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# Periodic solutions of almost linear Volterra integro-dynamic systems

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**Abstract**
In this paper, we use Krasnoselskii's fixed point theorem to establish new results on the existence of periodic solutions for the almost linear Volterra integro-dynamic system on periodic time scales of the form

$$\begin{cases} x^{\Delta}(t) = a(t)\,p(x(t)) + \int_{-\infty}^{t} C(t,s)\,h(y(s))\,\Delta s + e(t), \\ y^{\Delta}(t) = b(t)\,q(y(t)) + \int_{-\infty}^{t} D(t,s)\,g(x(s))\,\Delta s + f(t). \end{cases}$$

**Keywords**
Volterra integro-dynamic systems, time scales, Krasnoselskii's fixed point theorem, periodic solutions.

**AMS Subject Classification**
34K13, 34K20, 45J05, 45D05.

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## Contents



## 1. Introduction

Delay dynamic equations arise from a variety of applications including in various fields of science and engineering such as applied sciences, physics, chemistry, biology, medicine, etc. In particular, problems concerning qualitative analysis of delay dynamic equations have received the attention of many authors, see [1]–[20] and the references therein.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this article, we are interested in the analysis of qualitative theory of periodic solutions of almost linear Volterra integro-dynamic systems. Inspired and motivated by the references in this paper, we consider the following almost linear Volterra integro-dynamic system on time scales

$$\begin{cases} x^{\Delta}(t) = a(t)\,p(x(t)) + \int_{-\infty}^{t} C(t,s)\,h(y(s))\,\Delta s + e(t), \\ y^{\Delta}(t) = b(t)\,q(y(t)) + \int_{-\infty}^{t} D(t,s)\,g(x(s))\,\Delta s + f(t), \end{cases} \tag{1.1}$$

where $a$, $b$, $e$ and $f$ are rd-continuous functions, $p$, $q$, $f$ and $g$ are continuous functions. We assume that there exist constants $P$, $Q$, $H$, $G$ and positive constants $P^*$, $Q^*$, $H^*$, $G^*$ such that

$$|p(x) - Px| \leq P^*, \;\; |q(x) - Qx| \leq Q^*, \tag{1.2}$$

and

$$|h(x) - Hx| \leq H^*, \;\; |g(x) - Gx| \leq G^*. \tag{1.3}$$

To show the existence of periodic solutions of (1.1), we transform (1.1) into an integral system and then use Krasnoselskii's fixed point theorem. The obtained integral system is the sum of two mappings, one is a contraction and the other is compact. Our results generalize previous results due to Raffoul [20], from the one dimension to the two dimensions.

## 2. Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1]-[12], [16]-[19] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [8, 9, 19] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Kaufmann and Raffoul [18]. The following two definitions are borrowed from [18].

**Definition 2.1.** *We say that a time scale $\mathbb{T}$ is periodic if there exist a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive $\omega$ is called the period of the time scale.*

**Example 2.2.** *The following time scales are periodic.*

1. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih],\ h > 0$ *has period* $\omega = 2h$.
2. $\mathbb{T} = h\mathbb{Z}$ *has period* $\omega = h$.
3. $\mathbb{T} = \mathbb{R}$.
4. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z},\ m \in \mathbb{N}_0\}$ *where,* $0 < q < 1$ *has period* $\omega = 1$.

**Remark 2.3** ([18])**.** *All periodic time scales are unbounded above and below.*

**Definition 2.4.** *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period $\omega$. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period $T$ if there exists a natural number $n$ such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and $T$ is the smallest number such that $f(t \pm T) = f(t)$.*

*If $\mathbb{T} = \mathbb{R}$, we say that $f$ is periodic with period $T > 0$ if $T$ is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.*

**Remark 2.5** ([18])**.** *If $\mathbb{T}$ is a periodic time scale with period $\omega$, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function $\mu$ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period $\omega$.*

**Definition 2.6** ([8])**.** *$f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous function provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

