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Dominator chromatic number of grid graph

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Abstract

Let G be a graph. A dominator coloring of G is a proper coloring in which every vertex of G dominates at least one color class. The dominator chromatic number of G is denoted by $\chi_d(G)$ and is defined by the minimum number of colors needed in a dominator coloring of G. In this paper, we obtain dominator chromatic number of grid graphs.

Keywords

Dominator chromatic number, grid graph.

AMS Subject Classification

05C15, 05C69

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Contents

1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definition of graph theory as found in [1]. Let $G=(V,E)$ be a graph of order n. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. The path and cycle of order n are denoted by P_n and C_n respectively. For any two graphs G and H, we define the cartesian product, denoted by $G\times H$, to be the graph with vertex set $V(G)\times V(H)$ and edges between two vertices (u_1,v_1) and (u_2,v_2) iff either $u_1=u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1=v_2$. A grid graphs can be defined as $P_m \times P_n$ where m, $n \geq 2$ and denoted by $P_{m\times n}$.
A subset S of V is called a dominating set if every vertex in $V-S$ is adjacent to atleast one vertex in S. The dominating set is minimal dominating set if no proper subset of S is a dominating set of G. The domination number γ is the minimum cardinality taken over all minimal dominating set of G. A γ-set is any minimal dominating set with cardinality γ.
A proper coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. The minimum number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. A dominator coloring of G is a proper coloring of G in which every vertex of G dominates atleast one color class. The dominator chromatic number is denoted by $\chi_d(G)$ and is defined by the minimum number of colors needed in a dominator coloring of G .This concept was introduced by Ralucca Michelle Gera in 2006 [2].
In a proper coloring C of G, a color class of C is a set consisting of all those vertices assigned the same color. Let C^1 be a minimal dominator coloring of G. We say that a color class $c_i \in C^1$ is called a non-dominated color class ($n-d$ color class) if it is not dominated by any vertex of G. These color classes are also called repeated color classes. The dominator chromatic number of paths found in [2]. We have the following observation from [2].

Theorem 1.1. *[2] The path P_n of order $n \geq 2$ has*

$$\chi_d(p_n)=\begin{cases}\lceil \frac{n}{3} \rceil +1 & \text{if } n=2,3,4,5,7\\ \lceil \frac{n}{3} \rceil +2 & \text{otherwise}\end{cases}$$

In this paper, we obtain the least value for dominator chromatic number for grid graphs.

2. Main Results

Theorem 2.1. *If m and n both even then*

$$\chi_d(p_{m\times n}) = \begin{cases} \frac{mn}{4}+2 & \text{if } m,n \equiv 0(mod4) \\ \\ \frac{mn}{4}+3 & \text{otherwise} \end{cases}$$

Proof. Let the vertex set of

$$P_{m\times n} = V(P_{m\times n}) = \left\{ u_{ij} \middle| \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array} \right\}.$$

We consider two cases.
Case (1): Let $m,n \equiv 0(mod4)$. Let

$$D^{(1)} = \left\{ \left\{ u_{ij} \middle| \begin{array}{l} i = 1,5,9,\dots,(m-1) \\ j = 2,6,10,\dots,(n-2) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 2,6,10,\dots,(m-2) \\ j = 4,8,12,\dots,n \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 3,7,11,\dots,(m-1) \\ j = 1,5,9,,\dots,(n-3) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 4,8,12,,\dots,m \\ j = 3,7,11,\dots,(n-1) \end{array} \right\} \right\}$$

be any arbitrary γ-set of $P_{m\times n}$. We assign one new color say $3,4,5,\dots,(\frac{mn}{4}+2)$ to every vertex in $D^{(1)}$. Also we assign two repeated colors say $1,2$ to the vertices u_{ij} and $u_{kl} \in V(P_{m\times n}) - D^{(1)}$ such that $|i-k|+|j-l| = 1$. So $\chi_d(P_{m\times n}) = \frac{mn}{4}+2$.
Case (2): Let except $m,n \equiv 0(mod4)$. we have three sub cases.
Subcase (2.1): Let $m \equiv 0(mod4) \& n \equiv 2(mod4)$. Let

$$S^{(1)} = \left\{ \left\{ u_{ij} \middle| \begin{array}{l} i = 1,5,9,\dots,(m-1) \\ j = 2,6,10,\dots,n \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 2,6,10,\dots,(m-2) \\ j = 4,8,12,\dots,(n-2) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 3,7,11,\dots,(m-1) \\ j = 1,5,9,,\dots,(n-1) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 4,8,12,,\dots,m \\ j = 3,7,11,\dots,(n-3) \end{array} \right\} \bigcup \{u_{mn}\} \right\}$$

