Malaya Journal of Matematik, Vol. 8, No. 4, 1443-1453, 2020
https://doi.org/10.26637/MJM0804/0018

Strong and mild solutions of the system of fractional ordinary differential equation and it's applications

J.M. Sonawane[1*], B. R. Sontakke[2] and K. C. Takale[3]

Abstract
The purpose of this paper is to solve the system of fractional ordinary differential equations. Furthermore, we prove the solution obtained by using Laplace transform technique are mild and strong solutions. We established the existence and uniqueness of the solution. Also, we simulate strong solutions of the system of fractional order differential equations by maxima software.

Keywords
Fractional derivatives, Mittag-Leffler function, Strong and Mild Solutions, Green's Function, Maxima.

[1] *Department of Mathematics, R.H. Sapat College of Engineering, Management Studies and Research Center, Nashik, India.*
[2] *Department of Mathematics, Pratishthan Mahavidyalaya, Paithan, Aurangabad, India.*
[3] *Department of Mathematics, RNC Arts, JDB Commerce and NSC Science College, Nashik-Road, Nashik, India.*
***Corresponding author**: [1] jagdish.sonawane555@gmail.com; [2] brsontakke@rediffmail.com; [3]kalyanraotakale@rediffmail.com
Article History: Received **06** May **2020**; Accepted **29** August **2020**

Contents

1. Introduction

Fractional Calculus is field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently many applications of fractional calculus can be found in basic sciences, technical sciences, fluid dynamics,stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics etc.Various phenomenon were modelled with fractional differential equations. Fractional differential equations have appeared in many branches like physics,chemistry,biology,economics and engineering. There has been a considerable development in fractional differential equation in the last decades.

Recently, Many researcher solved ordinary fractional differential equation and obtained their mild and strong solutions. Zufeng Zhang, Bin Liu established sufficient conditions for the existence of mild solution of fractional differential evolution equation by using Banach fixed point theorem [14]. H.L.Tidke, M.B.Dhakne prove the existence and uniqueness of mild and strong solutions of a nonlinear Volterra-integro differential equations with non-local condition, and analysis is based on semigroup theory and Banach fixed point theorem[1, 3]. Adel Jawahdow is concerned with the existence of mild solutions for fractional semilinear differential equations with non-local conditions in separable Banach space and furthermore the result is obtained using the technique of measures of non-compactness in Banach space of continuous functions and Schauder fixed point theorem[1]. K.Bhalchandran, S. Ilamaran prove the existence and uniqueness of mild and strong solutions of a semilinear evolution equation with non-local initial conditions using method of semigroups and Banach fixed point theorem[5]. Lijuin Pan is concerned with the existence of mild solution for impulsive stochastic differential equation with non-local condition in pc-norm and approach is based on Kransnoselskii fixed point theorem[7]. J.Vanterler da c.Sousa, Leandro S. Tavares, E. Capelas de Oliveira investigate the existence and uniqueness of mild and strong solution of fractional semilinear evolution equation in the Hilfer sense by means of Banach fixed point theorem and Grunwall inequality[4]. Lizhen Chen , Zhenbin Fan prove new existence results of mild solution to fractional differential equation with non-local condition in Banach space[8]. Sayyedah A.Qasem, Rabha W.Ibrahim, ZailamSiri prove existence and uniqueness of bounded m-solutions and

s-solutions for fractional integro-differential equations with fractional resolvent and unbounded delay[12]. Uttam Ghosh, Susmita Sarkar, Shantanu Das developed analytical method to solve the system of fractional differential equation in terms of M-L function and generalized sine and cosine function where fractional derivative operator is of Jumarie fractional derivative which is modified RL frcational derivative[13]. Sabbavarapiu Nageswara Rao, Meshari Alesemil established the existence and uniqueness results for a non-linear coupled system of Caputo type fractional differential equation, supplemented with coupled fractional non-local non-separated boundary conditions by using Banach contraction principal and Leray Schauder fixed point theorem[11]. Mohamed A.E. Herzallah studied two fractional periodic boundary value problems and under some conditions the uniqueness of mild solution is proved for both problems. Finally these mild solutions will be strong solutions under certain conditions[10].
In this connection we solved system of fractional ordinary differential equations, and obtained their mild and strong solutions. The main Problem is

$$\left.\begin{array}{l} a_1\, {}_0^C D_t^\alpha x(t) + b_1 y(t) = f(t) \\ a_2 x(t) + b_2\, {}_0^C D_t^\alpha y(t) = g(t) \end{array}\right\}. \tag{1.1}$$

where $0<\alpha<1$ and $a_1,a_2,b_1,b_2 \in R-\{0\}$, initial conditions are

$$x(0)=x(1), y(0)=y(1).$$

Here $x(t)$ and $y(t)$ are unknown functions and $f(t), g(t)$ are known functions. Also, we prove that the mild solutions will strong solutions under certain conditions. Furthermore, we also established the existence and uniqueness by using Banach contraction principal and Schauder fixed point theorem.
We organized the paper as follows:
In section 2, we define the basic definitions and properties of fractional calculus. Section 3, is devoted for strong and mild solutions of the system of fractional ordinary differential equations of order $0<\alpha\leq 1$. Section 4, is devoted for existence and uniqueness of the strong and mild solutions of the system of fractional ordinary differential equations. In section 5, we solve some test problems and their solutions are represented graphically by Maxima software.

2. PRELIMINARIES

Definition 2.1. *Riemann-Liouville Fractional integral: If $f(t) \in C[a,b]$ and $a<t<b$ then*

$${}_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \tag{2.1}$$

where $\alpha \in (-\infty,\infty)$ is called the Riemann-Liouville fractional integral of order α.

Definition 2.2. *Riemann-Liouville Fractional Derivative: If $f(t) \in C[a,b]$ and $a<t<b$ then*

$${}_a^R D_t^\alpha f(t) = D_a^n I_t^{n-\alpha} f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \tag{2.2}$$

where $\alpha \in (n-1,n)$ is called the Riemann-Liouville fractional derivative of order α.

Definition 2.3. *M.Caputo Fractional Derivative: If $f(t) \in C[a,b]$ and $a<t<b$ then*

$${}_a^C D_t^\alpha f(t) = {}_a I_t^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s)ds, \tag{2.3}$$

where $\alpha \in (n-1,n)$ is called the Caputo fractional derivative of order α.