*The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by*

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

**Definition 2.7** ([8])**.** *For $f : \mathbb{T} \to \mathbb{R}$, we define $f^\Delta(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that*

$$\left| (f(\sigma(t)) - f(s)) - f^\Delta(t)\,(\sigma(t) - s) \right| < \varepsilon\, |\sigma(t) - s|,$$

*for all $s \in U$. The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of $f$ on $\mathbb{T}^k$.*

**Definition 2.8** ([8])**.** *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathscr{R} = \mathscr{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathscr{R}^+$ of all positively regressive elements of $\mathscr{R}$ by*

$$\mathscr{R}^+ = \mathscr{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathscr{R} : 1 + \mu(t)p(t) > 0,\ \forall t \in \mathbb{T}\}.$$

**Definition 2.9** ([8])**.** *Let $p \in \mathscr{R}$, then the generalized exponential function $e_p$ is defined as the unique solution of the initial value problem*

$$x^\Delta(t) = p(t)x(t),\ x(s) = 1,\ \text{where } s \in \mathbb{T}.$$

*An explicit formula for $e_p(t,s)$ is given by*

$$e_p(t,s) = \exp\left( \int_s^t \xi_{\mu(v)}(p(v))\Delta v \right),\ \text{for all } s,t \in \mathbb{T},$$

*with*

$$\xi_\mu(p) = \begin{cases} \frac{\log(1+\mu p)}{\mu} & \text{if } \mu \neq 0, \\ p & \text{if } \mu = 0, \end{cases}$$

*where* log *is the principal logarithm function.*

**Lemma 2.10** ([8])**.** *Let $p, q \in \mathscr{R}$. Then*

*(i)* $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$,

*(ii)* $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$,

*(iii)* $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

*(iv)* $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

*(v)* $e_p(t,s)e_p(s,r) = e_p(t,r)$,

*(vi)* $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e^\sigma_p(\cdot,s)}$.

**Lemma 2.11** ([1])**.** *If $p \in \mathscr{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left( \int_s^t p(v)\Delta v \right),\ \forall t \in \mathbb{T}.$$

The proof of the main results in the next section is based upon an application of the following Krasnoselskii fixed point theorem.

**Theorem 2.12** (Krasnoselskii's fixed point theorem [21])**.** *Let $\mathbb{M}$ be a closed, convex, nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $A$ and $B$ map $\mathbb{M}$ into $\mathbb{B}$ such that*

*i) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,*

*ii) $A$ is continuous and $A\mathbb{M}$ is contained in a compact subset of $\mathbb{B}$,*

*iii) $B$ is a contraction mapping.*

*Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.*

## 3. Periodic Solutions

Let $\mathbb{T}$ be a periodic time scale with period $\omega$. Let $T>0$ be fixed, and if $\mathbb{T}\neq\mathbb{R}$, then $T=n\omega$ for some $n\in\mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b]=\{t\in\mathbb{T}:a\leq t\leq b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and $(a,b)$ are defined similarly. Let $P_T$ be the set of all continuous scalar functions, periodic of period $T$. Then $(P_T,\|.\|)$ is a Banach space with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{T}}|x(t)|=\sup_{t\in[0,T]}|x(t)|.$$

In this section we investigate the existence of a periodic solution of (1.1) using Krasnoselskii's fixed point theorem.

The next lemma is essential to our next results. Its proof can be found in [18].

**Lemma 3.1.** *Let $x\in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\|=\|x\|$.*

In this section we assume that for all $(t,s)\in\mathbb{T}\times\mathbb{T}$,

$$\sup_{t\in\mathbb{T}}\int_{-\infty}^{t}|C(t,s)|\Delta s<\infty,\ \sup_{t\in\mathbb{T}}\int_{-\infty}^{t}|D(t,s)|\Delta s<\infty. \tag{3.1}$$

We assume $a,b\in\mathscr{R}^+$ with

$$e_{\ominus(Pa)}(t,t-T)\neq 1 \text{ and } e_{\ominus(Qb)}(t,t-T)\neq 1.$$

Suppose that

$$\begin{gathered} a(t+T)=a(t),\ b(t+T)=b(t),\\ e(t+T)=e(t),\ f(t+T)=f(t),\\ C(t+T,s+T)=C(t,s),\ D(t+T,s+T)=D(t,s). \end{gathered} \tag{3.2}$$

