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Total dominator chromatic number of $P_m \times C_n$

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Abstract
A total dominator coloring of a graph G with $\delta(G) \geq 1$ is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The total dominator chromatic number of G is the minimum number of colors needed in a total dominator coloring of G, denoted by $\chi_{td}(G)$. In this paper, we obtain total dominator chromatic number of $P_m \times C_n$.

Keywords
Total dominator chromatic number, ladder graph, grid graph and $P_m \times C_n$.

AMS Subject Classification
05C15, 05C69.

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Contents

1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definition of graph theory as found in [1]. Let $G = (V,E)$ be a graph of order n with $\delta(G) \geq 1$. The open neighborhood $N(v) = \{u \in V(G)/uv \in E(G)\}$. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The path and cycle of order n are denoted by P_n and C_n respectively. For any two graphs G and H, we define the cartesian product, denoted by $G \times H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices (u_1, v_1) and (u_2, v_2) iff either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 = v_2$. $P_m \times C_n$ is defined as the cartesian product of path and cycle. A grid graphs can be defined as $P_m \times P_n$ where $m, n \geq 2$.

A subset S of V is called a total dominating set if every vertex in V is adjacent to some vertex in S. The total dominating set is minimal total dominating set if no proper subset of S is a total dominating set of G. The total domination number γ_t is the minimum cardinality taken over all minimal total dominating set of G. A γ_t-set is any minimal total dominating set with cardinality γ_t.

A proper coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. The chromatic number, $\chi(G)$, is the minimum number of colors in a proper coloring of G. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The total dominator chromatic number of G is the minimum number of colors needed in a total dominator coloring of G denoted by $\chi_{td}(G)$. This concept was introduced by A. Vijiyalekshmi in [2]. This notion is also referred as a Smarandachely k-dominator coloring of $G(k \geq 1)$ and was introduced by A. Vijiyalekshmi in [4]. For an integer $k \geq 1$, a Smarandachely k-dominator coloring of G is a proper coloring of G such that every vertex in G properly dominates a k color class. The smallest number of colors for which there exist a Smarandachely k-dominator coloring of G is called the Smarandachely k-dominator chromatic number of G, and is denoted by $\chi^s_{td}(G)$. For further details on this theory and on its applications, we advice the reader to refer [6–9].

In a proper coloring $\mathscr{C}$ of G, a color class of $\mathscr{C}$ is a set consisting of all those vertices assigned the same color. Let $\mathscr{C}^1$ be a minimal td-coloring of G. We say that a color class $c_i \in\in \mathscr{C}^1$ is called a non-dominated color class ($n-d$ color class) if it is not dominated by any vertex of G. These color classes are also called repeated color classes.

2. Preliminaries

In this segment, we remember the critical [3] theorem which is quite helpful in our research. For the subsequent observation the total dominator chromatic number of a ladder graphs has been identified.

Theorem 2.1. *[3] Let G be p_n or C_n. Then*

$$\chi_{td}\ (p_n) = \chi_{td}\ (C_n) = \begin{cases} 2\lfloor \frac{n}{4} \rfloor + 2, & if n \equiv 0(mod4) \\ 2\lfloor \frac{n}{4} \rfloor + 3, & if n \equiv 1(mod4) \\ 2\lfloor \frac{n+2}{4} \rfloor + 2, & otherwise. \end{cases}$$

Theorem 2.2. *[3] For every $n \geq 2$, the total dominator chromatic number of a ladder graph L_n is*

$$\chi_{td}\ (L_n) = \begin{cases} 2\lfloor \frac{p}{6} \rfloor + 2, & if n \equiv 0(mod6) \\ 2\lfloor \frac{p-2}{6} \rfloor + 4, & \\ 2\lfloor \frac{p-4}{6} \rfloor + 4, & otherwise. \end{cases}$$

In this paper, we obtain the least value for total dominator chromatic number for $P_m \times C_n$.

3. Main Result

In this section, we present and establish the main results. For our convenience, we denote

$\mathscr{G}_{m,n} = P_m \times C_n$ and let $D = \{v_{(ij)}/1 \leq i \leq m$ and $1 \leq j \leq n\}$.

