, every ILFPSM of M can be represented in the following manner

$$f_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \alpha', & \text{otherwise.} \end{cases}$$

$\forall x \in M$, here α' is complement of the prime element α in L and N is a primary submodule of M. ☐

The above theorem is helpful in settling whether an ILFSM is primary or not. This fact is illustrated in the next example:

Example 3.9. *Let $M = Z = R$. Then M is a Z-module. Define*

$A \in ILFS(M)$ *as:*

$$f_A(x)=\begin{cases}1, & if\ x\in<p^k>\\ 0.25, & if\ otherwise\end{cases};\ g_A(x)=\begin{cases}0, & if\ x\in<p^k>\\ 0.75, & otherwise.\end{cases}$$

where p is a prime integer and $k>1$. Then A is an ILFPSM of Z, since $<p^k>$ is a primary submodule of M and 0.25 is a prime element in $[0,1]$. Notice that A is not an ILF-prime submodule of M.

In the two succeeding theorems we shall investigate both the image and inverse image of an ILFPSM under a R-module epimorphism.

Theorem 3.10. *Let $h: M\to M_1$ be an R-modules epimorphism, and suppose that L is distributive. If A is an ILFPSM of M such that $\chi_{kerh}\subseteq A$, then $h(A)$ is an ILFPSM of M_1.*

Proof. Now it is easy to see that $h(A)$ is an IFSM of M_1.

We show that $h(A)$ is an ILFPSM of M_1. Since A is an ILFPSM of M, so A is of the form

$$f_A(x)=\begin{cases}1, & \text{if } x\in N\\ \alpha, & \text{if otherwise}\end{cases};\quad g_A(x)=\begin{cases}0, & \text{if } x\in N\\ \alpha', & \text{otherwise.}\end{cases}$$

$\forall x\in M$, where α' is complement of the prime element α in L and $N=A_*$ is a primary submodule of M.

We first claim that if A_* is a primary submodule of M and $\chi_{kerh}\subseteq A$, then $h(A_*)$ is primary submodule of M_1.

Let $x\in\chi_{kerh}$. Then $f_{\chi_{kerh}}(x)=1\le f_A(x)$ and $g_{\chi_{kerh}}(x)=0\ge h_A(x)$ implies that $f_A(x)=f_A(\theta)$ and $g_A(x)=g_A(\theta)\Rightarrow x\in A_*$. Thus, $kerh\subseteq A_*$.

For all $r\in R, w\in M_1, rw\in h(A_*)$, $\exists\ z\in A_*$ such that $rw=h(z)$. Since h is an epimorphism $\exists\ m\in M$ such that $rw=rh(m)=h(rm)=h(z)$. Now, $rm\in A_*$ and A_* is a primary submodule of M, so either $m\in A_*$ or $r^nM\subseteq A_*$, for some $n\in\mathbb{N}$.

If $m\in A_*$, then $w=h(m)\in h(A_*)$ and if $r^nM\subseteq A_*$, then $r^nM_1=h(r^nM)\subseteq h(A_*)$.

Thus $h(A_*)$ is a primary submodule of M_1. Also, because α is a prime element in L, so by Theorem (3.6), for all $w\in M_1$,

$$f_{h(A)}(w)=\begin{cases}1, & \text{if } w\in h(A_*)\\ \alpha, & \text{if otherwise}\end{cases}$$

$$g_{h(A)}(w)=\begin{cases}0, & \text{if } w\in h(A_*)\\ \alpha', & \text{otherwise.}\end{cases}$$

Hence $h(A)$ is an ILFPSM of M_1. □

Theorem 3.11. *Let $h: M\to M_1$ be a R-module epimorphism. If B is an ILFPSM of M_1, then $h^{-1}(B)$ is an ILFPSM of M.*

Proof. Let B be an ILFPSM of M_1. Then

$$f_B(x)=\begin{cases}1, & \text{if } x\in B_*\\ \alpha, & \text{if otherwise}\end{cases};\quad g_B(x)=\begin{cases}0, & \text{if } x\in B_*\\ \alpha', & \text{otherwise.}\end{cases}$$

$\forall x\in M_1$, here α' is complement of the prime element α in L and B_* is a primary submodule of M_1.

We first show that $h^{-1}(B_*)$ is a primary submodule of M.

For all $r\in R, m\in M$, if $rm\in h^{-1}(B_*)\Rightarrow h(rm)\in B_*$, i.e., $rh(m)\in B_*$. As B_* is primary submodule of M_1. Therefore, either $h(m)\in B_*$ or $r^nM_1\subseteq B_*$, for some $n\in\mathbb{N}$.

If $h(m)\in B_*$, then $m\in h^{-1}(B_*)$ and if $r^nM_1\subseteq B_*$, then $r^nh(M)=h(r^nM)\subseteq B_*\Rightarrow r^nM\subseteq h^{-1}(B_*)$. Hence

$$f_{h^{-1}(B)}(x)=\begin{cases}1, & \text{if } x\in h^{-1}(B_*)\\ \alpha, & \text{if otherwise}\end{cases}$$

$$g_{h^{-1}(B)}(x)=\begin{cases}0, & \text{if } x\in h^{-1}(B_*)\\ \alpha', & \text{otherwise.}\end{cases}$$

Hence $h^{-1}(B)$ is an ILFPSM of M. □

4. Intuitionistic L-fuzzy P-primary submodules(ILFPPSM)

For any submodule N of a module M, the colon ideal of M into N is denoted by $(N:M)=\{r|r\in R, rM\subseteq N\}=Ann(M/N)$ and the radical of $(N:M)$ is denoted by $\sqrt{(N:M)}=\{r|r\in R, \exists n\in\mathbb{N}\text{ such that } r^nM\subseteq N\}$.

