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On the stability of character edgecut polynomial and associated edgecut polynomial of graphs

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Abstract

In this paper, we introduce two new polynomials viz character edgecut polynomial and associated edgecut polynomial of a graph, which are closely related to the edgecut polynomial of a simple finite connected graph. Moreover we analyse the relationship connecting these polynomials and study their stability.

Keywords

Character edgecut polynomial, Associated edgecut polynomial, Edgecut polynomial.

AMS Subject Classification

05C31, 05C39.

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1. Introduction

In social networking systems, in order to improve the efficiency of content delivery and to minimize the total cost, graph partitioning is a vital pre-processing step and the majority of multilevel graph partitioning formulations have primarily focused on edgecut based models and have tried to optimize edgecut related objectives. Also, the pandemic COVID-19 has revealed the necessity of social distancing in public domains so that there arises a need for an efficient machinery to break social gatherings for isolating people. Keeping this in mind, in [3] the authors introduced the *edgecut polynomial* of a simple graph as follows:

$$E[G;x] = \sum_{i=1}^{n-1} E(G,i)x^{i+1},$$

where $E(G,i)$ is the number of edges removed at the i^{th} step in the standard edgecut of G for $1 \leq i \leq n-1$. A polynomial $p(z)$ is said to be *stable*, or a *Hurwitz polynomial*, if all of its zeroes lie in the open left half-plane[5]. That is, all the zeros of $p(z)$ are either negative real numbers or must have negative real part. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems [2]. Thus the study of a graph polynomial is interesting only if it succeeds in predicting the behavior of some stable physical systems. This fact motivates the authors to study the stability of two polynomials which are closely related to the edgecut polynomial of a graph. Throughout this paper, G denotes a finite simple connected graph with vertex set and edgeset denoted by $V(G)$ and $E(G)$ respectively. All the graph theoretic terminology and notations used in the paper are as in [1].

2. Main Results

2.1 Character edgecut polynomial of a graph

In this section, we first introduce the notion of character edgecut polynomial of a finite simple connected graph and determine its stability for some well known graphs.

Definition 2.1. *Let $G = (V,E)$ be a graph with n vertices and let*

$$(\underbrace{a_1,\dots a_1}_{m_1\ times},\cdots \underbrace{a_k,\dots a_k}_{m_k\ times})$$

be its edgecut sequence, where $1 \leq m_i \leq n-1$ for all $1 \leq i \leq k$.

The character edgecut polynomial of G, denoted by $C_e[G;x]$, is defined as

$$C_e[G;x]: \;= \sum_{i=1}^{k} a_i^{m_i} x^{i-1}.$$

We observe the following simple properties of $C_e[G;x]$:

(i) $C_e[G;x]$ is a monic polynomial of degree at most $n-2$ and has degree $n-2$ iff $m_i = 1$ for all $1 \leq i \leq k$.

(ii) All the zeros of $C_e[G;x]$ are non-zero.

(iii) The graph G has cutedges if and only if $C_e[G;0] = 1$.

(iv) If $C_e[G;x]$ is a polynomial of degree m, then the coefficients of x^i is nonzero for all $0 \leq i \leq m$.

(v) If G is a tree on n vertices, then $C_e[G;x]$ is the constant polynomial 1.

Theorem 2.2. *(*Routh-Hurwitz Criteria *[5]) Given a polynomial,*

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where the coefficients a_i's are real constants. Then the n Hurwitz matrices using the coefficients a_i of the above polynomial are defines as

$$H_1 = \begin{bmatrix} a_1 \end{bmatrix}, H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \cdots,$$

$$H_n = \begin{bmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: $det(H_j) > 0, j = 1, 2, \cdots, n$.

Theorem 2.3. *Let G be a graph with n vertices. If $C_e[G;x]$ is a linear polynomial, then it is stable. Moreover, the unique zero of the polynomial is a negative integer and its modulus exceeds unity if and only if G has no cutedges.*

Proof. Since $C_e[G;x]$ is a monic linear polynomial with positve integral coefficients, its only zero is the negative of the constant term. Observe that the constant term is a negative integer. Thus $C_e[G;x]$ is stable. Also observe that the constant term of the polynomial $C_e[G;x]$ is unity iff the graph G has at least one cutedge. Thus the constant term of the polynomial is greater than 1 iff G has no cutedges.