Let $\mathbb{P}_T=P_T\times P_T$, then $\mathbb{P}_T$ is a Banach space when endowed with the maximum norm

$$\|(x,y)\|=\max\left\{\sup_{t\in[0,T]}|x(t)|,\ \sup_{t\in[0,T]}|y(t)|\right\}.$$

For any positive constant $m$ the set

$$\mathbb{M}=\{(x,y)\in\mathbb{P}_T:\|(x,y)\|\leq m\}. \tag{3.3}$$

is a bounded closed convex subset of $\mathbb{P}_T$.

**Lemma 3.2.** *If $(x,y)\in\mathbb{P}_T$, then $(x,y)$ is a solution of (1.1) if and only if*

$$x(t)=\eta_1\int_{t-T}^{t}[Pa(u)x^\sigma(u)+a(u)p(x(u)) +k(u)]e_{\ominus(Pa)}(t,u)\Delta u, \tag{3.4}$$

*and*

$$y(t)=\eta_2\int_{t-T}^{t}[Qb(u)y^\sigma(u)+b(u)q(y(u)) +l(u)]e_{\ominus(Qa)}(t,u)\Delta u, \tag{3.5}$$

*where*

$$\eta_1=\left[1-e_{\ominus(Pa)}(T,0)\right]^{-1},\ \eta_2=\left[1-e_{\ominus(Qa)}(T,0)\right]^{-1},$$

$$k(t)=e(t)+\int_{-\infty}^{t}C(t,s)[h(y(s))-Hy(s)]\Delta s +\int_{-\infty}^{t}C(t,s)Hy(s)\Delta s,$$

*and*

$$l(t)=f(t)+\int_{-\infty}^{t}D(t,s)[g(x(s))-Gx(s)]\Delta s +\int_{-\infty}^{t}D(t,s)Gy(s)\Delta s.$$

*Proof.* For convenience we put the first equation in (1.1) in the form

$$\begin{aligned} &x^\Delta(t)+Pa(t)x^\sigma(t)\\ &=Pa(t)x^\sigma(t)+a(t)p(x(t))+e(t)\\ &+\int_{-\infty}^{t}C(t,s)[h(y(s))-Hy(s)]\Delta s\\ &+\int_{-\infty}^{t}C(t,s)Hy(s)\Delta s. \end{aligned} \tag{3.6}$$

Let

$$k(t)=e(t)+\int_{-\infty}^{t}C(t,s)[h(y(s))-Hy(s)]\Delta s +\int_{-\infty}^{t}C(t,s)Hy(s)\Delta s.$$

Then we may write (3.6) as

$$x^\Delta(t)+Pa(t)x^\sigma(t)=Pa(t)x^\sigma(t)+a(t)p(x(t))+k(t). \tag{3.7}$$

Let $x\in P_T$ and assume (3.2). Multiply both sides of (3.7) by $e_{Pa}(t,0)$ and then integrate both sides from $t-T$ to $t$ to obtain

$$\begin{aligned} &e_{Pa}(t,0)x(t)-e_{Pa}(t-T,0)x(t-T)\\ &=\int_{t-T}^{t}[Pa(u)x^\sigma(u)+a(u)p(x(u))\\ &+k(u)]e_{Pa}(u,0)\Delta u. \end{aligned}$$

Divide both sides of the above equation by $e_{Pa}(t,0)$ and use the fact that $x(t-T)=x(t)$ to obtain

$$\begin{aligned} &x(t)\left[1-e_{\ominus(Pa)}(t,t-T)\right]\\ &=\int_{t-T}^{t}[Pa(u)x^\sigma(u)+a(u)p(x(u))\\ &+k(u)]e_{\ominus(Pa)}(t,u)\Delta u, \end{aligned}$$

where we have used Lemma 2.10 to simplify the exponentials. Since every step is reversible, the converse holds. The proof of (3.5) is similar and hence we omit it. □