In this segment we introduce and study the notion of residual quotients of ILFPSMs.

Definition 4.1. *([25]) For $P,Q\in IF_L(M)$ and $S\in ILFI(R)$. Then the residual quotient $(P:Q)$ and $(P:S)$ are given by*

(i) $(P:Q)=\bigcup\{r_{(\alpha,\beta)}: r\in R, \alpha,\beta\in L, \alpha\vee\beta\le 1$ such that $r_{(\alpha,\beta)}\cdot Q\subseteq P\}$

(ii) $(P:S)=\bigcup\{x_{(\alpha,\beta)}: x\in M, \alpha,\beta\in L, \alpha\vee\beta\le 1$ such that $S\cdot x_{(\alpha,\beta)}\subseteq P\}$.

Note that here $f_{(P:Q)}(r)=Sup\{\alpha\in L|r_{(\alpha,\beta)}\cdot Q\subseteq P\}$ and $g_{(P:Q)}(r)=Inf\{\beta\in L|r_{(\alpha,\beta)}\cdot Q\subseteq P\}$, $\forall r\in R$ and

$f_{(P:S)}(x)=Sup\{\alpha\in L|S\cdot x_{(\alpha,\beta)}\subseteq P\}$ and $g_{(P:S)}(x)=Inf\{\beta\in L|S\cdot x_{(\alpha,\beta)}\subseteq P\}$, $\forall x\in M$.

It is shown in ([25]) that $(P:Q)\in ILFI(R)$ and $(P:S)\in IF_L(M)$.

Theorem 4.2. *([25]) For $P,Q\in IF_L(M)$ and $S\in ILFI(R)$. Then we have* *(i)* $(P:Q)\cdot Q\subseteq P$ *;*

(ii) $S\cdot(P:S)\subseteq P$ *;*

(iii) $S\cdot Q\subseteq P\Leftrightarrow S\subseteq(P:Q)\Leftrightarrow Q\subseteq(P:S)$.

Theorem 4.3. *Let A be an ILFPSM of M, then* $(A:\chi_M)$ *is an ILFPI and hence* $\sqrt{(A:\chi_M)}$ *is an ILF-prime ideal of R.*

Proof. Assume that A be an ILFPSM of M.
Let $a_{(s_1,t_1)}, b_{(s_2,t_2)}\in ILFP(R)$ such that $a_{(s_1,t_1)}b_{(s_2,t_2)}\in(A:\chi_M)$. Then $a_{(s_1,t_1)}b_{(s_2,t_2)}\chi_M\subseteq A$. If $b_{(s_2,t_2)}\notin A$ then there exists $x_{(p,q)}\in ILFP(M)$ such that $a_{(s_1,t_1)}(b_{(s_2,t_2)}x_{(p,q)})\in A$, but $b_{(s_2,t_2)}x_{(p,q)}\notin A$. Thus $\exists$ a natural number $m\in\mathbb{N}$ such that $a^m_{(s_1,t_1)}\in A$. Hence $(A:\chi_M)$ is a ILFPI and so $\sqrt{(A:\chi_M)}$ is an ILF-prime ideal of R. □

Theorem 4.4. *Let* $B\in IF_L(M)$ *and A be an ILFPSM of M. Then*

(i) if $B\subseteq A$*, then* $(A:B)=\chi_R$ *and*

(ii) if $B\nsubseteq A$*, then* $\sqrt{(A:B)}=\sqrt{(A:\chi_M)}$.

Proof. *For(i)* Let $r_{(\alpha,\beta)}\in\chi_R\Rightarrow r_{(\alpha,\beta)}B\subseteq B\subseteq A\Rightarrow r_{(\alpha,\beta)}\in(A:B)$. This implies that $\chi_R\subseteq(A:B)$. Also $(A:B)\subseteq\chi_R$ always. Thus we get $(A:B)=\chi_R$.

For(ii) Suppose $B\nsubseteq A$. Let $r_{(\alpha,\beta)}\in\sqrt{(A:B)}$
$\Rightarrow r^n_{(\alpha,\beta)}\in(A:B)$, for some $n\in\mathbb{N}$ and so $r^n_{(\alpha,\beta)}B\subseteq A$.
As A is ILFPSM of M and $B\nsubseteq A$ implies that $r^n_{(\alpha,\beta)}\in(A:\chi_M)$, i.e., $r_{(\alpha,\beta)}\in\sqrt{(A:\chi_M)}$.
Therefore, $\sqrt{(A:B)}\subseteq\sqrt{(A:\chi_M)}$. But $\sqrt{(A:\chi_M)}\subseteq\sqrt{(A:B)}$ always. Hence $\sqrt{(A:B)}=\sqrt{(A:\chi_M)}$. □

Theorem 4.5. *Let* $A\in IF_L(M)$*,* $C\in ILFI(R)$ *and A be an ILFPSM of M.*

(i) If $C\nsubseteq\sqrt{(A:\chi_M)}$ *then* $(A:C)=A$*;*

(ii) If $C\subseteq(A:\chi_M)$ *then* $(A:C)=\chi_M$.