This completes the proof. □

Corollary 2.4. *Let C_n be a cycle on n vertices. Then $C_e[C_n;x]$ is stable.*

Proof. In [3], the authors proved that

$$E[C_n;x] = \sum_{i=1}^{n-2} x^{i+2} + 2x^2.$$

Consequently, we have $C_e[C_n;x] = x+2$, a linear polynomial and hence stable by theorem 2.3. □

Corollary 2.5. *If S_n is a shell graph on $n > 2$ vertices, then $C_e[S_n;x]$ is stable.*

Proof. In [3], the authors proved that

$$E[S_n;x] = x^n + \sum_{i=1}^{n-2} 2x^{i+1}.$$

Thus $C_e[S_n;x]$ is the linear polynomial $x + 2^{n-2}$. Therefore by theorem 2.3, $C_e[S_n;x]$ is stable . □

Theorem 2.6. *If $C_e[G;x]$ is a quadratic polynomial of a graph G on n vertices, then it is stable. Moreover, if G has at least one cutedge, then both the zeros of the polynomial are negative real numbers.*

Proof. Let $C_e[G;x] = x^2 + ax + b$. Since $a, b > 0$, it follows that the following matrices have positive determinants.

$$H_1 = \begin{bmatrix} a \end{bmatrix}, H_2 = \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}.$$

Then by the light of Routh-Hurwitz criteria, $C_e[G;x]$ is stable. If G has at least one cutedge, then $C_e[G;x] = x^2 + ax + 1$. Then by the definition of the character edgecut polynomial $a > 1$ and thus the discriminant of this polynomial is $a^2 - 4 \geq 0$. Since $C_e[G;x]$ is a polynomial with positive coefficients, both the zeros of the polynomial are negative real numbers.

This completes the proof. □

Corollary 2.7. *Let G be a graph with n vertices. If G has exactly $n-3$ cutedges, then $C_e[G;x]$ is stable.*

Proof. Since G has exactly $n-3$ cutedges, it is evident that $C_e[G;x]$ is a quadratic polynomial and thus its stability follows from theorem 2.6. □

Corollary 2.8. *Let W_n be a wheel graph on n vertices. Then $C_e[W_n;x]$ is stable.*

Proof. We have,

$$E[W_n;x] = x^n + \sum_{i=1}^{n-3} 2x^{i+2} + 3x^2$$

(see [3]) so that $C_e[W_n;x] = x^2 + 2^{n-3}x + 3$, a quadratic polynomial and hence stable. □

Corollary 2.9. *For a tadpole graph $T_{n,l}$ with $n > 2$, $C_e[T_{n,l};x]$ is stable.*

Proof. The edgecut polynomial of tadpole graph is given by

$$E[T_{n,l};x] = \sum_{i=l+1}^{n+l-1} x^{i+1} + \sum_{i=1}^{l+1} x^{i+1}$$

(see [3]). Thus $C_e[T_{n,l};x] = x^2 + 2x + 1$, a quadratic polynomial, and hence stable. □

Corollary 2.10. *For an armed crown $C_n \odot P_m$ with $n > 2$, $C_e[C_n \odot P_m;x]$ is stable.*

Proof. We have,

$$E[C_n \odot P_m;x] = \sum_{i=nm+1}^{nm+n-1} x^{i+1} + \sum_{i=1}^{nm+1} x^{i+1}$$

(see [3]). Thus $C_e[C_n \odot P_m;x] = x^2 + 2x + 1$ is stable being a quadratic polynomial. □

Theorem 2.11. *If $C_e[G;x]$ is a cubic polynomial with at least one cutedge, then it is stable.*

Proof. Let $C_e[G;x] = x^3 + ax^2 + bx + 1$. Then the Hurwitz matrices are given by

$$H_1 = \begin{bmatrix} a \end{bmatrix}, H_2 = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, H_3 = \begin{bmatrix} a & 1 & 0 \\ 1 & b & a \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that the determinant of all these three matrices are positive. Then Routh-Hurwitz criteria tells us that $C_e[G;x]$ is stable.