be any arbitrary γ-set of $P_{m\times n}$. We assign one new color say $3,4,5,\dots,(\frac{mn}{4}+3)$ to every vertex in $S^{(1)}$. Also we assign two repeated colors say $1,2$ to the vertices u_{ij} and $u_{kl} \in V(P_{m\times n}) - S^{(1)}$ such that $|i-k|+|j-l| = 1$. So $\chi_d(P_{m\times n}) = \frac{mn}{4}+3$.
Subcase (2.2): Let $m \equiv 2(mod4) \& n \equiv 0(mod4)$. Let

$$S^{(2)} = \left\{ \left\{ u_{ij} \middle| \begin{array}{l} i = 1,5,9,\dots,(m-1) \\ j = 2,6,10,\dots,(n-2) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 2,6,10,\dots,m \\ j = 4,8,12,\dots,n \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 3,7,11,\dots,(m-3) \\ j = 1,5,9,,\dots,(n-3) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 4,8,12,,\dots,(m-2) \\ j = 3,7,11,\dots,(n-1) \end{array} \right\} \bigcup \{u_{m1}\} \right\}$$

be any arbitrary γ-set of $P_{m\times n}$. We assign one new color say $3,4,5,\dots,(\frac{mn}{4}+3)$ to every vertex in $S^{(2)}$. Also we assign two repeated colors say $1,2$ to the vertices u_{ij} and $u_{kl} \in V(P_{m\times n}) - S^{(2)}$ such that $|i-k|+|j-l| = 1$. So $\chi_d(P_{m\times n}) = \frac{mn}{4}+3$.
Subcase (2.3): Let $m \equiv 2(mod4) \& n \equiv 2(mod4)$. Let

$$S^{(3)} = \left\{ \left\{ u_{ij} \middle| \begin{array}{l} i = 1,5,9,\dots,(m-1) \\ j = 2,6,10,\dots,n \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 2,6,10,\dots,m \\ j = 4,8,12,\dots,(n-2) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 3,7,11,\dots,(m-3) \\ j = 1,5,9,,\dots,(n-1) \end{array} \right\} \bigcup \left\{ u_{ij} \middle| \begin{array}{l} i = 4,8,12,,\dots,(m-2) \\ j = 3,7,11,\dots,(n-3) \end{array} \right\} \bigcup \{u_{mn}\} \right\}$$

be any arbitrary γ-set of $P_{m\times n}$. We assign one new color say $3,4,5,\dots,(\frac{mn}{4}+3)$ to every vertex in $S^{(3)}$. Also we assign two repeated colors say $1,2$ to the vertices u_{ij} and $u_{kl} \in V(P_{m\times n}) - S^{(3)}$ such that $|i-k|+|j-l| = 1$. So $\chi_d(P_{m\times n}) = \frac{mn}{4}+3$. Thus

$$\chi_d(P_{m\times n}) = \begin{cases} \frac{mn}{4}+2 & \text{if } m,n \equiv 0(mod4) \\ \frac{mn}{4}+3 & \text{otherwise} \end{cases}$$