Proof. *For(i)* Let $x_{(p,q)}\in(A:C)$ such that $r^n_{(\alpha,\beta)}x_{(p,q)}\in A$, where $r_{(\alpha,\beta)}\in ILFP(R)$. As A is ILFPSM of M, then either $x_{(p,q)}\in A$ or $r^n_{(\alpha,\beta)}\chi_M\subseteq A$. But $r^n_{(\alpha,\beta)}\chi_M\nsubseteq A$, for then $r_{(\alpha,\beta)}\in\sqrt{(A:\chi_M)}$ which is not possible. So we get $x_{(p,q)}\in A$. Thus $(A:C)\subseteq A$.

For other inclusion, now $C.A\subseteq\chi_R.A=A\Rightarrow C.A\subseteq A$ and so $A\subseteq(A:C)$. Hence $(A:C)=A$.
For(ii) Assume $C\subseteq(A:\chi_M)$. Then $C.\chi_M\subseteq(A:\chi_M).\chi_M\subseteq A$. Thus $C.B\subseteq C.\chi_M\subseteq A$, for every $B\in IF_L(M)$. Hence $(A:C)=\chi_M$. □

Definition 4.6. *For* $A,B\in IF_L(M)$ *with* $A\subseteq B$*, the ILF-radical of A in B is defined as* $\bigcap\{C\in IF_LSpec(R)|(A:B)\subseteq C\}$*. It is denoted by* $IFrad_B(A)$ *or* $\sqrt{(A:B)}$.

Thus $IFrad_B(A)=\bigcap\{C\in IF_LSpec(R)|(A:B)\subseteq C\}$.

Theorem 4.7. *Let A be an ILFPSM of M. Then* $IFrad_{\chi_M}(A)$ *is an ILFPI of R.*

Proof. Since A be an ILFPSM of M, then by Theorem (3.8), A is of the form

$$f_A(x)=\begin{cases}1, & \text{if } x\in A_*\\ \alpha, & \text{if otherwise}\end{cases};\quad g_A(x)=\begin{cases}0, & \text{if } x\in A_*\\ \alpha', & \text{otherwise.}\end{cases}$$

for all $x\in M$, where A_* is a primary submodule of M and α' is complement of the prime element α in L.

Let $\mathbf{p}=rad_M(A_*)=\sqrt{(A_*:M)}$. As defined in ([9], p.68). So $\mathbf{p}$ is a primary ideal of R. Now define $P\in ILFI(R)$ as follows:

$$f_P(x)=\begin{cases}1, & \text{if } x\in\mathbf{p}\\ \alpha, & \text{if otherwise}\end{cases};\quad g_P(x)=\begin{cases}0, & \text{if } x\in\mathbf{p}\\ \alpha', & \text{otherwise.}\end{cases}$$

So by Theorem (2.14) of [26], P is an ILFPI of R. We show that $IFrad_{\chi_M}(A)=P$. For this we show that
(i) $(A:\chi_M)\subseteq P$ and
(ii) P is the smallest ILFPI containing $(A:\chi_M)$.

For (i)
Obviously $f_{(A:\chi_M)}(x)\le 1=f_P(x)$ and $g_{(A:\chi_M)}(x)\ge 0=g_P(x)$, for all $x\in\mathbf{p}$ ...(1)
Now let $x\notin\mathbf{p}=rad_M(A_*)$. Since $rad_M(A_*)=\{r\in R|r^nM\subseteq A_*$ for some $n\in\mathbb{N}\}$. It follows that $x^nM\nsubseteq A_*$, for all $n\in\mathbb{N}$. Thereby there is $m\in M$ such that $xm\notin A_*$ and so $f_A(xm)=\alpha$, $g_A(xm)=\alpha'$ .....(2)
Suppose $(p,q)\in L\times L$ such that $p\vee q\le 1$ and $x_{(p,q)}\circ\chi_M\subseteq A$. By lemma (2.4) we have

$$f_{x_{(p,q)}\circ\chi_M}(w)=\begin{cases}p, & \text{if } w=xm, x\in R, m\in M\\ 0, & \text{if } w \text{ is not expressible as } w=xm\end{cases}\quad and$$

$$g_{x_{(p,q)}\circ\chi_M}(w)=\begin{cases}q, & \text{if } w=xm, x\in R, m\in M\\ 1, & \text{if } w \text{ is not expressible as } w=xm.\end{cases}$$

For $w=xm$, by (1) we have $p=f_{x_{(p,q)}\circ\chi_M}(xm)\le f_A(xm)=\alpha$, $q=g_{x_{(p,q)}\circ\chi_M}(xm)\ge g_A(xm)=\alpha'$. Hence $p\le\alpha, q\ge\alpha'$, for all $(p,q)\in L\times L$ such that $x_{(p,q)}\circ\chi_M\subseteq A$. ......(3)

But $f_{(A:\chi_M)}(x)=Sup\{p: f_{x_{(p,q)}\circ\chi_M}(x)\le f_A(x)\}$ and $g_{(A:\chi_M)}(x)=Inf\{q: g_{x_{(p,q)}\circ\chi_M}(x)\ge g_A(x)\}$. By (3) we get

$f_{(A:\chi_M)}(x)\le\alpha=f_P(x)$, $g_{(A:\chi_M)}(x)\ge\alpha'=g_P(x)$, for all $x\notin\mathbf{p}$ ....(4).