This completes the proof. □

Corollary 2.12. *Let G be a graph with n vertices. If G has exactly $n-4$ cutedges, then $C_e[G;x]$ is stable.*

Proof. Since the graph possesses exactly $n-4$ cutedges, the remaining four vertices must be a part of a cycle of length 3 or 4. It is easy to verify that there are only three such graphs viz C_4, K_4 and $K_4 - \{e\}$, where e is an edge of K_4. The edgecut sequence of C_4 and $K_4 - \{e\}$ are $(2,1,1)$ and $(2,2,1)$ respectively and thus $C_e[G;x]$ is a quadratic polynomial and hence stable. Now K_4 is a graph on four vertices, it follows that $C_e[G;x]$ is a cubic polynomial and since G has at least one cutedge, its stability follows from theorem 2.11. □

Corollary 2.13. *If H_n is a helm with $n \geq 4$ vertices, then $C_e[H_n;x]$ is stable.*

Proof. We have,

$$E[H_n;x] = x^{2n-1} + \sum_{i=2}^{n-1} 2x^{2n-i} + \sum_{i=1}^{n} x^{i+1}$$

(see [3]). Thus $C_e[H_n;x] = x^3 + 2^{n-3}x^2 + 3x + 1$, a cubic polynomial. Since the constant term in $C_e[H_n;x]$ is one, it can be inferred that H_n has at least one cutedge. Therefore by theorem 2.11, $C_e[H_n;x]$ is stable. □

Theorem 2.14. *If B_N is a bow graph with $N > 4$ vertices, then $C_e[B_N;x]$ is unstable.*

Proof. We have,

$$E[B_N;x] = \sum_{i=N-2}^{N-1} x^{i+1} + \sum_{i=N-3}^{N-2} x^{i+1} + \sum_{i=1}^{N-4} 2x^{i+1}$$

(see [3]). Therefore $C_e[B_N;x] = x^3 + 2x^2 + x + 2^{N-4}$ is a cubic polynomial. The Hurwitz matrices are given by

$$H_1 = \begin{bmatrix} 2 \end{bmatrix}, H_2 = \begin{bmatrix} 2 & 1 \\ 2^{N-4} & 1 \end{bmatrix}, H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 2^{N-4} & 1 & 2 \\ 0 & 0 & 2^{N-4} \end{bmatrix}.$$

The determinant of H_2 is less than or equal to zero since $N > 4$. Therfore by Routh-Hurwitz criteria, $C_e[B_N;x]$ is unstable.

This completes the proof. □

Theorem 2.15. *If BF_n is a butterfly graph with $n > 6$ vertices, then $C_e[BF_n;x]$ is unstable.*

Proof. From [3], we have

$$E[BF_n;x] = \sum_{i=n-2}^{n-1} x^{i+1} + \sum_{i=n-3}^{n-2} x^{i+1} + \sum_{i=3}^{n-4} 2x^{i+1} + \sum_{i=1}^{2} x^{i+1}.$$

Consequently, $C_e[BF_n;x] = x^4 + 2x^3 + x^2 + 2^{n-6}x + 1$. The first two Hurwitz matrices of this polynomial are as follows:

$$H_1 = \begin{bmatrix} 2 \end{bmatrix}, H_2 = \begin{bmatrix} 2 & 1 \\ 2^{n-6} & 1 \end{bmatrix}.$$

Since $n > 6$, $\det(H_2) \leq 0$. Hence by Routh-Hurwitz criteria, $C_e[BF_n;x]$ is unstable.

This completes the proof. □

Theorem 2.16. *If WB_n is a webgraph on $n > 4$ vertices, then $C_e[WB_n;x]$ is unstable.*

Proof. The edgecut polynomial of the webgraph WB_n is given by

$$E[WB_n;x] = x^{3n-2} + \sum_{i=1}^{n-3} 2x^{2n+i} + 3x^{2n} + x^{2n-1} + \sum_{i=1}^{n-2} 2x^{n+i} + \sum_{i=1}^{n} x^{i+1}$$

(see [3]). From this we obtain

$$C_e[WB_n;x] = x^6 + 3x^5 + 2^{n-3}x^4 + x^3 + 3x^2 + 2^{n-3}x + 1.$$

The first four Hurwitz matrices corresponding $C_e[WB_n;x]$ are

$$H_1 = \begin{bmatrix} 3 \end{bmatrix}, H_2 = \begin{bmatrix} 3 & 1 \\ 1 & 2^{n-3} \end{bmatrix}, H_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2^{n-3} & 3 \\ 2^{n-3} & 3 & 1 \end{bmatrix},$$

$$H_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2^{n-3} & 3 & 1 \\ 2^{n-3} & 3 & 1 & 2^{n-3} \\ 0 & 1 & 2^{n-3} & 3. \end{bmatrix}.$$

Note that

$$\det(H_4) = -((2^{2n-6}-3)(3.2^{n-3}-1)+3.2^{n-3}+2^{2n-6}+75)$$

is less than zero. Hence by Routh-Hurwitz criteria, $C_e[WB_n;x]$ is unstable.