Lemma 3.1. *For every $n, \chi_d(\mathscr{G}_{2,n}) = 2\lceil \frac{m}{3} \rceil + 2$.*

Proof. Since the td-colouring of $\mathscr{G}_{2,n}$ is same as td-colouring of $L_n, \chi_{td}(\mathscr{G}_{2,n}) = \chi_{td}(L_n)$. From Theorem 2.2, we get

$$\chi_d(\mathscr{G}_{2,n}) = 2\lceil \frac{m}{3} \rceil + 2.$$

□

Illustration: Consider $\mathscr{G}_{2,11}$

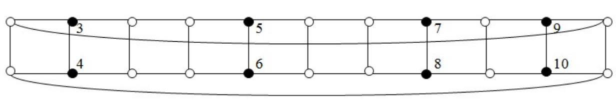

Figure 1

Therefore

$$\chi_{td}(\mathscr{G}_{2,11}) = 10.$$

Theorem 3.2. *If $m, n \equiv 0(mod3)$, then $\chi_d(\mathscr{G}_{m,n}) = \frac{mn}{3} + 2$.*

Proof. Let $D = \{v_{(ij)}/1 \leq i \leq m$ and $j = 2, 5, 8, ..(n-1)\}$ be a unique γ_t-set of $\mathscr{G}_{m,n}$. We assign $\frac{mn}{3}$ distinct colors say $3, 4, 5, \ldots, \frac{mn}{3}, \frac{mn}{3}+1, \frac{mn}{3}+2$ to vertices of D. Set $S = V(\mathscr{G}_{m,n}) - D$, we assign two repeated colors say 1, 2 to the vertices v_{ij} and $v_{kl} \in S$ such that $|i-k|+|j-l| = 1$ and adjacent vertices in S received different colors, we get a td-coloring of $\mathscr{G}_{m,n}$.

So

$$\chi_d(\mathscr{G}_{m,n}) = \frac{mn}{3} + 2.$$

□

Illustration: Consider $\mathscr{G}_{6,9}$

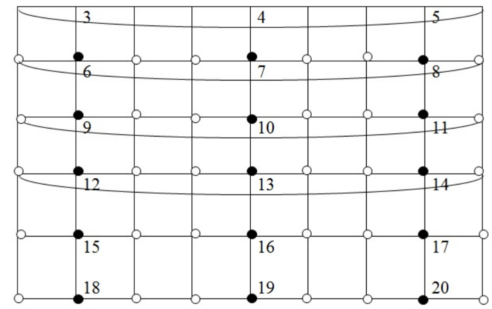

Figure 2

Therefore

$$\chi_{td}(\mathscr{G}_{6,9}) = 20.$$

Theorem 3.3. *For $m \equiv 0(mod3)$ and $n \equiv 1, 2(mod3)$,*

$$\chi_{td}(\mathscr{G}_{m,n}) = \begin{cases} \frac{mn}{3} + 2, & if \quad n \quad is\ even \\ \frac{mn}{3} + 3, & if \quad n \quad is\ odd. \end{cases}$$

Proof. Let $D_1 = \{v_{(ij)}/i = 2, 5, 8, \ldots, (n-1)$ and $1 \leq j \leq n\}$ be a unique γ_t-set of $\mathscr{G}_{m,n}$. We consider two cases.

Case (i): When n is even. The td-coloring of $\mathscr{G}_{m,n}$ is same as the td-coloring of Theorem 3.1. So $\chi_{td}(\mathscr{G}_{m,n}) = \frac{mn}{3} + 2$.

Case (ii): When n is odd. Assign $\frac{mn}{3}$ distinct colors say $4, 5, 6, \ldots, \frac{mn}{3}, \frac{mn}{3}+1, \frac{mn}{3}+2, \frac{mn}{3}+3$ to vertices of D_1. Set $S_1 = \{V(\mathscr{G}_{m,n}) - D_1\}$, we assign two repeated colors say 1, 2 to the vertices v_{ij} and $v_{kl} \in S_1$ such that $|i-k|+|j-l| = 1$ and adjacent vertices in S_1 received two different repeated colors. Now there several vertices in S_1, which are not received repeated colors either 1 or 2, we assign another repeated color say 3 to the those vertices in S_1, we get a td-coloring of $\mathscr{G}_{m,n}$. So $\chi_{td}(\mathscr{G}_{m,n}) = \frac{mn}{3} + 3$.