Define mappings $A$ and $B$ from $\mathbb{M}$ into $\mathbb{P}_T$ as follows. For $(\varphi_1,\varphi_2)\in\mathbb{M}$,

$$A(\varphi_1,\varphi_2)(t)=(A_1(\varphi_1,\varphi_2)(t),A_2(\varphi_1,\varphi_2)(t)),$$

such that

$$\begin{aligned}&A_1(\varphi_1,\varphi_2)(t)\\&=\eta_1\left\{\int_{t-T}^{t}a(u)\left[p(\varphi_1(u))+P\varphi_1^{\sigma}(u)\right]e_{\ominus(Pa)}(t,u)\Delta u\right.\\&+\int_{t-T}^{t}\int_{-\infty}^{u}C(t,s)\left[h(\varphi_2(s))-H\varphi_2(s)\right]\Delta s\\&\left.\times e_{\ominus(Pa)}(t,u)\Delta u\right\},\end{aligned}$$

$$\begin{aligned}&A_2(\varphi_1,\varphi_2)(t)\\&=\eta_2\left\{\int_{t-T}^{t}b(u)\left[q(\varphi_2(u))+Q\varphi_2^{\sigma}(u)\right]e_{\ominus(Qb)}(t,u)\Delta u\right.\\&+\int_{t-T}^{t}\int_{-\infty}^{u}D(t,s)\left[g(\varphi_1(s))-G\varphi_1(s)\right]\Delta s\\&\left.\times e_{\ominus(Qb)}(t,u)\Delta u\right\},\end{aligned}$$

and for $(\psi_1,\psi_2)\in\mathbb{M}$,

$$B(\psi_1,\psi_2)(t)=(B_1(\psi_1,\psi_2)(t),B_2(\psi_1,\psi_2)(t)),$$

such that

$$\begin{aligned}&B_1(\psi_1,\psi_2)(t)\\&=\eta_1\left\{\int_{t-T}^{t}\int_{-\infty}^{u}C(u,s)H\psi_2(s)\Delta s e_{\ominus(Pa)}(t,u)\Delta u\right.\\&\left.+\int_{t-T}^{t}e(u)e_{\ominus(Pa)}(t,u)\Delta u\right\}.\end{aligned}$$

$$\begin{aligned}&B_2(\psi_1,\psi_2)(t)\\&=\eta_2\left\{\int_{t-T}^{t}\int_{-\infty}^{u}D(u,s)G\psi_1(s)\Delta s e_{\ominus(Qb)}(t,u)\Delta u\right.\\&\left.+\int_{t-T}^{t}f(u)e_{\ominus(Qb)}(t,u)\Delta u\right\}.\end{aligned}$$

It can be easily verified that both $A(\varphi_1,\varphi_2)$ and $B(\psi_1,\psi_2)$ are $T$-periodic and continuous. Assume

$$|\eta_1|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}\int_{-\infty}^{u}|C(u,s)||H|\Delta s e_{\ominus(Pa)}(t,u)\Delta u\le\alpha_1<1, \tag{3.8}$$

$$|\eta_2|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}\int_{-\infty}^{u}|D(u,s)||G|\Delta s e_{\ominus(Qb)}(t,u)\Delta u\le\alpha_2<1, \tag{3.9}$$

$$\begin{aligned}&|\eta_1|\sup_{t\in\mathbb{T}}\left\{\int_{t-T}^{t}|a(u)|P^*e_{\ominus(Pa)}(t,u)\Delta u\right.\\&\left.+\int_{t-T}^{t}\int_{-\infty}^{u}|C(t,s)|H^*\Delta s e_{\ominus(Pa)}(t,u)\Delta u\right\}\\&\le\beta_1<\infty,\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&|\eta_2|\sup_{t\in\mathbb{T}}\left\{\int_{t-T}^{t}|b(u)|Q^*e_{\ominus(Qb)}(t,u)\Delta u\right.\\&\left.+\int_{t-T}^{t}\int_{-\infty}^{u}|D(t,s)|G^*\Delta s e_{\ominus(Qb)}(t,u)\Delta u\right\}\\&\le\beta_2<\infty.\end{aligned} \tag{3.11}$$