Definition 2.4. *Mittag-Leffler function of one parameter: The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as*

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}, \alpha \in C, Re(\alpha)>0 \tag{2.4}$$

Definition 2.5. *Mittag-Leffler function of two parameter: The Mittag-Leffler function of two parameter is denoted by $E_{\alpha,\beta}(z)$ and defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta \in C\, Re(\alpha)>0, Re(\beta)>0. \tag{2.5}$$

Definition 2.6. *Laplace transform of Caputo Fractional derivative: The Laplace Transform of Caputo Fractional derivative of order $\alpha(\alpha>0)$ is defined as*

$$L\{{}_0^C D_t^\alpha f(t)\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \tag{2.6}$$

Definition 2.7. *Mild and Strong Solutions: Let $J=[0,1]$, the function $x,y \in C(J)$ is called*
(i) A strong solution of system (1.1) if $x,y \in AC(J)$ and (1.1) holds on J.
(ii) A mild solution of problem (1.1) on J if

$$x(t) = x(0) - \frac{b_1}{a_1}\, {}_0 I_t^\alpha y(t) + \frac{1}{a_1}\, {}_0 I_t^\alpha f(t) \tag{2.7}$$

$$y(t) = y(0) - \frac{a_2}{b_2}\, {}_0 I_t^\alpha x(t) + \frac{1}{b_2}\, {}_0 I_t^\alpha g(t) \tag{2.8}$$

Proposition 2.8. *Convolution integral: The Convolution integral of two functions $f(t)$ and $g(t)$ is denoted by $f(t)*g(t)$ and is defined as*

$$f(t)*g(t) = \int_0^t f(t-s)\, g(s)\, ds \tag{2.9}$$

Proposition 2.9. *Laplace transform of Convolution integral: The Laplace transform of Convolution integral of two functions $f(t)$ and $g(t)$ is given as*

$$L\{f(t)*g(t)\} = L\{f(t)\}\cdot L\{g(t)\} \tag{2.10}$$

Proposition 2.10. *Laplace Transform of Mittag-Leffler function*

$$L\{t^{\alpha k+\beta-1}E^{k}_{\alpha,\beta}(at^{\alpha})\} = \frac{k!s^{\alpha-\beta}}{(s^{\alpha}-a)^{k+1}} \tag{2.11}$$

Hence we have following inverse Laplace Transform

$$L^{-1}\{\frac{s^{\alpha-\beta}}{(s^{\alpha}-a)^{k+1}}\} = \frac{t^{\alpha k+\beta-1}}{k!}E^{k}_{\alpha,\beta}(at^{\alpha}) \tag{2.12}$$

Proposition 2.11. *Integral of Mittag-Leffler function*

$$_0I^{\alpha}_t[t^{q-1}E_{p,q}(\lambda t^{p})] = t^{q+\alpha-1}E_{p,q+\alpha}(\lambda t^{p}) \tag{2.13}$$

Proposition 2.12. *Derivative of Mittag-Leffler function*

$$\frac{d}{dt}[t^{q-1}E_{p,q}(\lambda t^{p})] = t^{q-2}E_{p,q-1}(\lambda t^{p}) \tag{2.14}$$

3. MAIN RESULTS

Lemma 3.1. *Let $0<\alpha<1$ and $u\in C(0,T)$. If there exist $f\in C(0,T)$ such that $u = {}_0I^{\alpha}_t f$ then the function u has fractional derivative ${}_0D^{\alpha}_t u = f$*

Theorem 3.2. *If $f(t)$, $g(t)$ are continuous function and a_1, b_1, a_2 and b_2 are non-zero real constant then the system (1.1) has continuous mild solution are given by*

$$\begin{aligned} x(t) =& \frac{1}{a_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)\\ &-\frac{\lambda}{a_2}t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*g(t)\\ &-\frac{b_2\lambda}{a_2}t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0) \end{aligned} \tag{3.1}$$

$$\begin{aligned} y(t) =& \frac{-\lambda}{b_1}t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*f(t)\\ &-\frac{a_1\lambda}{b_1}t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\ &+\frac{1}{b_2}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)+E_{2\alpha,1}(\lambda t^{2\alpha})y(0) \end{aligned} \tag{3.2}$$

This solutions become strong if x, $y\in AC(0,T)$.

Proof. We have to find solution of system(1.1) by using Laplace transform, therefore taking Laplace transform of both sides of the system (1.1), we get

$$\begin{aligned} a_1L\{{}^C_0D^{\alpha}_t x(t)\}+b_1L\{y(t)\} =& L\{f(t)\}\\ a_2L\{x(t)\}+b_2L\{{}^C_0D^{\alpha}_t y(t)\} =& L\{g(t)\}\\ a_1s^{\alpha}X(s)-a_1s^{\alpha-1}x(0)+b_1Y(s) =& F(s)\\ a_2X(s)+b_2s^{\alpha}Y(s)-b_2s^{\alpha-1}y(0) =& G(s) \end{aligned}$$

$$\begin{aligned} a_1s^{\alpha}X(s)+b_1Y(s) =& F(s)+a_1s^{\alpha-1}x(0) \\ a_2X(s)+b_2s^{\alpha}Y(s) =& G(s)+b_2s^{\alpha-1}y(0) \end{aligned} \tag{3.3}$$

Solving above equations simultaneously for elimination of $Y(s)$, we get

$$\begin{aligned} (a_1b_2s^{2\alpha}-a_2b_1)X(s) =& b_2s^{\alpha}F(s)+a_1b_2s^{2\alpha-1}x(0)\\ &-b_1G(s)-b_1b_2s^{\alpha-1}y(0). \end{aligned}$$

$$\begin{aligned} X(s) =& \frac{1}{a_1}\frac{s^{\alpha}}{s^{2\alpha}-\lambda}F(s)+\frac{s^{2\alpha-1}}{s^{2\alpha}-\lambda}x(0)-\frac{\lambda}{a_2}\frac{G(s)}{s^{2\alpha}-\lambda}\\ &-\frac{b_2\lambda}{a_2}\frac{s^{\alpha-1}}{s^{2\alpha}-\lambda}y(0) \end{aligned}$$

where $\lambda = \frac{a_2b_1}{a_1b_2}$
Taking inverse laplace transform, we get

$$\begin{aligned} x(t) =& \frac{1}{a_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)\\ &-\frac{\lambda}{a_2}t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*g(t)\\ &-\frac{b_2\lambda}{a_2}t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0) \end{aligned} \tag{3.4}$$

From equation (3.3), we have

$$\begin{aligned} Y(s) =& \frac{1}{b_1}\frac{-\lambda}{s^{2\alpha}-\lambda}F(s)-\frac{a_1\lambda}{b_1}\frac{s^{\alpha-1}}{s^{2\alpha}-\lambda}x(0)\\ &+\frac{1}{b_2}\frac{s^{\alpha}}{s^{2\alpha}-\lambda}G(s)+\frac{s^{2\alpha-1}}{s^{2\alpha}-\lambda}y(0). \end{aligned}$$