Hence (1) and (4) imply (i)
For (ii)

Let $P^{'}$ be an ILFPSM of M such that $P^{'} \supseteq (A:\chi_M)$. It is easy to see that $r_{(\alpha,\alpha')} \circ \chi_M \subseteq A$ for all $r \in R$,

So, $\alpha_1 \in \{s \in L | r_{(s,t)} \circ \chi_M \subseteq A\}$ and $\alpha_1^{'} \in \{t \in L | r_{(s,t)} \circ \chi_M \subseteq A\}$ ...(5)
Theorem (2.14) of [26], we have

$$f_{P'}(x) = \begin{cases} 1, & \text{if } x \in P_*^{'} \\ \alpha_1, & \text{if otherwise} \end{cases} ; \quad g_{P'}(x) = \begin{cases} 0, & \text{if } x \in P_*^{'} \\ \alpha_1^{'}, & \text{otherwise.} \end{cases}$$

where $\alpha_1^{'}$ is a complement of the prime element α_1 in L and $P_*^{'}$ is an ILFPI of R.
Let $r \notin P_*^{'}$. Then by (5) we get $\alpha_1 = f_{P'}(r) \geq f_{(A:\chi_M)}(r) \geq \alpha$ and $\alpha_1^{'} = g_{P'}(r) \leq g_{(A:\chi_M)}(r) \leq \alpha^{'}$.

Thus we get $\alpha_1 \geq \alpha$ and $\alpha_1^{'} \leq \alpha^{'}$ ....(6)

For $y \in (A_*:M)$, by lemma (2.4), we have

$$f_{y_{(1,0)}\circ\chi_M}(w) = \begin{cases} 1, & \text{if } w = ym, y \in R, m \in M \\ 0, & \text{if } w \text{ is not expressible as } w = ym \end{cases} \textit{and}$$

$$g_{y_{(1,0)}\circ\chi_M}(w) = \begin{cases} 0, & \text{if } w = ym, y \in R, m \in M \\ 1, & \text{if } w \text{ is not expressible as } w = ym. \end{cases} \quad ......(7)$$

Also $yM \subseteq A_*$ and therefore $f_A(ym) = 1; g_A(ym) = 0, \forall m \in M$ .....(8)

so by (7) and (8) we have
$f_{y_{(1,0)}\circ\chi_M}(w) \leq f_A(w)$ and $g_{y_{(1,0)}\circ\chi_M}(w) \geq g_A(w)$ for all $w \in M$. Hence $y_{(1,0)} \circ \chi_M \subseteq A$ and thus $(1,0) \in \{(s,t) \in L \times L | y_{(1,0)} \circ \chi_M \subseteq A\}$....(9)
Since $P^{'} \supseteq (A:\chi_M)$, by (9) we have $f_{P'}(y) \geq f_{(A:\chi_M)}(y) \geq 1$ and $g_{P'}(y) \leq g_{(A:\chi_M)}(y) \leq 0$. Therefore, $f_{P'}(y) = 1$ and $g_{P'}(y) = 0$, i.e., $y \in P_*^{'}$. Hence $P_*^{'} \supseteq (A_*:M)$. So $P_*^{'} \supseteq \sqrt{(A_*:M)} = \mathbf{p}$....(10)
Now (6) and (10) imply that $IFrad_{\chi_M}(A) = \sqrt{(A:\chi_M)} = P$. This complete (ii). Hence $IFrad_{\chi_M}(A)$ is an ILFPI of R. □

Theorem 4.8. *If A is an ILFPSM of M, then $IFrad_{\chi_M}(A)$ is an ILFPI of R iff $rad_M(A_*)$ is primary ideal of R.*

Proof. This follows from Theorem (4.7) and Theorem (3.8) □

Definition 4.9. *Let A be an ILFPSM of M and $P = IFrad_{\chi_M}(A)$. Then A is said to be an intuitionistic L-fuzzy P-primary submodule (ILFPPSM) of M.*

Proposition 4.10. *Let A be an ILFPPSM of M. Then $r_{(s,t)}x_{(p,q)} \in A$ implies that either $x_{(p,q)} \in A$ or $r_{(s,t)} \in P$, where $r \in R, x \in M, (p,q),(r,s) \in L \times L$.*

Proof. Let $r_{(s,t)}x_{(p,q)} \in A$ and $x_{(p,q)} \notin A$. Then $r^n_{(s,t)}\chi_M \subseteq A$ for some $n \in \mathbb{N}$....(11)
But $P = IFrad_{\chi_M}(A) = \sqrt{(A:\chi_M)} = \bigcap\{C \in IF_LSpec(R) | (A:\chi_M) \subseteq C\}$.