□

3. Illustration

Consider $P_{8\times 12}$ and $P_{8\times 10}$

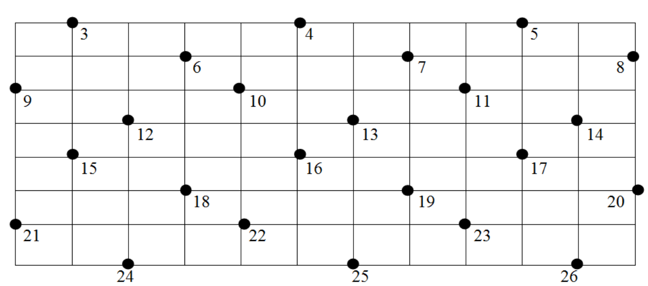

Figure 1. $\chi_d(P_{8\times 12}) = 26$

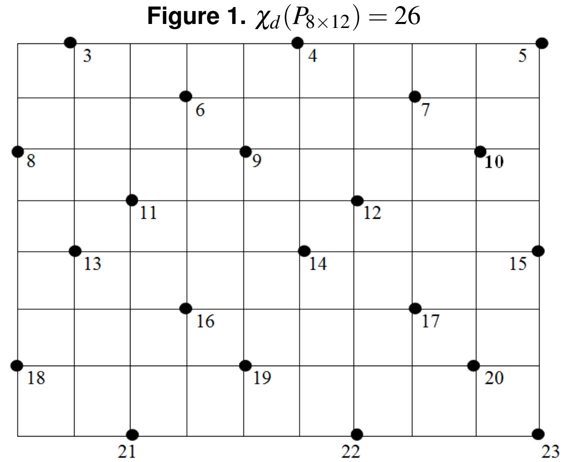

Figure 2. $\chi_d(P_{8\times 10}) = 23$

Consider $P_{6\times 12}$ and $P_{6\times 10}$

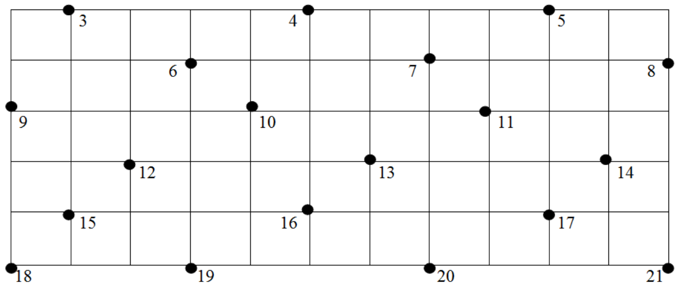

Figure 3. $\chi_d(P_{6\times 12}) = 21$

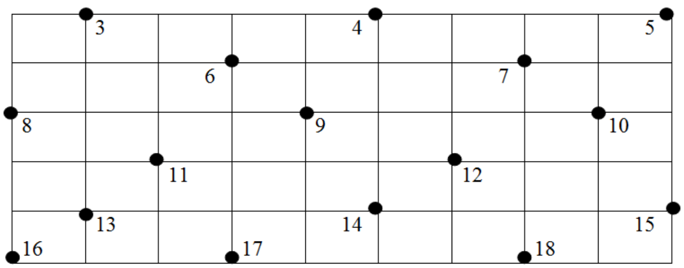

Figure 4. $\chi_d(P_{6\times 10}) = 18$

Theorem 3.1. *If m is odd and n is even then*

$$\chi_d(P_{m\times n}) = \begin{cases} \frac{(m-1)n}{4} + \lceil \frac{n}{3} \rceil + 2 & \text{if } m-1, n \equiv 0(mod4) \\ \frac{(m-1)n}{4} + \lceil \frac{n}{3} \rceil + 3 & \text{otherwise} \end{cases}$$

Proof. We have $P_{m\times n}$ is obtained by $P_{(m-1)\times n}$ followed by P_n. Since in a dominator coloring of $P_{m\times n}$ we cannot use the non-repeated colors of vertices in P_n and we can use the same repeated colors of vertices in the graphs $P_{(m-1)\times n}$ and P_n. Since $m-1$ is even, we get by Theorem 1.1

$$\chi_d(P_{(m-1)\times n}) = \begin{cases} \frac{(m-1)n}{4} + 2 & \text{if } m-1, n \equiv 0(mod4) \\ \frac{(m-1)n}{4} + 3 & \text{otherwise} \end{cases}$$

So

$$\chi_d(P_{m\times n}) = \begin{cases} \frac{(m-1)n}{4} + \lceil \frac{n}{3} \rceil + 2 & \text{if } m-1, n \equiv 0(mod4) \\ \frac{(m-1)n}{4} + \lceil \frac{n}{3} \rceil + 3 & \text{otherwise} \end{cases}$$

□

Let us consider $P_{7\times 8}$

Theorem 3.2. *If m is even and n is odd then*

$$\chi_d(P_{m\times n}) = \begin{cases} \frac{m(n-1)}{4} + \lceil \frac{m}{3} \rceil + 2 & \text{if } m, n-1 \equiv 0(mod4) \\ \frac{m(n-1)}{4} + \lceil \frac{m}{3} \rceil + 3 & \text{otherwise} \end{cases}$$

Proof. Since m and $n-1 \equiv 0(mod4)$ and $P_{m\times n}$ is obtained by $P_{m\times(n-1)}$ followed by P_m. By Theorem 3.1,

$$\chi_d(P_{m\times n}) = \chi_d(P_{m\times(n-1)}) + \chi_d(P_m) - 2.$$

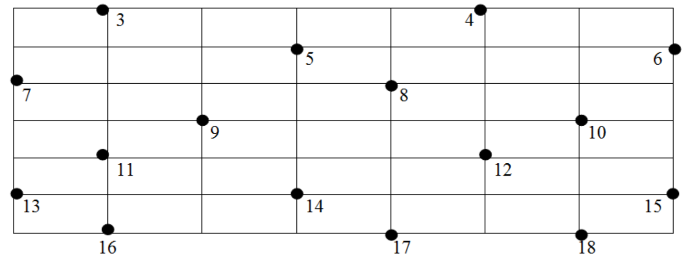

Figure 5. $\chi_d(P_{7\times 8}) = 18$

By Theorem 1.1

$$\chi_d(P_{m\times(n-1)}) = \begin{cases} \frac{m(n-1)}{4} + 2 & \text{if } m, n-1 \equiv 0(mod4) \\ \frac{m(n-1)}{4} + 3 & \text{otherwise} \end{cases}$$