Taking inverse laplace transform, we get

$$y(t) = \frac{-\lambda}{b_1}t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*f(t) \tag{3.5}$$

$$\begin{aligned} &-\frac{a_1\lambda}{b_1}t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\ &+\frac{1}{b_2}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)+E_{2\alpha,1}(\lambda t^{2\alpha})y(0) \end{aligned} \tag{3.6}$$

Using equations (2.7) and (2.8), to prove that the solution of the system (1.1) are mild.

$$
\begin{aligned}
&x(0)-\frac{b_1}{a_1}\,{}_0I_t^{\alpha}y(t)+\frac{1}{a_1}\,{}_0I_t^{\alpha}f(t)\\
&=x(0)-\frac{b_1}{a_1}\,{}_0I_t^{\alpha}\Big\{\frac{-\lambda}{b_1}\,t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*f(t)\\
&\quad-\frac{a_1\lambda}{b_1}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)+\frac{1}{b_2}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})\\
&\quad*g(t)+E_{2\alpha,1}(\lambda t^{2\alpha})y(0)\Big\}+\frac{1}{a_1}\,{}_0I_t^{\alpha}f(t)\\
&=x(0)+\frac{\lambda}{a_1}\,t^{3\alpha-1}E_{2\alpha,3\alpha}(\lambda t^{2\alpha})*f(t)\\
&\quad+\lambda\,\frac{1}{\lambda}\Big[E_{2\alpha,1}(\lambda t^{2\alpha})-1\Big]x(0)\\
&\quad-\frac{b_1}{a_1 b_2}\Big[t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})\Big]*g(t)\\
&\quad-\frac{b_1}{a_1}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)+\frac{1}{a_1}\,{}_0I_t^{\alpha}f(t)\\
&=x(0)+\frac{\lambda}{a_1}\Big[\frac{1}{\lambda}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})-\frac{1}{\lambda}\,\frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big]*f(t)\\
&\quad+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)-x(0)-\frac{\lambda}{a_2}\,[t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})]\\
&\quad*g(t)-\frac{b_2\lambda}{a_2}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)+\frac{1}{a_1}\,{}_0I_t^{\alpha}f(t)\\
&=\frac{1}{a_1}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)-\frac{1}{a_1}\,{}_0I_t^{\alpha}f(t)\\
&\quad+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)-\frac{\lambda}{a_2}\,[t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})]*g(t)\\
&\quad-\frac{b_2\lambda}{a_2}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)+\frac{1}{a_1}\,{}_0I_t^{\alpha}f(t)\\
&=\frac{1}{a_1}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)\\
&\quad-\frac{\lambda}{a_2}\,t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*g(t)\\
&\quad-\frac{b_2\lambda}{a_2}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)\\
&=x(t)
\end{aligned}
$$

Hence, equation (2.7) is proved.

$$
\begin{aligned}
&y(0)-\frac{a_2}{b_2}\,{}_0I_t^{\alpha}x(t)+\frac{1}{b_2}\,{}_0I_t^{\alpha}g(t)\\
&=y(0)-\frac{a_2}{b_2}\,{}_0I_t^{\alpha}\Big\{\frac{1}{a_1}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)\\
&\quad+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)-\frac{\lambda}{a_2}\,t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*g(t)\\
&\quad-\frac{b_2\lambda}{a_2}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)\Big\}+\frac{1}{b_2}\,{}_0I_t^{\alpha}g(t)\\
&=y(0)-\frac{a_2}{b_2 a_1}\Big[t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})\Big]*f(t)\\
&\quad-\frac{a_2}{b_2}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)+\frac{\lambda}{b_2}\,t^{3\alpha-1}E_{2\alpha,3\alpha}(\lambda t^{2\alpha})\\
&\quad*g(t)+\lambda\,\frac{1}{\lambda}\Big[E_{2\alpha,1}(\lambda t^{2\alpha})-1\Big]y(0)+\frac{1}{b_2}\,{}_0I_t^{\alpha}g(t)\\
&=y(0)-\frac{\lambda}{b_1}\Big[t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})\Big]*f(t)\\
&\quad-\frac{a_1\lambda}{b_1}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\
&\quad+\frac{\lambda}{b_2}\Big[\frac{1}{\lambda}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})-\frac{1}{\lambda}\,\frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big]*g(t)\\
&\quad+E_{2\alpha,1}(\lambda t^{2\alpha})y(0)-y(0)+\frac{1}{b_2}\,{}_0I_t^{\alpha}g(t)\\
&=-\frac{\lambda}{b_1}\,t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*f(t)\\
&\quad-\frac{a_1\lambda}{b_1}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\
&\quad+\frac{1}{b_2}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)-\frac{1}{b_2}\,{}_0I_t^{\alpha}g(t)\\
&\quad+E_{2\alpha,1}(\lambda t^{2\alpha})y(0)+\frac{1}{b_2}\,{}_0I_t^{\alpha}g(t)\\
&=y(t)
\end{aligned}
$$

Hence, equation (2.8) is Proved.
To prove $x(t)$ is strong solution of the system(1.1)

$$
\begin{aligned}
x'(t)&=\frac{d}{dt}x(t)\\
&=\frac{d}{dt}\Big[\frac{1}{a_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)\\
&\quad-\frac{\lambda}{a_2}\,t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*g(t)\\
&\quad-\frac{b_2\lambda}{a_2}\,t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)\Big]\\
&=\frac{1}{a_1}t^{\alpha-2}E_{2\alpha,\alpha-1}(\lambda t^{2\alpha})*f(t)+t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})x(0)\\
&\quad-\frac{\lambda}{a_2}\,t^{2\alpha-2}E_{2\alpha,2\alpha-1}(\lambda t^{2\alpha})*g(t)\\
&\quad-\frac{b_2\lambda}{a_2}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})y(0)
\end{aligned}
$$

$$
\begin{aligned}
x'(t)&=\frac{1}{a_1}\,t^{\alpha-2}E_{2\alpha,\alpha-1}(\lambda t^{2\alpha})*f(t)\\
&\quad+\lambda t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})x(0)\\
&\quad-\frac{\lambda}{a_2}\,t^{2\alpha-2}E_{2\alpha,2\alpha-1}(\lambda t^{2\alpha})*g(t)\\
&\quad-\frac{b_1}{a_1}\,t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})y(0)
\end{aligned}
$$