Choose the constant $m$ of (3.3) satisfying

$$|\eta_1|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}|e(u)|e_{\ominus(Pa)}(t,u)\Delta u+\alpha_1 m+\beta_1\le m, \tag{3.12}$$

and

$$|\eta_2|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}|f(u)|e_{\ominus(Qb)}(t,u)\Delta u+\alpha_2 m+\beta_2\le m. \tag{3.13}$$

**Lemma 3.3.** *Assume (3.1), (3.2) and (3.8)-(3.13) hold. Then $B$ is a contraction from $\mathbb{M}$ into $\mathbb{M}$.*

*Proof.* For $(\psi_1,\psi_2)\in\mathbb{M}$,

$$\begin{aligned}&|B_1(\psi_1,\psi_2)(t)|\\&\le m|\eta_1|\int_{t-T}^{t}\int_{-\infty}^{u}|C(u,s)||H|\Delta s e_{\ominus(Pa)}(t,u)\Delta u\\&+|\eta_1|\int_{t-T}^{t}|e(u)|e_{\ominus(Pa)}(t,u)\Delta u\\&\le|\eta_1|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}|e(u)|e_{\ominus(Pa)}(t,u)\Delta u+\alpha_1 m\le m,\end{aligned}$$

and

$$\begin{aligned}&|B_2(\psi_1,\psi_2)(t)|\\&=m|\eta_2|\int_{t-T}^{t}\int_{-\infty}^{u}|D(u,s)||G|\Delta s e_{\ominus(Qb)}(t,u)\Delta u\\&+|\eta_2|\int_{t-T}^{t}|f(u)|e_{\ominus(Qb)}(t,u)\Delta u\\&\le|\eta_2|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}|f(u)|e_{\ominus(Qb)}(t,u)\Delta u+\alpha_2 m\le m,\end{aligned}$$

then

$$\|B(\psi_1,\psi_2)\|\le m.$$

For $(\phi_1,\phi_2),(\psi_1,\psi_2)\in\mathbb{M}$, we obtain

$$\begin{aligned}&|B_1(\phi_1,\phi_2)(t)-B_1(\psi_1,\psi_2)(t)|\\&\le|\eta_1|\int_{t-T}^{t}\int_{-\infty}^{u}|C(u,s)||H||\phi_2(s)-\psi_2(s)|\Delta s\\&\times e_{\ominus(Pa)}(t,u)\Delta u\\&\le\alpha_1\|(\phi_1,\phi_2)-(\psi_1,\psi_2)\|,\end{aligned}$$

and in a similar way one can easily show that

$$|B_2(\phi_1,\phi_2)(t)-B_2(\psi_1,\psi_2)(t)|\le\alpha_2\|(\phi_1,\phi_2)-(\psi_1,\psi_2)\|.$$

Therefore

$$\|B(\phi_1,\phi_2)(t)-B(\psi_1,\psi_2)(t)\|\le\alpha\|(\phi_1,\phi_2)-(\psi_1,\psi_2)\|.$$

where $\alpha=\max\{\alpha_1,\alpha_2\}<1$. This proves that $B$ is a contraction mapping from $\mathbb{M}$ into $\mathbb{M}$. □

**Lemma 3.4.** *Assume (1.2), (1.3), (3.1), (3.2) and (3.10)-(3.13). Then $A$ from $\mathbb{M}$ into $\mathbb{M}$ is continuous, and $A\mathbb{M}$ is contained in a compact subset of $\mathbb{P}_T$.*