So

$$\chi_d(P_{m\times n}) = \begin{cases} \frac{m(n-1)}{4} + \lceil \frac{m}{3} \rceil + 2 & \text{if } m, n-1 \equiv 0(mod4) \\ \frac{m(n-1)}{4} + \lceil \frac{m}{3} \rceil + 3 & \text{otherwise} \end{cases}$$

□

Now consider $P_{8\times 11}$

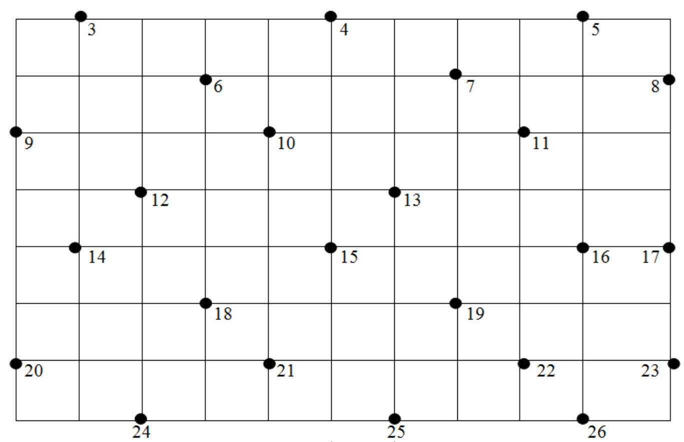

Figure 6. $\chi_d(P_{8\times 11}) = 26$

Theorem 3.3. *If m is odd and n is odd then*

$$\chi_d(P_{m\times n}) = \begin{cases} \frac{(m-1)(n-1)}{4} + \lceil \frac{m+n-1}{3} \rceil + 2 & \text{if } m-1, n-1 \equiv 0(mod4) \\ \frac{(m-1)(n-1)}{4} + \lceil \frac{m+n-1}{3} \rceil + !3 & \text{otherwise} \end{cases}$$

Proof. Since $(m-1)$ and $(n-1) \equiv 0(mod4)$ and $P_{m\times n}$ is obtained by $P_{(m-1)\times(n-1)}$ followed by $P + m + n - 1$. By theorem 3.1,

$$\chi_d(P_{m\times n}) = \chi_d(P_{(m-1)\times(n-1)}) + \chi_d(P_{m+n-1}) - 2.$$

By Theorem 1.1,

$$\chi_d(P_{(m-1)\times(n-1)}) = \begin{cases} \frac{(m-1)(n-1)}{4} + 2 & \text{if } m-1, n-1 \equiv 0(mod4) \\ \frac{(m-1)(n-1)}{4} + 3 & \text{otherwise} \end{cases}$$

So

$$\chi_d(P_{m\times n}) = \begin{cases} \frac{(m-1)(n-1)}{4} + \left\lceil \frac{m+n-1}{3} \right\rceil + 2 & \text{if } m-1, n-1 \equiv 0(mod4) \\ \frac{(m-1)(n-1)}{4} + \left\lceil \frac{m+n-1}{3} \right\rceil + 3 & \text{otherwise} \end{cases}$$

□

Let us consider $P_{11\times 7}$

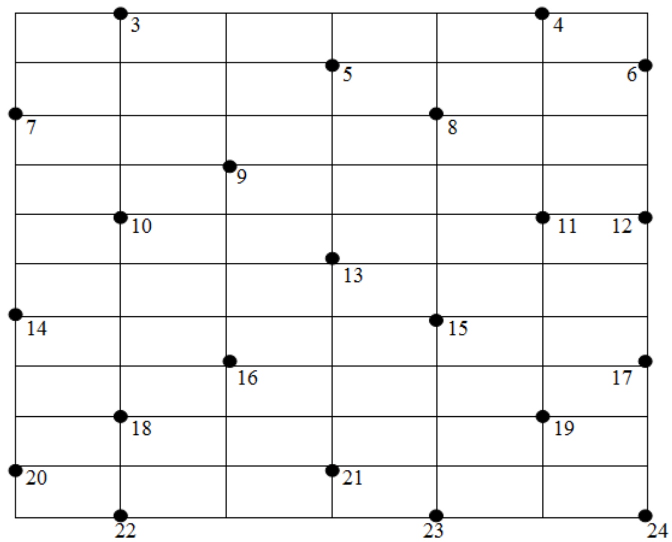

Figure 7. $\chi_d(P_{11\times 7}) = 24$

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