$$
\begin{aligned}
{}_0^C D_t^{\alpha}x(t)&={}_0I_t^{1-\alpha}x'(t)\\
&={}_0I_t^{1-\alpha}\Big[\frac{1}{a_1}\,t^{\alpha-2}E_{2\alpha,\alpha-1}(\lambda t^{2\alpha})*f(t)
\end{aligned}
$$

$$+\lambda t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})x(0) - \frac{\lambda}{a_2}t^{2\alpha-2}E_{2\alpha,2\alpha-1}(\lambda t^{2\alpha})*g(t) - \frac{b_1}{a_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})y(0)\Big]$$

$$\begin{aligned}{}_0^C D_t^{\alpha}x(t) &= \frac{1}{a_1}t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})*f(t)+\lambda t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\ &\quad -\frac{\lambda}{a_2}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)-\frac{b_1}{a_1}E_{2\alpha,1}(\lambda t^{2\alpha})y(0)\end{aligned} \tag{3.7}$$

$$\begin{aligned}\frac{1}{a_1}f(t)-\frac{b_1}{a_1}y(t) &= \frac{1}{a_1}f(t)-\frac{b_1}{a_1}\Big[-\frac{\lambda}{b_1}t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})\\ &\quad *f(t)-\frac{a_1\lambda}{b_1}t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{1}{b_2}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)+E_{2\alpha,1}(\lambda t^{2\alpha})y(0)\Big]\\ &= \frac{1}{a_1}f(t)+\frac{\lambda}{a_1}\Big[\frac{1}{\lambda}t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{1}{\lambda}\frac{t^{-1}}{\Gamma(0)}*f(t)\Big]+\lambda t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\ &\quad -\frac{b_1}{a_1b_2}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)-\frac{b_1}{a_1}E_{2\alpha,1}(\lambda t^{2\alpha})y(0)\end{aligned}$$

$$\begin{aligned}\frac{1}{a_1}f(t)-\frac{b_1}{a_1}y(t) &= \frac{1}{a_1}t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})*f(t)\\ &\quad +\lambda t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\ &\quad -\frac{\lambda}{a_2}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)-\frac{b_1}{a_1}E_{2\alpha,1}(\lambda t^{2\alpha})y(0)\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we get

$${}_0^C D_t^{\alpha}x(t) = \frac{1}{a_1}f(t)-\frac{b_1}{a_1}y(t)$$

Thus, $x(t)$ is strong solution of the system (1.1).
To prove $y(t)$ is strong solution of the system(1.1)

$$\begin{aligned}y'(t) &= \frac{d}{dt}y(t)\\ &= \frac{d}{dt}\Big[-\frac{\lambda}{b_1}t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{a_1\lambda}{b_1}t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{1}{b_2}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*g(t)+E_{2\alpha,1}(\lambda t^{2\alpha})y(0)\Big]\\ &= -\frac{\lambda}{b_1}t^{2\alpha-2}E_{2\alpha,2\alpha-1}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{a_1\lambda}{b_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{1}{b_2}t^{\alpha-2}E_{2\alpha,\alpha-1}(\lambda t^{2\alpha})*g(t)+t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})y(0)\\ &= -\frac{\lambda}{b_1}t^{2\alpha-2}E_{2\alpha,2\alpha-1}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{a_1\lambda}{b_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{1}{b_2}t^{\alpha-2}E_{2\alpha,\alpha-1}(\lambda t^{2\alpha})*g(t)\\ &\quad +\lambda t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})y(0)\end{aligned}$$

$$\begin{aligned}{}_0^C D_t^{\alpha}y(t) &= {}_0I_t^{1-\alpha}y'(t)\\ &= {}_0I_t^{1-\alpha}\Big[-\frac{\lambda}{b_1}t^{2\alpha-2}E_{2\alpha,2\alpha-1}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{a_1\lambda}{b_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{1}{b_2}t^{\alpha-2}E_{2\alpha,\alpha-1}(\lambda t^{2\alpha})*g(t)\\ &\quad +\lambda t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})y(0)\Big]\end{aligned}$$

$$\begin{aligned}{}_0^C D_t^{\alpha}y(t) &= -\frac{\lambda}{b_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{a_1\lambda}{b_1}E_{2\alpha,1}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{1}{b_2}t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})*g(t)\\ &\quad +\lambda t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{1}{b_2}g(t)-\frac{a_2}{b_2}x(t) &= \frac{1}{b_2}g(t)-\frac{a_2}{b_2}\Big[\frac{1}{a_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})\\ &\quad *f(t)+E_{2\alpha,1}(\lambda t^{2\alpha})x(0)-\frac{\lambda}{a_2}t^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda t^{2\alpha})\\ &\quad *g(t)-\frac{b_2\lambda}{a_2}t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)\Big]\\ &= \frac{1}{b_2}g(t)-\frac{a_2}{b_2a_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{a_2}{b_2}E_{2\alpha,1}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{\lambda}{b_2}\Big[\frac{1}{\lambda}t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})*g(t)-\frac{1}{\lambda}\frac{t^{-1}}{\Gamma(0)}*g(t)\Big]\\ &\quad +\lambda t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)\end{aligned}$$

$$\begin{aligned}\frac{1}{b_2}g(t)-\frac{a_2}{b_2}x(t) &= -\frac{\lambda}{b_1}t^{\alpha-1}E_{2\alpha,\alpha}(\lambda t^{2\alpha})*f(t)\\ &\quad -\frac{a_1\lambda}{b_1}E_{2\alpha,1}(\lambda t^{2\alpha})x(0)\\ &\quad +\frac{1}{b_2}t^{-1}E_{2\alpha,0}(\lambda t^{2\alpha})*g(t)\\ &\quad +\lambda t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})y(0)\end{aligned} \tag{3.10}$$