This completes the proof. □

2.2 Associated edgecut polynomial of a graph

In this section, we first introduce the notion of associated edgecut polynomial of a finite simple connected graph and determine its stability for some well known graphs.

Definition 2.17. *Let $G = (V,E)$ be a graph with n vertices and let*

$$(\underbrace{a_1,\dots a_1}_{m_1\ times},\cdots \underbrace{a_k,\dots a_k}_{m_k\ times})$$

be its edgecut sequence, where $1 \leq m_i \leq n-1 \quad for\ all\ 1 \leq i \leq k$. The associated edgecut polynomial of G, denoted by $A_e[G;x]$, is defined as

$$A_e[G;x] = \sum_{i=1}^{k} m_i a_i x^{i-1}.$$

We observe the following simple properties of $A_e[G;x]$:

(i) $A_e[G;x]$ is a polynomial of degree at most $n-2$ and has degree $n-2$ iff $m_i = 1$ for all $1 \leq i \leq k$.

(ii) All the zeros of $A_e[G;x]$ are non-zero.

(iii) $A_e[G;1] = |E|$ and $A_e[G;0] = 1$ if and only if G has a unique cutedge.

(iv) If $A_e[G;x]$ is a polynomial of degree m, then the coefficients of x^i is nonzero for all $0 \leq i \leq m$.

(v) If G is a tree on n vertices, then $A_e[G;x]$ is the constant polynomial $n-1$.

(vi) If G has cutedges, then the constant term of $A_e[G;x]$ gives the total number of cutedges of G.

Theorem 2.18. *Let G be a graph with n vertices. If $A_e[G;x]$ is a linear polynomial, then it is stable. More precisely, G will be free of cutedges.*

Proof. Since $A_e[G;x]$ is a linear polynomial $ax+b$, its only zero is $x = -b/a$, where a and b are positve integers. Thus $A_e[G;x]$ is stable.
If possible G has cutedges, then the constant term of $A_e[G;x]$ is the sum total of cutedges of G and since $E[G;x]$ is monic, the associated edgecut polynomial of G must be $n-1$. This contradicts the fact that $A_e[G;x]$ is linear polynomial. Therefore G has no cutedges. This completes the proof. □

Corollary 2.19. *Let C_n be a cycle on n vertices. Then $A_e[C_n;x]$ is stable.*

Proof. We have,

$$E[C_n;x] = \sum_{i=1}^{n-2} x^{i+2} + 2x^2$$

(see [3]). Thus $A_e[C_n;x] = (n-2)x+2$, a linear polynomial and hence its stability follows from theorem 2.18. □

Corollary 2.20. *If S_n is a shell graph on $n > 2$ vertices, then $C_e[S_n;x]$ is stable.*

Proof. We have,

$$E[S_n;x] = x^n + \sum_{i=1}^{n-2} 2x^{i+1}$$

(see [3]). Hence it follows that $C_e[S_n;x]$ is the linear polynomial $x+2^{n-2}$ and hence stable. □

Theorem 2.21. *Let G be a graph with n vertices. If $A_e[G;x]$ is a quadratic polynomial, then it is stable. Moreover, if $A_e[G;x]$ is monic and G has exactly one cutedge, then both the zeros of the polynomial are negative real numbers.*

Proof. Let $A_e[G;x] = ax^2+bx+c$. The Hurwitz matrices of this polynomial are given by,

$$H_1 = \begin{bmatrix} b \end{bmatrix}, H_2 = \begin{bmatrix} b & a \\ 0 & c \end{bmatrix}.$$

Since the determinants of both these matrices are positive, by Routh-Hurwitz criteria, $A_e[G;x]$ is stable.
Now suppose that $A_e[G;x]$ is monic and G has exactly one cutedge. Then $A_e[G;x] = x^2+bx+1$. Its discriminant b^2-4 is non-negative since $b \geq 2$ and thus both the zeros of the polynomial are negative real numbers.