By (11) we have $f_P(r^n) \geq f_{(A:\chi_M)}(r^n) = Sup\{s_1 | r^n_{(s_1,t_1)}\chi_M \subseteq A\} \geq s$ and $g_P(r^n) \leq g_{(A:\chi_M)}(r^n) = Inf\{t_1 | r^n_{(s_1,t_1)}\chi_M \subseteq A\} \leq t$. So $r^n_{(s,t)} \in P$. Since P is an ILF-prime ideal by using Theorem (2.14) of [26] and some manipulation we get $r_{(s,t)} \in P$. □

Theorem 4.11. *Let A be an ILFPPSM of M and $C \in ILFI(R)$, $B \in IF_L(M)$. If $CB \subseteq A$ inferred that $C \subseteq P$ or $B \subseteq A$.*

Proof. Suppose $B \subseteq A$. Then $\exists\, x \in M$ such that $f_B(x) \leq f_A(x)$ and $g_B(x) \geq g_A(x)$. This imply $x_{(f_B(x),g_B(x))} \notin A$.
Since $CB \subseteq A$ we get $r_{(s,t)}x_{(f_B(x),g_B(x))} = (rx)_{(s\wedge f_B(x), t\vee g_B(x))} \subseteq A$, $\forall r \in R$. So by Proposition (4.10) $r_{(s,t)} \in P$ for all $r \in R$. Thus $r_{(s,t)} \in C$ implies $r_{(s,t)} \in P\ \forall r \in R$. Therefore $C \subseteq P$. □

Corollary 4.12. *Let A be an ILFPPSM of M and $C \in ILFI(R)$, $C \subseteq P$. Then $(A:C) = A$.*

Proof. By Theorem (4.2)(ii) we have $C(A:C) \subseteq A$. Since $C \subseteq P$, by Theorem (4.11) $(A:C) \subseteq A$. But $A \subseteq (A:C)$ is obvious. Hence $(A:C) = A$. □

Theorem 4.13. *Let $A \in IF_L(M)$ be a non-constant and $P \in ILFI(R)$. Then A is an ILFPPSM of M iff*

1. *$r_{(s,t)}x_{(p,q)} \in A$ and $x_{(p,q)} \notin A$, then $r_{(s,t)} \in P$ for all ILFPs $r_{(s,t)}$ of R and $x_{(p,q)}$ of M*
2. *if $r_{(s,t)} \in P$ then $r^n_{(s,t)}\chi_M \subseteq A$ for some $n \in \mathbb{N}$.*

Proof. Let A be an ILFPPSM of M. Then by Theorem (4.12)(i) holds. Let $r_{(s,t)} \in P$ we show that $r^n_{(s,t)}\chi_M \subseteq A$ for some $n \in \mathbb{N}$. By considering definition (4.9) and Theorem (4.7) if $f_p(r) = \alpha$, $g_p(r) = \beta$, then $r_{(s,t)} \in P$ implies $s \leq f_P(r) = \alpha$, $t \geq g_P(r) = \beta$. Therefore by using lemma(2.4) we get
$f_{r^n_{(s,t)}\chi_M}(w) \leq s \leq \alpha \leq f_A(w)$ and $g_{r^n_{(s,t)}\chi_M}(w) \geq t \geq \beta \geq g_A(w)$ for all $w \in M$. So in this case we get $r_{(s,t)}\chi_M \subseteq A$, otherwise $f_P(r) = 1$, $g_P(r) = 0$. Thus $r \in Rad_M(A_*)$ which implies $r^nM \subseteq A_*$, for some $n \in \mathbb{N}$.
Hence $\mu_A(r^nx) = 1$ and $\nu_A(r^nx) = 0$, for all $x \in M$ ......(1)

Now for arbitrary $w \in M$, if $r^n_{(s,t)}\chi_M(w) = (0,1)$, then $f_{r^n_{(s,t)}\chi_M}(w) \leq g_A(w)$ and $g_{r^n_{(s,t)}\chi_M}(w) \geq g_A(w)$ otherwise by using lemma(2.4) we have $f_{r^n_{(s,t)}\chi_M}(w) = s$ and $g_{r^n_{(s,t)}\chi_M}(w) = t$ and $w = r^nx$ for some $x \in M$. So from (1) we get
$f_{r^n_{(s,t)}\chi_M}(w) = s \leq 1 = f_A(w)$ and $g_{r^n_{(s,t)}\chi_M}(w) = t \leq 0 = g_A(w)$, for all $w \in M$ implies $r^n_{(s,t)}\chi_M \subseteq A$, so (ii) proved.

Conversely, assume that (i) and (ii) holds. We claim that A is an ILFPPSM of M. Obviously (i) and (ii) infer that A is an ILFPSM of M. So it is sufficient to show that $P = IFrad_{\chi_M}(A)$. Let $r_{(s,t)}$ be an arbitrary $ILFP$ of R. We show that $r_{(s,t)} \in P$ iff $r_{(s,t)} \in IFrad_{\chi_M}(A)$. Now let $r_{(s,t)} \in P$. Then by (ii) we have $r^n_{(s,t)}\chi_M \subseteq A$ for some $n \in \mathbb{N}$....(2)