From(3.9) and (3.10), we get

$${}_0^C D_t^{\alpha}y(t) = \frac{1}{b_2}g(t)-\frac{a_2}{b_2}x(t)$$

Thus, $y(t)$ is strong solution of the system (1.1). □

Theorem 3.3. *The solution of the system(1.1) is written in the form of Greens functions as follow*

$$x(t)=\frac{1}{a_1}\int_0^1 G_{\lambda,\alpha}(t,s)f(s)\,ds-\frac{\lambda}{a_2}\int_0^1 \mathscr{G}_{\lambda,\alpha}(t,s)g(s)\,ds$$

and

$$y(t)=\frac{1}{b_2}\int_0^1 G_{\lambda,\alpha}(t,s)g(s)\,ds-\frac{\lambda}{b_1}\int_0^1 \mathscr{G}_{\lambda,\alpha}(t,s)f(s)\,ds$$

where $G_{\lambda,\alpha}(t,s)$ and $\mathscr{G}_{\lambda,\alpha}(t,s)$ are Green's function given as follow

$$G_{\lambda,\alpha}(t,s)=\begin{cases}\phi(\alpha,\lambda,t)(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]\\ +\lambda\psi(\alpha,\lambda,t)(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]\\ +(t-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(t-s)^{2\alpha}], \quad 0\le s\le t\\ \phi(\alpha,\lambda,t)(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]\\ +\lambda\psi(\alpha,\lambda,t)(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]\\ t\le s\le 1\end{cases} \tag{3.11}$$

and

$$\mathscr{G}_{\lambda,\alpha}(t,s)=\begin{cases}\phi(\alpha,\lambda,t)(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]\\ +\psi(\alpha,\lambda,t)(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{\alpha}]\\ +(t-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(t-s)^{2\alpha}], 0\le s\le t\\ \phi(\alpha,\lambda,t)(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]\\ +\psi(\alpha,\lambda,t)(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{\alpha}]\\ t\le s\le 1\end{cases} \tag{3.12}$$

Proof. Using initial condition $x(1)=x(0)$ in (3.4), we get

$$\begin{aligned}x(1)=&\frac{1}{a_1}\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &-\frac{\lambda}{a_2}\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\ &+E_{2\alpha,1}(\lambda)\,x(0)-\frac{b_2\lambda}{a_2}E_{2\alpha,\alpha+1}(\lambda)\,y(0)\end{aligned}$$

$$\begin{aligned}&[1-E_{2\alpha,1}(\lambda)]x(0)+\frac{b_2\lambda}{a_2}E_{2\alpha,\alpha+1}(\lambda)\,y(0)\\ &=\frac{1}{a_1}\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &-\frac{\lambda}{a_2}\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\end{aligned} \tag{3.13}$$

Using initial condition $y(1)=y(0)$ in (3.5), we get

$$\begin{aligned}y(1)=&-\frac{\lambda}{b_1}\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &+\frac{1}{b_2}\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\ &-\frac{a_1\lambda}{b_1}E_{2\alpha,\alpha+1}(\lambda)\,x(0)+E_{2\alpha,1}(\lambda)\,y(0)\end{aligned}$$

$$\frac{a_1\lambda}{b_1}E_{2\alpha,\alpha+1}(\lambda)x(0)+[1-E_{2\alpha,1}(\lambda)]\,y(0) \tag{3.14}$$

$$\begin{aligned}&=-\frac{\lambda}{b_1}\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &+\frac{1}{b_2}\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\end{aligned} \tag{3.15}$$

we, elliminate $y(0)$ from (3.13) and (3.14), we get

$$\begin{aligned}&[1-E_{2\alpha,1}(\lambda)]^2x(0)-\frac{\lambda b_2}{a_2}\frac{\lambda a_1}{b_1}[E_{2\alpha,\alpha+1}(\lambda)]^2x(0)\\ &=\frac{1-E_{2\alpha,1}(\lambda)}{a_1}\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &-\frac{\lambda[1-E_{2\alpha,1}(\lambda)]}{a_2}\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\ &+\frac{\lambda}{b_1}\frac{b_2\lambda}{a_2}E_{2\alpha,\alpha+1}(\lambda)\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &-\frac{1}{b_2}\frac{b_2\lambda}{a_2}E_{2\alpha,\alpha+1}(\lambda)\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\end{aligned}$$

$$\begin{aligned}x(0)=&\frac{1-E_{2\alpha,1}(\lambda)}{a_1\triangle(\alpha,\lambda)}\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &-\frac{\lambda[1-E_{2\alpha,1}(\lambda)]}{a_2\triangle(\alpha,\lambda)}\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\ &+\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_1\triangle(\alpha,\lambda)}\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\ &-\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2\triangle(\alpha,\lambda)}\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\end{aligned} \tag{3.16}$$

where $\triangle(\alpha,\lambda)=[1-E_{2\alpha,1}(\lambda)]^2-\lambda[E_{2\alpha,\alpha+1}(\lambda)]^2$.

From equation (3.14), we get

$$
\begin{aligned}
[1-E_{2\alpha,1}(\lambda)]y(0) = & -\frac{\lambda}{b_1}\int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{1}{b_2}\int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& - \frac{a_1\lambda}{b_1}E_{2\alpha,\alpha+1}(\lambda)\left[\frac{1-E_{2\alpha,1}(\lambda)}{a_1\triangle(\alpha,\lambda)}\right] \\
& \qquad \times \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{a_1\lambda}{b_1}E_{2\alpha,\alpha+1}(\lambda)\left[\frac{\lambda[1-E_{2\alpha,1}(\lambda)]}{a_2\triangle(\alpha,\lambda)}\right] \\
& \qquad \times \int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& - \frac{a_1\lambda}{b_1}E_{2\alpha,\alpha+1}(\lambda)\left[\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_1\triangle(\alpha,\lambda)}\right] \\
& \qquad \times \int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{a_1\lambda}{b_1}E_{2\alpha,\alpha+1}(\lambda)\left[\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2\triangle(\alpha,\lambda)}\right] \\
& \qquad \times \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
= & \frac{-\lambda[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,\alpha+1}(\lambda)}{b_1\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{\lambda[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,\alpha+1}(\lambda)}{b_2\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& - \frac{\lambda[1-E_{2\alpha,1}(\lambda)]^2}{b_1\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{\lambda[1-E_{2\alpha,1}(\lambda)]^2}{b_2\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds
\end{aligned}
$$

$$
\begin{aligned}
y(0) = & \frac{-\lambda E_{2\alpha,\alpha+1}(\lambda)}{b_1\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{b_2\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& - \frac{\lambda[1-E_{2\alpha,1}(\lambda)]}{b_1\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{\lambda[1-E_{2\alpha,1}(\lambda)]}{b_2\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds
\end{aligned}
\tag{3.17}
$$