*Proof.* For any $(\varphi_1,\varphi_2)\in\mathbb{M}$, it follows from (1.2) and (1.3) that

$$\begin{aligned}&|A_1(\varphi_1,\varphi_2)(t)|\\&\le|\eta_1|\left\{\int_{t-T}^{t}|a(u)|\,|p(\varphi_1(u))+P\varphi_1^{\sigma}(u)|\right.\\&\times e_{\ominus(Pa)}(t,u)\Delta u\\&+\int_{t-T}^{t}\int_{-\infty}^{u}|C(t,s)|\,|h(\varphi_2(s))-H\varphi_2(s)|\,\Delta s\\&\times e_{\ominus(Pa)}(t,u)\Delta u\}\\&\le|\eta_1|\left\{\int_{t-T}^{t}|a(u)|P^{*}e_{\ominus(Pa)}(t,u)\Delta u\right.\\&\left.+\int_{t-T}^{t}\int_{-\infty}^{u}|C(t,s)|H^{*}\Delta s e_{\ominus(Pa)}(t,u)\Delta u\right\},\end{aligned}$$

Using (3.10) and (3.12), we get

$$|A_1(\varphi_1,\varphi_2)(t)|\le\beta_1\le m.$$

and in a similar way we have

$$|A_2(\varphi_1,\varphi_2)(t)|\le\beta_2\le m.$$

Therefore

$$\|A(\varphi_1,\varphi_2)\|\le m. \tag{3.14}$$

So, $A$ maps $\mathbb{M}$ into $\mathbb{M}$, and the set $\{A(\phi_1,\phi_2)\}$ for $(\phi_1,\phi_2)\in\mathbb{M}$ is uniformly bounded. To show that $A$ is a continuous we let $\{(\phi_1^n,\phi_2^n)\}$ be any sequence of functions in $\mathbb{M}$ with $\|(\phi_1^n,\phi_2^n)-(\phi_1,\phi_2)\|\to 0$ as $n\to\infty$. Since $\mathbb{M}$ is closed, we have $(\phi_1,\phi_2)\in\mathbb{M}$. Then by the definition of $A$ we have

$$\begin{aligned}&\|A(\phi_1^n,\phi_2^n)-A(\phi_1,\phi_2)\|\\&=\max\left\{\sup_{t\in[0,T]}|A_1(\phi_1^n,\phi_2^n)(t)-A_1(\phi_1,\phi_2)(t)|,\right.\\&\left.\sup_{t\in[0,T]}|A_2(\phi_1^n,\phi_2^n)(t)-A_2(\phi_1,\phi_2)(t)|\right\},\end{aligned}$$

in which

$$\begin{aligned}&|A_1(\phi_1^n,\phi_2^n)(t)-A_1(\phi_1,\phi_2)(t)|\\&=\left|\eta_1\left\{\int_{t-T}^{t}a(u)[p(\phi_1^n(u))+P\phi_1^{n\sigma}(u)]\right.\right.\\&\times e_{\ominus(Pa)}(t,u)\Delta u\\&-\int_{t-T}^{t}a(u)[p(\phi_1(u))+P\phi_1^{\sigma}(u)]e_{\ominus(Pa)}(t,u)\Delta u\\&+\int_{t-T}^{t}\int_{-\infty}^{t}C(t,s)[h(\phi_2^n(s))-H\phi_2^n(s)]\Delta s\\&\times e_{\ominus(Pa)}(t,u)\Delta u\\&-\int_{t-T}^{t}\int_{-\infty}^{t}C(t,s)[h(\phi_2(s))-H\phi_2(s)]\Delta s\\&\left.\times e_{\ominus(Pa)}(t,u)\Delta u\}\right|\\&\le|\eta_1|\left\{\int_{t-T}^{t}|a(u)|[|p(\phi_1^n(u))-p(\phi_1(u))|\right.\\&+|P\phi_1^{n\sigma}(u)-P\phi_1^{\sigma}(u)|]e_{\ominus(Pa)}(t,u)\Delta u\\&+\int_{t-T}^{t}\int_{-\infty}^{t}|C(t,s)|[|h(\phi_2^n(s))-h(\phi_2(s))|\\&+|H\phi_2^n(s)-H\phi_2(s)|]\Delta s e_{\ominus(Pa)}(t,u)\Delta u\}.\end{aligned}$$