If $f_{IFrad_{\chi_M}(A)}(r) = 1$, $g_{IFrad_{\chi_M}(A)}(r) = 0$, then $s \leq 1 = f_{IFrad_{\chi_M}(A)}(r)$ and $t \geq 0 = g_{IFrad_{\chi_M}(A)}(r)$ and therefore $r_{(s,t)} \in IFrad_{\chi_M}(A)$ else by observing Theorem (3.8) and (4.5) we get
$f_{IFrad_{\chi_M}(A)}(r) = \alpha$ and $g_{IFrad_{\chi_M}(A)}(r) = \alpha'$ ....(3)
Thus $r \notin IFrad_{\chi_M}(A)$, and so $r^m M \subseteq A_*$ for all $m \in \mathbb{N}$, especially, $r^n M \subseteq A_*$. Therefore $\exists' s\ x \in M$ such that $r^n x \notin A_*$; thus $f_A(r^n x) = \alpha$ and $g_A(r^n x) = \alpha'$. But (2) and corollary (2.17)(i) imply that $s = f_{r^n_{(s,t)}\chi_M}(r^n x) \leq f_A(r^n x) = \alpha$ and $t = g_{r^n_{(s,t)}\chi_M}(r^n x) \geq g_A(r^n x) = \alpha'$ and so by (3) we get
$s \leq \alpha \leq f_{IFrad_{\chi_M}(A)}(r)$ and $t \geq \alpha' \geq g_{IFrad_{\chi_M}(A)}(r)$.
Hence $r_{(s,t)} \in IFrad_{\chi_M}(A)$.

Next suppose that $r_{(s,t)} \in IFrad_{\chi_M}(A)$. If $r = 0$ then $s \leq 1 = f_P(0)$ and $t \geq 0 = g_P(0)$ so $r_{(s,t)} \in P$. Thus we presuppose that $r \neq 0$. Now by considering Theorems (3.8) and (4.7) if $f_{IFrad_{\chi_M}(A)}(r) = \alpha$ and $g_{IFrad_{\chi_M}(A)}(r) = \alpha'$ then $r \notin IFrad_{\chi_M}(A)$ and so $r^n M \subseteq A_*$ for all $n \in \mathbb{N}$, especially $rM \subseteq A_*$. So there exists $x \in M$ such that $rx \in A_*$. Thus $f_A(x) = \alpha$ and $g_A(x) = \alpha'$ and since A_* is a submodule of M, $x \notin A_*$ and hence $x_{(1,0)} \notin A$. But $r_{(s,t)} \in IFrad_{\chi_M}(A)$ implies that $s \leq f_{IFrad_{\chi_M}(A)}(r) = \alpha = f_A(rx)$ and $t \geq g_{IFrad_{\chi_M}(A)}(r) = \alpha' = g_A(rx)$.

Therefore $r_{(s,t)} x_{(1,0)} = (rx)_{(s,t)} \in A$. Since $x_{(1,0)} \notin A$, it follows that $r_{(s,t)} \in P$ by part(1).
If $f_{IFrad_{\chi_M}(A)}(r) = 1$ and $g_{IFrad_{\chi_M}(A)}(r) = 0$, then $r \in rad_M(A_*)$ and so $r^n M \subseteq A_*$ for some $n \in \mathbb{N}$. Since $r \neq 0$ we select the smallest natural number m such that $r^m M \subseteq A_*$ and $r^{m-1} M \nsubseteq A_*$. Therefore $\exists\ x \in M$ such that $r^m x \in A_*$ and $r^{m-1} x \notin A_*$, so that $r^{m-1} x)_{(1,0)} \notin A$. Since $r^m x \in A_*$, so $f_A(r^m x) = 1$ and $g_A(r^m x) = 0$ and thus $r_{(s,t)}(r^{m-1} x)_{(1,0)} = (r^m x)_{(s,t)} \in A$. Therefore by part (1) $r_{(s,t)} \in P$. Hence $P = IFrad_{\chi_M}(A)$. □

Theorem 4.14. *Let A be an ILFPPSM of M and $C \in ILFI(R)$ and $(A : C)$ be non-constant. Then $(A : C)$ is an ILFPPSM of M.*

Proof. Suppose $r_{(s,t)} \in R$ and $x_{(p,q)} \in M$ be an arbitrary $ILFPs$ such that $r_{(s,t)} x_{(p,q)} \in (A : C)$ and $x_{(p,q)} \notin (A : C)$ imply that $f_{(A:C)}(x) \geq p$ and $g_{(A:C)}(x) \leq q$.
But $f_{(A:C)}(x) = Sup\{\alpha \in L | C.x_{(\alpha,\beta)} \subseteq A\}$ and $g_{(A:C)}(x) = Inf\{\beta \in L | C.x_{(\alpha,\beta)} \subseteq A\}$.
We conclude that $C.x_{(\alpha,\beta)} \subseteq A$. Therefore $\exists\ w \in M$ such that $f_{C.x_{(\alpha,\beta)}}(w) \leq f_A(w)$ and $g_{C.x_{(\alpha,\beta)}}(w) \geq g_A(w)$, so $f_A(w) \neq 1$ and $g_A(w) \neq 0$ as such $C.x_{(\alpha,\beta)}(w) \neq (1,0)$. Then by lemma (2.16)(ii) we get

$Sup_{w=ax}\{inf\{f_C(a), p\}\} \leq f_A(w)$ and
$Inf_{w=ax}\{sup\{g_C(a), q\}\} \geq g_A(w)$. Thus $\exists\ a \in R$ such that $w = ax$ and $Inf\{f_C(a), p\} \leq f_A(w)$ and $Sup\{g_C(a), q\} \geq g_A(w)$ $\Rightarrow a_{(f_C(a), g_C(a))} x_{(p,q)} = (ax)_{(f_C(a) \wedge p, g_C(a) \vee q)} \notin A$.
Since $a_{(f_C(a), g_C(a))} \in C$ and $r_{(s,t)} x_{(p,q)} \in (A : C)$ we get $a_{(f_C(a), g_C(a))}(r_{(s,t)} x_{(p,q)}) \in C(A : C)$ and so by lemma (3.4)(ii) we get $r_{(s,t)}(a_{(f_C(a), g_C(a))} x_{(p,q)}) \in C(A : C)$. Now by () and Theorem (4.12) we get $r_{(s,t)} \in P$. This set-up (i) of Theorem (4.15).