Using equation (3.16) and (3.17) in equation (3.1), we get

$$
\begin{aligned}
x(t) = & \frac{1}{a_1}\int_0^t (t-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(t-s)^{2\alpha}]f(s)ds \\
& + \frac{[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& - \frac{\lambda[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{2\alpha-1} \\
& \qquad \times E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& + \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)}\int_0^1 (1-s)^{2\alpha-1} \\
& \qquad \times E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& - \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& - \frac{\lambda}{a_2}\int_0^t (t-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(t-s)^{2\alpha})]g(s)ds \\
& + \frac{b_2\lambda}{a_2}\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{b_1\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& - \frac{b_2\lambda}{a_2}\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{b_2\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& + \frac{b_2\lambda}{a_2}\frac{\lambda[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{b_1\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& - \frac{b_2\lambda}{a_2}\frac{\lambda[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{b_2\triangle(\alpha,\lambda)} \\
& \qquad \times \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
= & \left[\frac{[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)} + \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)}\right] \\
& \int_0^1 (1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds \\
& + \frac{1}{a_1}\int_0^t (t-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(t-s)^{2\alpha}]f(s)ds \\
& + \left[\frac{-\lambda[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)} - \frac{\lambda\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)}\right] \\
& \int_0^1 (1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds \\
& - \frac{\lambda}{a_2}\int_0^t (t-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(t-s)^{2\alpha})]g(s)ds \\
& + \left[\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)}\right.
\end{aligned}
$$

$$+\frac{\lambda[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)}\Bigg]$$
$$\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds$$
$$+\Bigg[\frac{-\lambda E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)}-\frac{\lambda[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)}\Bigg]$$
$$\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds$$

$$\begin{aligned}x(t)=&\frac{1}{a_1}\left[\phi(\alpha,\lambda,t)\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds+\int_0^t(t-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(t-s)^{2\alpha}]f(s)ds\right]\\&-\frac{\lambda}{a_2}\left[\phi(\alpha,\lambda,t)\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds+\int_0^t(t-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(t-s)^{2\alpha}]g(s)ds\right]\\&+\frac{\lambda}{a_1}\psi(\alpha,\lambda,t)\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\&-\frac{\lambda}{a_2}\psi(\alpha,\lambda,t)\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\end{aligned}$$

where

$$\phi(\alpha,\lambda,t)=\frac{[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})+\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{\triangle(\alpha,\lambda)} \quad (3.18)$$

$$\psi(\alpha,\lambda,t)=\frac{E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})+[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{\triangle(\alpha,\lambda)} \quad (3.19)$$

$$\therefore\ x(t)=\frac{1}{a_1}\int_0^1 G_{\lambda,\alpha}(t,s)f(s)\,ds-\frac{\lambda}{a_2}\int_0^1\mathscr{G}_{\lambda,\alpha}(t,s)g(s)\,ds \quad (3.20)$$

where $G_{\lambda,\alpha}(t,s)$ and $\mathscr{G}_{\lambda,\alpha}(t,s)$ are Green's function as defined in equation (3.11) and (3.12), respectively. Using equation (3.16) and (3.17) in equation (3.2), we get

$$\begin{aligned}y(t)=&\frac{-\lambda}{b_1}\int_0^t(t-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(t-s)^{2\alpha}]f(s)ds\\&-\frac{a_1\lambda}{b_1}\frac{[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\&+\frac{a_1\lambda}{b_1}\frac{\lambda[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,2\alpha+1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\&-\frac{a_1\lambda}{b_1}\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})}{a_1\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\&+\frac{a_1\lambda}{b_1}\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})}{a_2\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\&+\frac{1}{b_2}\int_0^t(t-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(t-s)^{2\alpha}]g(s)ds\\&-\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})}{b_1\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\&+\frac{\lambda E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})}{b_2\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\end{aligned}$$

$$\begin{aligned}&-\frac{\lambda[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})}{b_1\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\&+\frac{\lambda[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})}{b_2\triangle(\alpha,\lambda)}\times\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\=&\frac{-\lambda}{b_1}\left[\frac{[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})+\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})}{\triangle(\alpha,\lambda)}\right]\times\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\&-\frac{\lambda}{b_1}\int_0^t(t-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(t-s)^{2\alpha}]f(s)ds\\&+\frac{1}{b_2}\left[\frac{[1-E_{2\alpha,1}(\lambda)]E_{2\alpha,1}(\lambda t^{2\alpha})+\lambda E_{2\alpha,\alpha+1}(\lambda)t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})}{\triangle(\alpha,\lambda)}\right]\times\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\\&+\frac{1}{b_2}\int_0^t(t-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(t-s)^{2\alpha}]g(s)ds\\&-\frac{\lambda}{b_1}\left[\frac{E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})+[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})}{\triangle(\alpha,\lambda)}\right]\times\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\\&+\frac{\lambda}{b_2}\left[\frac{E_{2\alpha,\alpha+1}(\lambda)E_{2\alpha,1}(\lambda t^{2\alpha})+[1-E_{2\alpha,1}(\lambda)]t^{\alpha}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})}{\triangle(\alpha,\lambda)}\right]\times\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\end{aligned}$$

$$y(t)=\frac{-\lambda}{b_1}\left[\phi(\alpha,\lambda,t)\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds\right.$$
$$\left.+\int_0^t(t-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(t-s)^{2\alpha}]f(s)ds\right]$$
$$+\frac{1}{b_2}\left[\phi(\alpha,\lambda,t)\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds\right.$$
$$\left.+\int_0^t(t-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(t-s)^{2\alpha}]g(s)ds\right]$$
$$-\frac{\lambda}{b_1}\psi(\alpha,\lambda,t)\int_0^1(1-s)^{\alpha-1}E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}]f(s)ds$$
$$+\frac{\lambda}{b_2}\psi(\alpha,\lambda,t)\int_0^1(1-s)^{2\alpha-1}E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}]g(s)ds$$

where $\phi(\alpha,\lambda,t)$ and $\psi(\alpha,\lambda,t)$ as given in equation (3.18) and equation (3.19) respectively.

$$\therefore\ y(t)=\frac{1}{b_2}\int_0^1 G_{\lambda,\alpha}(t,s)g(s)\,ds-\frac{\lambda}{b_1}\int_0^1\mathcal{G}(\lambda,\alpha)(t,s)f(s)\,ds \tag{3.21}$$

where $G_{\lambda,\alpha}(t,s)$ and $\mathcal{G}_{\lambda,\alpha}(t,s)$ are Green's function as defined in equation (3.11) and (3.12), respectively. □

4. Uniqueness and Existence of solution

Theorem 4.1. *Schauder fixed point theorem: Let X be a normed linear space and let the operator $T:X\to X$ be compact, then either*
(i) the operator T has a fixed point in X, or
(ii) the set $B=\{u\in X: u=\mu T(u),\mu\in(0,1)\}$ is unbounded.