The continuity of $p$ and $h$ along with the Lebesgue dominated convergence theorem implies that

$$\sup_{t\in[0,T]}|A_1(\phi_1^n,\phi_2^n)(t)-A_1(\phi_1,\phi_2)(t)|\to 0 \text{ as } n\to\infty.$$

By a similar argument one can easily argue that

$$\sup_{t\in[0,T]}|A_2(\phi_1^n,\phi_2^n)(t)-A_2(\phi_1,\phi_2)(t)|\to 0 \text{ as } n\to\infty.$$

Thus,

$$\|A(\phi_1^n,\phi_2^n)-A(\phi_1,\phi_2)\|\to 0 \text{ as } n\to\infty.$$

This proves that $A$ is a continuous mapping.

It is trivial to show that for all $(\phi_1,\phi_2)\in\mathbb{M}$, there exist constants $L_1,L_2>0$ such that $\left|A_1(\phi_1,\phi_2)^{\Delta}(t)\right|\le L_1$ and $\left|A_2(\phi_1,\phi_2)^{\Delta}(t)\right|\le L_2$. This means $\left|A(\phi_1,\phi_2)^{\Delta}(t)\right|\le L$ where $L=\max\{L_1,L_2\}$. Therefore that the set $\{A(\phi_1,\phi_2)\}$ for $(\phi_1,\phi_2)\in\mathbb{M}$ is equicontinuous. Hence, by the Arzela-Ascoli theorem, $A\mathbb{M}$ is contained in a compact subset of $\mathbb{P}_T$. □

**Theorem 3.5.** *Suppose the assumptions of Lemmas 3.3 and 3.4 hold. Then (1.1) has a continuous $T$-periodic solution.*

*Proof.* For $(\varphi_1,\varphi_2),(\psi_1,\psi_2)\in\mathbb{M}$, we get

$$\begin{aligned}
&|A_1(\varphi_1,\varphi_2)(t)+B_1(\psi_1,\psi_2)(t)|\\
&=\bigg|\eta_1\bigg\{\int_{t-T}^{t}a(u)\left[p(\varphi_1(u))+P\varphi_1^{\sigma}(u)\right]\\
&\times e_{\ominus(Pa)}(t,u)\Delta u\\
&+\int_{t-T}^{t}\int_{-\infty}^{u}C(t,s)\left[h(\varphi_2(s))-H\varphi_2(s)\right]\Delta s\\
&\times e_{\ominus(Pa)}(t,u)\Delta u\bigg\}\\
&+\eta_1\bigg\{\int_{t-T}^{t}\int_{-\infty}^{u}C(u,s)H\psi_2(s)\Delta s e_{\ominus(Pa)}(t,u)\Delta u\\
&+\int_{t-T}^{t}e(u)e_{\ominus(Pa)}(t,u)\Delta u\bigg\}\bigg|\\
&\le|\eta_1|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}|e(u)|e_{\ominus(Pa)}(t,u)\Delta u+\alpha_1 m+\beta_1\\
&\le m.
\end{aligned}$$

and

$$\begin{aligned}
&|A_2(\varphi_1,\varphi_2)(t)+B_2(\psi_1,\psi_2)(t)|\\
&\le|\eta_1|\sup_{t\in\mathbb{T}}\int_{t-T}^{t}|f(u)|e_{\ominus(Qb)}(t,u)\Delta u+\alpha_2 m+\beta_2\\
&\le m.
\end{aligned}$$

This implies that

$$\|A(\varphi_1,\varphi_2)+B(\psi_1,\psi_2)\|\le m,$$

which proves that $A(\varphi_1,\varphi_2)+B(\psi_1,\psi_2)\in\mathbb{M}$.

Therefore, by Krasnoselskii's theorem there exists a function $(x,y)$ in $\mathbb{M}$ such that

$$(x,y)=A(x,y)+B(x,y).$$

This proves that (1.1) has a continuous $T$-periodic solution $(x,y)$. □

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