This completes the proof. □

Corollary 2.22. *Let W_n be a wheel graph on n vertices. Then $A_e[W_n;x]$ is stable.*

Proof. We have,

$$E[W_n;x] = x^n + \sum_{i=1}^{n-3} 2x^{i+2} + 3x^2$$

(see [3]). Thus $A_e[W_n;x] = x^2+2^{n-3}x+3$, a quadratic polynomial and hence stable. □

Corollary 2.23. *For a tadpole graph $T_{n,l}$ with $n > 2$, $A_e[T_{n,l};x]$ is stable.*

Proof. The edgecut polynomial of tadpole graph is given by

$$E[T_{n,l};x] = \sum_{i=l+1}^{n+l-1} x^{i+1} + \sum_{i=1}^{l+1} x^{i+1}$$

(see [3]). That is $A_e[T_{n,l};x] = (n-2)x^2+2x+l$ is a quadratic polynomial, which is always stable. □

Corollary 2.24. *For an armed crown $C_n \odot P_m$ with $n > 2$, $A_e[C_n \odot P_m; x]$ is stable.*

Proof. We have,

$$E[C_n \odot P_m; x] = \sum_{i=nm+1}^{nm+n-1} x^{i+1} + \sum_{i=1}^{nm+1} x^{i+1}$$

(see [3]). Thus $A_e[C_n \odot P_m; x] = (n-2)x^2 + 2x + nm$ is stable being a quadratic polynomial. □

Theorem 2.25. *If H_n is a helm with $n \geq 4$ vertices, then $A_e[H_n; x]$ is stable.*

Proof. We have,

$$E[H_n; x] = x^{2n-1} + \sum_{i=2}^{n-1} 2x^{2n-i} + \sum_{i=1}^{n} x^{i+1}$$

(see [3]). Thus $A_e[H_n; x] = x^3 + 2(n-3)x^2 + 3x + n - 1$, a cubic polynomial. The Hurwitz matrices of $A_e[H_n; x]$ are given by,

$$H_1 = \left[2(n-3)\right], H_2 = \begin{bmatrix} 2(n-3) & 1 \\ n-1 & 3 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 2(n-3) & 1 & 0 \\ n-1 & 3 & 2(n-3) \\ 0 & 0 & n-1 \end{bmatrix}.$$

Since $n \geq 4$, $\det(H_1) = 2(n-3)$, $\det(H_2) = 5n - 17$ and $\det(H_3) = 5n^2 - 8n + 17$ are all positive. Therefore by Routh-Hurwitz criteria, $A_e[H_n; x]$ is stable.

This completes the proof. □

Theorem 2.26. *If B_N is a bow graph with $N > 4$ vertices, then $A_e[B_N; x]$ is unstable.*

Proof. We have,

$$E[B_N; x] = \sum_{i=N-2}^{N-1} x^{i+1} + \sum_{i=N-3}^{N-2} x^{i+1} + \sum_{i=1}^{N-4} 2x^{i+1}$$

(see [3]). Therefore $A_e[B_N; x] = x^3 + 2x^2 + x + 2(N-4)$, a cubic polynomial. The Hurwitz matrices are given by:

$$H_1 = \left[2\right], H_2 = \begin{bmatrix} 2 & 1 \\ 2(N-4) & 1 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 2(N-4) & 1 & 2 \\ 0 & 0 & 2(N-4) \end{bmatrix}$$

Here $\det(H_2) = 10 - 2N \leq 0$ since $N \geq 5$. Thus by Routh Hurwitz criteria, $A_e[B_N; x]$ is unstable.

This completes the proof. □

Theorem 2.27. *If BF_n is a butterfly graph with $n > 6$ vertices, then $A_e[BF_n; x]$ is unstable.*

Proof. In [3] the authors proved that

$$E[BF_n; x] = \sum_{i=n-2}^{n-1} x^{i+1} + \sum_{i=n-3}^{n-2} x^{i+1} + \sum_{i=3}^{n-4} 2x^{i+1} + \sum_{i=1}^{2} x^{i+1}.$$

Consequently,

$$A_e[BF_n; x] = x^4 + 2x^3 + x^2 + 2(n-6)x + 2.$$

The first two Hurwitz matrices of this polynomial are as follows:

$$H_1 = \left[2\right], H_2 = \begin{bmatrix} 2 & 1 \\ 2(n-6) & 1 \end{bmatrix}.$$

Since $n > 6$, $\det(H_2) = 2 - 2(n-6) \leq 0$. Hence by Routh-Hurwitz criteria, $A_e[BF_n; x]$ is unstable.