Theorem 4.2. *Contraction Principal: Suppose $G_{\lambda,\alpha}(t,s)$ and $\mathcal{G}(\lambda,\alpha)$ are continuous on $[0,1]\times[0,1]$ and $|f(s)|<k_1$, $|g(s)|<k_2$, $|G_{\lambda,\alpha}|\le M_1$, $|\mathcal{G}_{\lambda,\alpha}|\le M_2$, $0\le t\le s\le 1$, and if*

$$\left|\frac{k_1M1}{a_1}-\frac{k_2M2}{a_2}\right|<1 \text{ and } \left|\frac{k_2M1}{b_2}-\frac{k_1M2}{b_1}\right|<1$$

then there is unique $x(t)$ and $y(t)$ in $C[0,1]$ such that

$$x(t)=\frac{1}{a_1}\int_0^1 G_{\lambda,\alpha}(t,s)f(s)\,ds-\frac{\lambda}{a_2}\int_0^1\mathcal{G}_{\lambda,\alpha}(t,s)g(s)\,ds$$

$$y(t)=\frac{1}{b_2}\int_0^1 G_{\lambda,\alpha}(t,s)g(s)\,ds-\frac{\lambda}{b_1}\int_0^1\mathcal{G}(\lambda,\alpha)(t,s)f(s)\,ds$$

for $0\le t\le 1$

Proof. If $x(t)\in C[0,1]$, let $T_1(x)=u$, therefore

$$u(t)=\frac{1}{a_1}\int_0^1 G_{\lambda,\alpha}(t,s)f(s,x(s))\,ds$$
$$-\frac{\lambda}{a_2}\int_0^1\mathcal{G}_{\lambda,\alpha}(t,s)g(s,x(s))\,ds$$

for $0<s<1$ since $u\in C[0,1]$, $T_1: C[0,1]\to C[0,1]$ if $x_1, x_2\in C[0,1]$, then

$$|u_1-u_2|\le\frac{1}{a1}\int_0^1|G_{\lambda,\alpha}(t,s)||f(s,x_1(s))-f(s,x_2(s))|ds$$
$$-\frac{1}{a2}\int_0^1|\mathcal{G}_{\lambda,\alpha}(t,s)||g(s,x_1(s))-g(s,x_2(s))|ds$$
$$\le\frac{k_1}{a1}\int_0^1|G_{\lambda,\alpha}(t,s)||x_1(s)-x_2(s)|ds$$
$$-\frac{k_2}{a2}\int_0^1|\mathcal{G}_{\lambda,\alpha}(t,s)||x_1(s)-x_2(s)|ds$$

$$|u_1-u_2|\le\left[\frac{k_1}{a1}\int_0^1|G_{\lambda,\alpha}(t,s)|ds-\frac{k_2}{a2}\int_0^1|\mathcal{G}_{\lambda,\alpha}(t,s)|ds\right]$$
$$\times|x_1(s)-x_2(s)|$$
$$\le\left[\frac{k_1M1}{a_1}-\frac{k_2M2}{a_2}\right]|x_1(s)-x_2(s)|$$
$$\le|x_1(s)-x_2(s)|$$

Since $|\frac{k_1M1}{a_1}-\frac{k_2M2}{a_2}|<1$, Therefore T_1 is contraction. Hence, there is unique x in $[0,1]$ such that $T_1(x)=x$. □

5. Test Problems

Example 5.1. *Consider the system of the system of fractional ordinary differential equations*

$${}_0^CD_t^\alpha x(t)+y(t)=t \tag{5.1}$$

$$x(t)+{}_0^CD_t^\alpha y(t)=1 \tag{5.2}$$

Solution: Taking Laplace transform, we get

$$L\{{}_0^CD_t^\alpha x(t)\}+L\{y(t)\}=L\{t\}$$
$$L\{x(t)\}+L\{{}_0^CD_t^\alpha y(t)\}=L\{1\}$$

$$s^\alpha X(s)+Y(s)=\frac{1}{s^2} \tag{5.3}$$

$$X(s)+s^\alpha Y(s)=\frac{1}{s} \tag{5.4}$$

Now,eliminating $Y(s)$ from above equations, we get

$$(s^{2\alpha}-1)X(s)=\frac{s^\alpha}{s^2}-\frac{1}{s}$$

$$X(s)=\frac{s^{\alpha-2}}{s^{2\alpha}-1}-\frac{s^{-1}}{s^{2\alpha}-1}$$

Taking inverse laplace, we get

$$x(t)=t^{\alpha+1}E_{2\alpha,\alpha+2}(t^{2\alpha})-t^{2\alpha}E_{2\alpha,2\alpha+1}(t^{2\alpha}) \tag{5.5}$$

putting $X(s)$ in (5.3), we get

$$Y(s) = \frac{1}{s^2} - s^{\alpha} X(s) = \frac{1}{s^2} - s^{\alpha}[\frac{s^{\alpha-2}}{s^{2\alpha}-1} - \frac{s^{-1}}{s^{2\alpha}-1}]$$

$$\therefore Y(s) = \frac{s^{\alpha-1}}{s^{2\alpha}-1} - \frac{s^{-2}}{s^{2\alpha}-1}$$

Taking inverse laplace, we get

$$y(t) = t^{\alpha} E_{2\alpha,\alpha+1}(t^{2\alpha}) - t^{2\alpha+1} E_{2\alpha,2\alpha+2}(t^{2\alpha}) \qquad (5.6)$$

The solution of the system is represented graphically by Maxima software as follows:

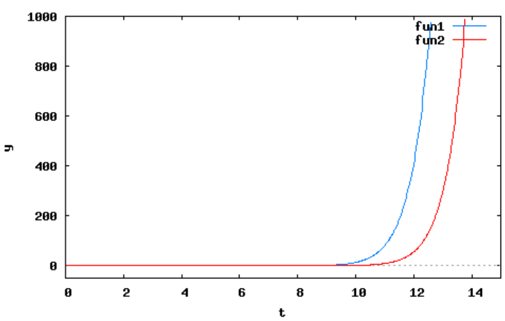

Figure 1. Strong Solution $x(t)$ for $\alpha = 0.8, 0.9$

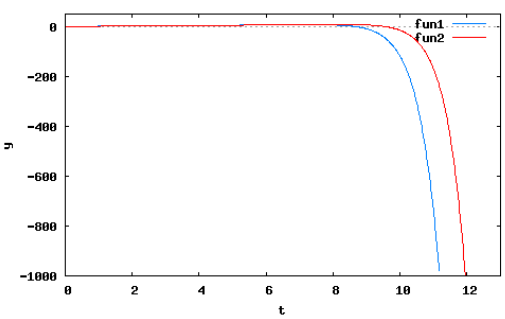

Figure 2. Strong Solution $y(t)$ for $\alpha = 0.8, 0.9$

Example 5.2. *Consider the system of the system of fractional ordinary differential equations*

$$ {}_0^C D_t^{\alpha} x(t) + y(t) = t \qquad (5.7)$$

$$x(t) + {}_0^C D_t^{\alpha} y(t) = \delta(t) \qquad (5.8)$$

where $\delta(t)$ is Dirac delta function.