Next suppose $r_{(s,t)} \in P$, so that by hypothesis Theorem (4.15)(ii) we have $r^n_{(s,t)} \chi_M \subseteq A$, for some $n \in \mathbb{N}$. Then by using lemma (3.4)(i) we get $r^n_{(s,t)} \chi_M \subseteq (A : C)$, for some $n \in \mathbb{N}$. This set-up (ii) of Theorem (4.12). □

Theorem 4.15. *Let A be an ILFPPSM of M, $B \in IF_L(M)$ and $(A : B)$ be non-constant. Then $(A : B)$ is an ILF-P-primary ideal of R.*

Proof. Moving parallel to the proof of Theorem (4.14) making corresponding changes and using Theorem(4.2)(i) instead of Theorem(4.2)(ii) One can checked that

(i) $r_{(s,t)} a_{(\alpha,\beta)} \in (A : B)$ and $a_{(\alpha,\beta)} \notin (A : B)$, then $r_{(s,t)} \in P$ for all $ILFPs$ $r_{(s,t)}, a_{(\alpha,\beta)}$ of R
(ii)if $r_{(s,t)} \in P$ then $r^n_{(s,t)} \chi_R \subseteq (A : B)$ for some $n \in \mathbb{N}$.

Thus, by Theorem (4.13), $(A : B)$ is a P-primary submodule of R. So by Theorem (3.3), $(A : B)$ is a P-primary ideal of R. □

Theorem 4.16. *Let $h : M \to M_1$ be an R-module epimorphism and suppose that L is distributive. If A is an ILFPPSM of M and h-invariant. Then $h(A)$ is also an ILFPPSM of M_1.*

Proof. Assume that A be an ILFPPSM of M and h-invariant, then by Theorem (3.10) $h(A)$ is an ILFPPSM of M_1. Now we claim that $\sqrt{(h(A) : \chi_{M_1})} = \sqrt{(A : \chi_M)}$.

Let $r \in \sqrt{(h(A_*) : M_1)}$ implies that $r^n \in (h(A_*) : M_1)$, for some $n \in \mathbb{N}$. Then $r^n M_1 \subseteq h(A_*) \Rightarrow r^n h(M) \subseteq h(A_*) \Rightarrow h(r^n M) \subseteq h(A_*)$.
Let $x = r^n m \in r^n M, m \in M$. Then $h(r^n m) \in h(r^n M)$. Then $h(x) = h(r^n m) \subseteq h(A_*)$. This implies that $h(x) = h(z)$, for some $z \in A_*$. As A is h-invariant so $f_A(x) = f_A(z) = f_A(\theta)$ and $g_A(x) = g_A(z) = g_A(\theta) \Rightarrow x \in A_*$. From this we get $r^n M \subseteq A_*$ and this implies that $r^n \in (A_* : M)$, and so $r \in \sqrt{(A_* : M)}$. Thus $\sqrt{(h(A_*) : M_1)} \subseteq \sqrt{(A_* : M)}$.

Again let $p \in \sqrt{(A_* : M)}$ this implies that $p^t \in (A_* : M)$ for some $t \in \mathbb{N}$. Then $p^t M \subseteq A_*$. From this we get $h(p^t M) \subseteq h(A_*)$. So, $p^t h(M) \subseteq h(A_*)$, i.e., $p^t M_1 \subseteq h(A_*)$, as h is an epimorphism. This implies that $p^t \in (h(A_*) : M_1)$, i.e., $p \in \sqrt{(h(A_*) : M_1)}$. Therefore we get $\sqrt{(A_* : M)} = \sqrt{(h(A_*) : M_1)}$ and so $\sqrt{(A : \chi_M)} = \sqrt{(h(A) : \chi_{M_1})}$. This complete the proof. □

Theorem 4.17. *Let $h : M \to M_1$ be a R-module epimorphism. If B is an ILFPPSM of M_1, then $h^{-1}(B)$ is an ILFPPSM of M.*

Proof. Assume that B be an ILFPPSM of M_1, then by Theorem (3.11) $h^{-1}(B)$ is an ILFPSM of M. Now we claim that

$\sqrt{(B:\chi_{M_1})} = \sqrt{(h^{-1}(B):\chi_M)}$.