□

Illustration: Consider $\mathscr{G}_{6,10}$

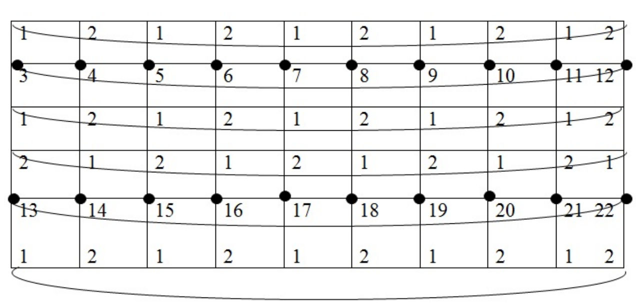

Figure 3

Therefore

$$\chi_{td}(\mathscr{G}_{6,10}) = 22.$$

Illustration: Consider $\mathscr{G}_{6,7}$

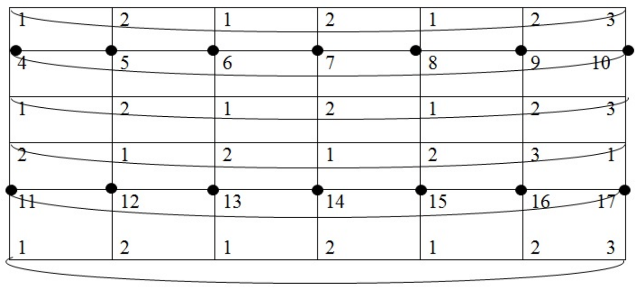

Figure 4

Therefore
$$\chi_{td}(\mathscr{G}_{6,7}) = 17.$$

Theorem 3.4. *If $m \equiv 1(mod3)$ then*

$$\chi_{td}(\mathscr{G}_{m,n}) = \begin{cases} \frac{(m-1)n}{3} + 2\lfloor \frac{n}{4} \rfloor + 2 & if n \equiv 0(mod4) \\ \frac{(m-1)n}{3} + 2\lfloor \frac{n}{4} \rfloor + 4 & if n \equiv 1(mod4) \\ \frac{(m-1)n}{3} + 2\lfloor \frac{n+2}{4} \rfloor + 2 & if n \equiv 2(mod4) \\ \frac{(m-1)n}{3} + 2\lfloor \frac{n+2}{4} \rfloor + 3 & otherwise. \end{cases}$$

Proof. Since $m-1 \equiv 0(mod3), \mathscr{G}_{m,n}$ is obtained by $\mathscr{G}_{m-1,n}$ is followed by $\mathscr{G}_{1,n}$. In a td-coloring of $\mathscr{G}_{m,n}, \chi_{td}(\mathscr{G}_{m,n}) = \chi_{td}(\mathscr{G}_{m-1,n}) + \chi_{td}(\mathscr{G}_{1,n})$. Also the used repeated colors are same the td-coloring of $\mathscr{G}_{1,n}$. So $\chi_{td}(\mathscr{G}_{m,n}) = \chi_{td}(\mathscr{G}_{m-1,n}) + \chi_{td}(\mathscr{G}_{1,n}) - 2$. By Theorem 2.1 , we get

$$\chi_{td}(\mathscr{G}_{m,n}) = \begin{cases} \frac{(m-1)n}{3} + 2\lfloor \frac{n}{4} \rfloor + 2 & if n \equiv 0(mod4) \\ \frac{(m-1)n}{3} + 2\lfloor \frac{n}{4} \rfloor + 4 & if n \equiv 1(mod4) \\ \frac{(m-1)n}{3} + 2\lfloor \frac{n+2}{4} \rfloor + 2 & if n \equiv 2(mod4) \\ \frac{(m-1)n}{3} + 2\lfloor \frac{n+2}{4} \rfloor + 3 & otherwise. \end{cases}$$