Solution: Taking Laplace transform, we get

$$L\{{}_0^C D_t^{\alpha} x(t)\} + L\{y(t)\} = L\{t\}$$

$$L\{x(t)\} + L\{{}_0^C D_t^{\alpha} y(t)\} = L\{\delta(t)\}$$

$$s^{\alpha} X(s) + Y(s) = \frac{1}{s^2} \qquad (5.9)$$

$$X(s) + s^{\alpha} Y(s) = 1 \qquad (5.10)$$

Now,eliminating $Y(s)$ from above equations, we get

$$(s^{2\alpha} - 1)X(s) = \frac{s^{\alpha}}{s^2} - 1$$

$$X(s) = \frac{s^{\alpha-2}}{s^{2\alpha}-1} - \frac{1}{s^{2\alpha}-1}$$

Taking inverse laplace, we get

$$x(t) = t^{\alpha+1} E_{2\alpha,\alpha+2}(t^{2\alpha}) - t^{2\alpha-1} E_{2\alpha,2\alpha}(t^{2\alpha}) \qquad (5.11)$$

putting $X(s)$ in (5.9), we get

$$Y(s) = \frac{1}{s^2} - s^{\alpha} X(s) = \frac{1}{s^2} - s^{\alpha}[\frac{s^{\alpha-2}}{s^{2\alpha}-1} - \frac{1}{s^{2\alpha}-1}]$$

$$\therefore Y(s) = \frac{s^{\alpha}}{s^{2\alpha}-1} - \frac{s^{-2}}{s^{2\alpha}-1}$$

Taking inverse laplace, we get

$$y(t) = t^{\alpha-1} E_{2\alpha,\alpha}(t^{2\alpha}) - t^{2\alpha+1} E_{2\alpha,2\alpha+2}(t^{2\alpha}) \qquad (5.12)$$

The solution of the system is represented graphically by Maxima software as follows:

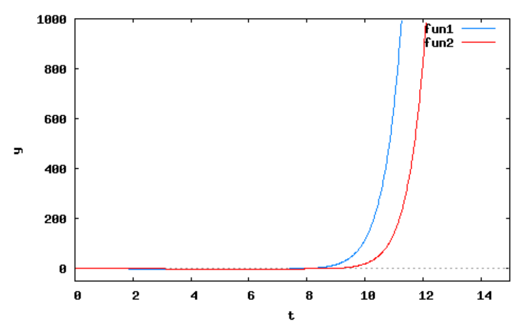

Figure 3. Strong Solution $x(t)$ for $\alpha = 0.8, 0.9$

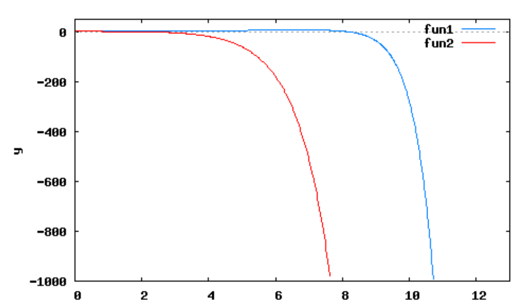

Figure 4. Strong Solution $y(t)$ for $\alpha = 0.8, 0.9$

6. Conclusion

We successfully solved the system of fractional ordinary differential equations using Laplace transform technique. Theoretically, we proved the solution of the given system is strong and mild. Also, we prove that the existence and uniqueness of the strong and mild solutions of the system of fractional ordinary differential equations. We solved some test problems and their solutions are represented graphically by maxima software.

References

[1] Adel Jawahdow , Existence of mild solutions for fractional differential equations in separable Banach space , *Differential Equation and Applications*, 7(4)(2015), 489-501.

[2] A.P.
tional calculus and its application, *Bulletin of Marathwada Mathematical Society*, 12(2)(2011), 1-7.

[3] H.L.Tidake, M. B. Dhakane, Existence and Uniqueness of mild and strong solution of non-linear Volterra integrodifferential equations in Banach space, *Demonstratio Mathematica*, XLIII(3)(2010), 1–10.

[4] J. Vanterler da c. Sousa, Leandro S. Tavares, E. Capelas de Oliveira, Mild and Strong solutions for Hilfer evolution equation, arXiv:1907.02019 [math.CA].

[5] K. Bhalchandran and S. ILamaran , Existence and Uniqueness of mild and strong solution of a semilinear evolution equation with non-local conditions, *Indian Journal Pure Appl. Math.,* 25(4)(1994), 411–418.

[6] K.C.Takale, V.R.Nikam, A.S.Shinde, Mittag Leffler functions, Its computations and application to differential equation of Fractional order, *International Conference on Mathematical Modelling and Applied Soft Computing*, (1)(2012), 561–575.

[7] Lijuin Pan , Existence of mild solution for Impulsive stochastic differential equation with non-local conditions, *Differential Equations and Applications*, 4(3)(2012), 485-494.

[8] Lizhen Chen, Zhenbin Fan, On mild solutions to fractional differential equations with non-local conditions, *Electronic Journal of Qualitative Theory of Differential Equations*, 53(2011), 1-53.

[9] I. Podlubny, *Fractional Differential equations*, Academic Press, San Diago (1999).

[10] Mohamed A. E. Herzallah, Mild and Strong solution of few types of fractional order non-linear equations with periodic boundary conditions, *Indian Journal Pure Appl. Math.,* 43(6)(2012), 619–635.

[11] Sabbavarapiu Nageswara Rao, Meshari Alesemil, On a coupled system of fractional differential equations with non-local, non-separated boundary conditions, *Advances in Difference Equation*, 2019, Article number: 97 (2019).

[12] Sayyedah A. Qasem, Rabha W. Ibrahim, ZailamSiri, On mild and strong solution of fractional differential equation with delay, *Conference on Mathematical Analysis and Its Applications*, 2015.

[13] Uttam Ghosh, Susmita Sarkar, Shantanu Das, Solution of system of fractional differential equation with modified derivative of Jumarie type, *American Journal of Mathematical Analysis*, 3(3)(2015), 72–84.

[14] Zufeng Zhang, Bin Liu, Existence of mild solutions for fractional evolution equations *Journal of Fractional Calculus and Applications*, 2(10)(2012), 1–10.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