Let $r \in \sqrt{(h^{-1}(B_*):M)}$, then $r^n \in (h^{-1}(B_*):M)$ for some $n \in \mathbb{N}$. So $r^n M \subseteq h^{-1}(B_*) \Rightarrow h(r^n M) \subseteq B_*$, i.e., $r^n h(M) \subseteq B_*$ as h is epimorphism, which infer that $r^n M_1 \subseteq B_*$, i.e., $r^n \in (B_*:M_1)$ and so $r \in \sqrt{(B_*:M_1)}$.
Thus, $\sqrt{(h^{-1}(B):M)} \subseteq \sqrt{(B_*:M_1)}$.

Again, let $p \in \sqrt{(B_*:M_1)}$ this infer that $p^t \in (B_*:M_1)$ for some $t \in \mathbb{N}$. Then $p^t M_1 \subseteq B_*$. So $p^t h(M) \subseteq B_*$, as h is epimorphism, so $h(p^t M) \subseteq B_*$, i.e., $p^t M \subseteq h^{-1}(B_*)$
$\Rightarrow p^t \in (h^{-1}(B_*):M)$, i.e. $p \in \sqrt{(h^{-1}(B_*):M)}$.
Thus $\sqrt{(B_*:M_1)} \subseteq \sqrt{(h^{-1}(B_*):M)}$. So $\sqrt{(h^{-1}(B_*):M)} = \sqrt{(B_*:M_1)}$. Hence, $\sqrt{(B:\chi_{M_1})} = \sqrt{(h^{-1}(B):\chi_M)}$.
This complete the proof of the theorem. □

Theorem 4.18. *If $A_1, A_2, ...,A_k$ be ILFPPSMs of M. Then $\bigcap_{i=1}^{k} A_i$ is also ILFPPSM of M.*

Proof. Let $A = \bigcap_{i=1}^{k} A_i$, where $A_1, A_2, ...,A_k$ be ILFPPSMs of M, then
$\sqrt{(A_1:\chi_M)} = \sqrt{(A_2:\chi_M)} = ... = \sqrt{(A_k:\chi_M)} = P$.

Let $r_{(s,t)} \in ILFI(R)$ and $x_{(p,q)} \in IF_L(M)$ such that $r_{(s,t)}x_{(p,q)} \in A = \bigcap_{i=1}^{k} A_i$ and $r_{(s,t)} \notin \sqrt{(A:\chi_M)}$. Since $\sqrt{(A:\chi_M)} = \sqrt{(\bigcap_{i=1}^{k} A_i:\chi_M)} = \sqrt{\bigcap_{i=1}^{k}(A_i:\chi_M)} = \bigcap_{i=1}^{k}\sqrt{(A_i:\chi_M)}$, by using Theorem (4.6) of [12] and Theorem (3.4) of [25]. Thus we get $r_{(s,t)}x_{(p,q)} \in A_i$ and $r_{(s,t)} \notin \sqrt{(A_i:\chi_M)}$, then since each A_i are ILFPPSMs of M, we have $x_{(p,q)} \in A_i, \forall i = 1,2,....,k$, so $x_{(p,q)} \in \bigcap_{i=1}^{k} A_i = A$. It remain to show that $\sqrt{(A:\chi_M)} = P$.

If $r_{(s,t)} \in P$ then $\exists\ n_i \in \mathbb{N}$ such that $r_{(s,t)}^{n_i}\chi_M \subseteq A_i$, $\forall i \in \{1,2,....,k\}$. Let $n = \sum_{i=1}^{k} n_i$, then $r_{(s,t)}^{n}\chi_M \subseteq A_i, \forall i \in \{1,2,....,k\}$. So we have $r_{(s,t)}^{n}\chi_M \subseteq \bigcap_{i=1}^{k} A_i = A$. Thus $r_{(s,t)} \in \sqrt{(A:\chi_M)}$.
So we have $P \subseteq \sqrt{(A:\chi_M)}$....(1)
Conversely, if $r_{(s,t)} \in \sqrt{(A:\chi_M)}$, then
$r_{(s,t)} \in \bigcap_{i=1}^{k}\sqrt{(A_i:\chi_M)} = P$, so $\sqrt{(A:\chi_M)} \subseteq P$......(2).
From (1) and (2) we get $\sqrt{(A:\chi_M)} = P$. This complete the result. □

5. Conclusion

In this paper we have explored the fundamental ideas of intuitionistic L-fuzzy primary and P-primary submodules. We proved that intuitionistic L-fuzzy primary submodule is a two valued intuitionistic L-fuzzy subset and base set is a primary submodule. (The base set of intuitionistic L-fuzzy primary submodule A is defined as the set $\{x \in M | f_A(x) = f_A(\theta); g_A(x) = g_A(\theta)\}$ and vice versa. We also investigated the effect on intuitionistic L-fuzzy primary submodules under module homomorphism. The radical of intuitionistic L-fuzzy primary submodule has been explored completely which has been used to define the notion of intuitionistic L-fuzzy P-primary submodules of M, i.e., if A is an intuitionistic L-fuzzy primary submodule and if $P = IFrad_{\chi_M}(A)$. Then A is termed as intuitionistic L-fuzzy P-primary submodule of M. Many properties of intuitionistic L-fuzzy submodules have been studied in terms of residual quotients. We have also laid down the foundation of the most important property in module theory: decomposition of submodules in terms of primary submodules in the intuitionistic fuzzy setting.

Acknowledgment

The second author wishes to acknowledge I.K. Gujral Punjab Technical University Jalandhar, for giving a chance to carry out her research.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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