This completes the proof. □

Theorem 2.28. *If WB_n is a webgraph on $n > 4$ vertices, then $A_e[WB_n; x]$ is unstable.*

Proof. The edgecut polynomial of WB_n is given by(see [3])

$$E[WB_n; x] = x^{3n-2} + \sum_{i=1}^{n-3} 2x^{2n+i} + 3x^{2n} + x^{2n-1} + \sum_{i=1}^{n-2} 2x^{n+i} + \sum_{i=1}^{n} x^{i+1}.$$

Therefore,

$$A_e[WB_n; x] = x^6 + 3x^5 + 2(n-3)x^4 + x^3 + 3x^2 + 2(n-3)x + n - 1.$$

The first four Hurwitz matrices are as follows:

$$H_1 = \left[3\right], H_2 = \begin{bmatrix} 3 & 1 \\ 1 & 2(n-3) \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2(n-3) & 3 \\ 2(n-3) & 3 & 1 \end{bmatrix},$$

$$H_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2(n-3) & 3 & 1 \\ 2(n-3) & 3 & 1 & 2(n-3) \\ 0 & (n-1) & 2(n-3) & 3. \end{bmatrix}$$

It can be calculated that,

$$det(H_4) = -24n^3 + 234n^2 - 669n + 459$$

and the maximum value of the function occurs at $n = \frac{13}{4} + \frac{\sqrt{183}}{12}$, which is less than 5. Also the value $det(H_4)$ for $n = 5$ is negative. From these inferences, we can conclude that $\det(H_4) < 0$ for $n > 4$ and hence by Routh-Hurwitz criteria, $A_e[WB_n; x]$ is unstable.

This completes the proof. □

2.3 Relation between Edgecut polynomial, Characteristic edgecut polynomial and Associated edgecut polynomial of a graph

In this section, we establish the relationship connecting the edgecut polynomial, character edgecut polynomial and associated edgecut polynomial of a graph.

Let G be a graph on n vertices. The following are some of the relevant facts that are obtained by comparing both $C_e[G;x]$ and $A_e[G;x]$ together.

(i) For any graph G, both the polynomials $C_e[G;x]$ and $A_e[G;x]$ have the same degree.

(ii) The polynomials $C_e[G;x]$ and $A_e[G;x]$ are equal iff their degree is $n-2$.

(iii) If $C_e[G;x] = A_e[G;x]$, then their coefficient of x^i is same as the coefficient of x^{i+2} in $E[G;x]$ for $0 \leq i \leq n-2$.

(iv) One added to the ratio of the coefficient of x^i in $A_e[G;x]$ to the base of the corresponding coefficient in $C_e[G;x]$ gives the number of newly formed components of G by removing same number of edges consecutively.

For example, consider the complete graph K_n. In this case,

$$E[K_n;x] = \sum_{i=1}^{n-1} (n-i)x^{i+1}.$$

Observe that

$$C_e[K_n;x] = \sum_{i=1}^{n-1} (n-i)x^{i-1} = A_e[K_n;x].$$

Moreover, degree of each polynomial is $n-2$.

Theorem 2.29. *Let G be a graph on n vertices. Then the polynomials $C_e[G;x]$ and $A_e[G;x]$ together determines the edgecut polynomial $E[G;x]$ completely.*

Proof. Without loss of generality, let

$$C_e[G;x] = \sum_{i=1}^{k} a_i^{m_i} x^{i-1}, A_e[G;x] = \sum_{i=1}^{k} m_i a_i x^{i-1}$$

be the character edgecut polynomial and associated edgecut polynomial of G respectively. Let p_i be the ratio of the coefficient of x^i in $A_e[G;x]$ to the base of the corresponding coefficient in $C_e[G;x]$ and let q_i be obtained by dividing the coefficients of $A_e[G;x]$ by p_i, where $1 \leq i \leq k$. It can be easily verified that $p_i = m_i$ and $q_i = a_i$ for each i and thus

$$(\underbrace{q_1, \dots q_1}_{p_1 \text{ times}}, \cdots \underbrace{q_k, \dots q_k}_{p_k \text{ times}})$$

represents the edgecut sequence of the graph G. Thus we have established that the polynomials $C_e[G;x]$ and $A_e[G;x]$ explicitly defines the edgecut polynomial $E[G;x]$ of the graph G. In other words these two polynomials together determines $E[G;x]$ completely.

This completes the proof. □

3. Conclusion

In this paper, we have characterized the edgecut polynomial of a graph in terms of two polynomials, which are comparatively simple in the sense that they are having degree less than that of the former polynomial. Thus even without analyzing the edgecut polynomial explicitly, we could deduce all the information provided by it from the corresponding character edgecut polynomial and associated edgecut polynomial. In fact, we have decomposed the original polynomial into simpler ones without losing any information.

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