□

Illustration: Consider $\mathscr{G}_{4,10}$

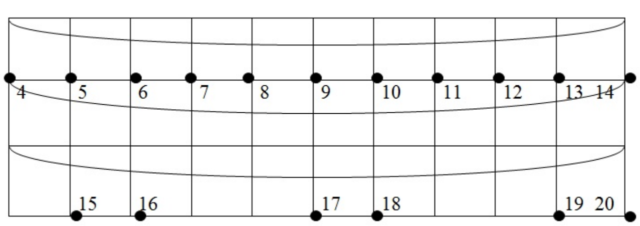

Figure 5

Therefore
$$\chi_{td}(\mathscr{G}_{4,10}) = 18.$$

Illustration: Consider $\mathscr{G}_{4,11}$

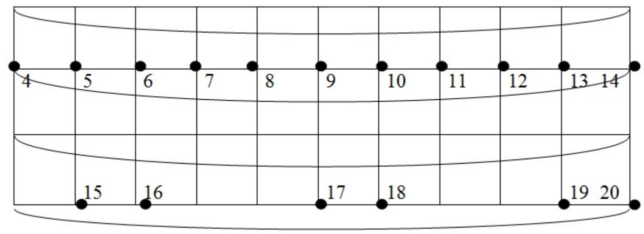

Figure 6

Therefore
$$\chi_{td}(\mathscr{G}_{4,11}) = 20.$$

Theorem 3.5. *If $m \equiv 2(mod3)$, then*

$$\chi_{td}(\mathscr{G}_{m,n}) = \begin{cases} \frac{(m-2)n}{3} + 2\lceil \frac{n}{3} \rceil + 2, & if \quad n \quad is\ even \\ \frac{(m-2)n}{3} + 2\lceil \frac{n}{3} \rceil + 3, & if \quad n \quad is\ odd. \end{cases}$$

Proof. Given $m-2 \equiv 0(mod3)$. We consider two cases.
Case (i): When n is even. We have $\mathscr{G}_{m,n}$ is obtained by $\mathscr{G}_{m-2,n}$ followed by $\mathscr{G}_{2,n}$. From Theorem 3.4, $\chi_{td}(\mathscr{G}_{m,n}) = \chi_{td}(\mathscr{G}_{m-2,n}) + \chi_{td}(\mathscr{G}_{2,n}) - 2$. By Theorem 3.3 and Lemma 3.1, we get
$$\chi_{td}(\mathscr{G}_{m,n}) = \frac{(m-2)n}{3} + 2\lceil \frac{n}{3} \rceil + 2.$$
Case (ii): When n is odd. We have $\mathscr{G}_{m,n}$ is obtained by $\mathscr{G}_{m-2,n}$ followed by $\mathscr{G}_{2,n}$. From Theorem 3.4, $\chi_{td}(\mathscr{G}_{m,n}) = \chi_{td}(\mathscr{G}_{m-2,n}) + \chi_{td}(\mathscr{G}_{2,n}) - 2$.

By Theorem 3.3 and Lemma 3.1, we get
$$\chi_{td}(\mathscr{G}_{m,n}) = \frac{(m-2)n}{3} + 2\lceil \frac{n}{3} \rceil + 3.$$
Thus
$$\chi_{td}(\mathscr{G}_{m,n}) = \begin{cases} \frac{(m-2)n}{3} + 2\lceil \frac{n}{3} \rceil + 2, & \text{if} \quad n \quad \text{is even} \\ \frac{(m-2)n}{3} + 2\lceil \frac{n}{3} \rceil + 3, & \text{if} \quad n \quad \text{is odd.} \end{cases}$$

□

Illustration: Consider $\mathscr{G}_{5,8}$

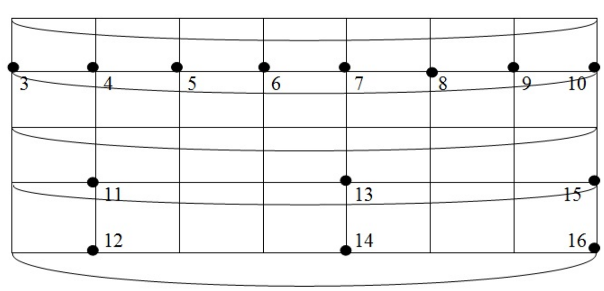

Figure 7

Therefore
$$\chi_{td}(\mathscr{G}_{5,8}) = 16.$$

Illustration: Consider $\mathscr{G}_{5,7}$

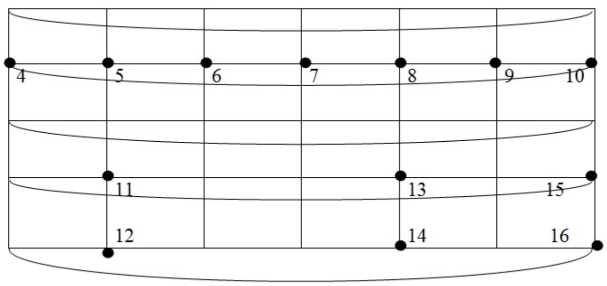

Figure 8

Therefore

$$\chi_{td}(\mathscr{G}_{5,7}) = 16.$$

4. Conclusion

In this paper, we obtain total dominator chromatic number of $P_m \times C_n$.

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