Perfect domination separation on square chessboard

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Abstract
This paper focuses on reducing the perfect domination number(γ_{pf}) of the chess pieces rooks, bishops and kings on an $n \times n$ board. Here we reduce this parameter by the separation problem which separates the board by placing a minimum number of chess pieces of a particular type with a minimum number of pawns. A subset D of $V(G)$ is said to be a *Perfect Dominating Set* (PDS) if every vertex in $V - D$ is dominated by exactly one vertex of D. Among all the perfect dominating sets the cardinality of the one with the minimum number of vertices is the *Perfect Domination Number* (γ_{pf}).

Keywords
Chessboard graphs, separation problem, perfect domination.

AMS Subject Classification
05C69.

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1. Introduction

A chessboard graph is obtained by taking each cell of the board as a vertex or node and joining these vertices by edges or links if they are adjacent to one another from the movement of the chess piece taken. Thus, a Bishops graph B_n on an $n \times n$ board has n^2 vertices with two squares adjacent if they lie on the same diagonal, whereas, in Kings graph K_n two squares u and v are adjacent if they are within a distance of one (i.e., $d(u,v) = 1$). In Rooks graph R_n two squares are adjacent if they lie on the same row or column.

A dominating set $D \subset V(G)$ is a perfect dominating set (*PDS*) if every vertex in $V - D$ is adjacent to exactly one vertex in D i.e,$|N(v) \cap D| = 1$ for every $v \in V - D$. The perfect domination number $\gamma_{pf}(G)$ is the cardinality of the perfect dominating set with minimum number of vertices.

The study of dominating sets in graphs started with the chessboard Domination Problem [5]. The *queens domination problem* was considered as one of the most interesting and difficult among the chessboard problems. In [7] Yaglom and Yaglom, found some domination parameters and their total number of solutions (i.e., number of different ways of arrangement) related to different chess pieces on a square board. Several papers in this area and their extensions were carried out, where Zhao [8] in 1998 extended it further by showing that more than n independent queens can be placed on an $n \times n$ board if enough blocking pieces, such as pawns, are placed between queens. Chatham et al. further extended the work of Zhao, and brought the concept of separated chessboard graphs with various domination parameters on an $n \times n$ board starting with independence domination in [2, 3]. Chatham et al. defined the separation problem as the legal placement of minimum number of pawns with the maximum number of independent chess pieces chosen on an $n \times n$ board which results in a separated board. A legal placement is the separation of attacking queens by pawns (Since the vertices in an independent dominating set are all non-adjacent).

Here we consider the domination parameter perfect domination introduce in [4] by Cockayne et al., in 1993. Work related to this parameter is done on various classes of graphs. In [6] the perfect domination number for rooks, bishops and kings on square chessboards were given as $\gamma_{pf}(R_n) = n$;
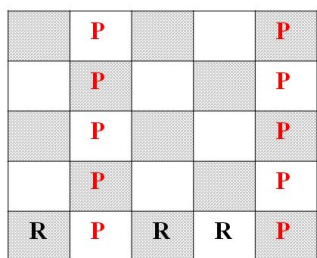
Figure 1. $s_R(\gamma_{pf}, 5-2, 5) = 10$

$\gamma_{pf}(B_n) = 2n-1$ for $n \geq 4$ and $\gamma_{pf}(K_n) = \left\lfloor \frac{n+2}{3} \right\rfloor^2$ for $n \geq 3$.

In this paper, we find the perfect domination separation on square chessboard by placing the pawns in the place of non-dominating cells to decrease γ_{pf} values obtained in [6]. We denote the perfect domination separation number for any chess piece $\mathbb{C}$ as $s_{\mathbb{C}}(\gamma_{pf}, \gamma_{pf}(\mathbb{C})-k, n)$, where k denotes the number of pawns. This separation number gives the minimum number of pawns required to decrease the perfect domination number of a particular chess piece.

For any positive integer n, a chessboard of order n has columns running from left to right and rows running from bottom to top, where the cell in the lower left-hand corner is numbered $(1,1)$ and the other cells are numbered as (r,c). Here r denote a row and c denote a column. The sum diagonal s_i and difference diagonal d_j are the diagonals that consists of all the cells (r,c) of the form $r+c = i$ and $r-c = j$ respectively. The empty cells are the cells that are not occupied by an chess piece.

2. Rooks perfect domination separation

Theorem 2.1. *For all* $n \in N$, $s_R(\gamma_{pf}, n-k, n) = kn$

Proof. To show that kn pawns are sufficient to separate $n-k$ rooks on an $n \times n$ board, place n rooks either in a column or a row. From the movement of rooks we can say that, each rook can cover $2n-1$ cells along with the cell in which it is placed. As we know that the perfect domination number of rooks is n, by placing n rooks in a row, each rook dominates $n-1$ empty cells. Now by reducing the perfect domination number by one we have $n-1$ rooks in that row and n pawns in the respective column from which the rook is removed. Thus, n-k rooks can be separated by nk pawns. See Fig. 1   □

3. Bishops perfect domination separation

Theorem 3.1. *For* $n \geq 4$,

(i) $s_B(\gamma_{pf}, 2n-1-k, n) = \sum_{x=1}^{\lfloor \frac{k}{2} \rfloor} 2x + k - \lfloor \frac{k}{2} \rfloor$, *when k is odd*

(ii) $s_B(\gamma_{pf}, 2n-1-k, n) = \sum_{x=1}^{\lfloor \frac{k}{2} \rfloor} 2x$, *when k is even*

Proof. First consider the placement of $2n-1$ bishops on an nn board as shown in Fig. 2(a). Since the board has $2n-1$ sum diagonals we place one bishop in each of the diagonals
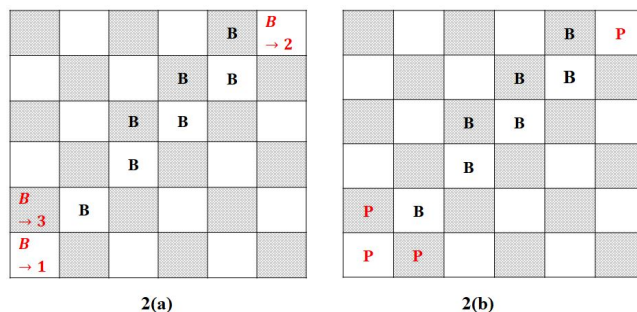

Figure 2. (a)Bishops to be removed, (b) $s_B(\gamma_{pf}, 11-3, 6) = 4$

such that every cell in a particular sum diagonal is dominated by exactly one bishop satisfying the property of perfect domination. Thus, we place n bishops in the positive main diagonal (i.e., the diagonal running from bottom left to top right) and n-1 bishop in the diagonal above as mentioned in [6]. Now, we decrease γ_{pf} by removing the bishops one after the other from the sum diagonals alternatively moving inward from either side of the diagonally opposite corners. Then, place pawns in each cell of the sum diagonal from which the bishop is removed.

Here, the board has one main sum diagonal with n cells and the remaining sum diagonals on either side of it. Now, we remove first bishop from the bottom left most corner (i.e., the sum diagonal with one cell) and replace it with a pawn and then continue removing as mentioned earlier.

(i) When number of bishops removed k is odd (i.e., when γ_{pf} is decreased by k number of bishops, where k is odd), we have $k - \lfloor \frac{k}{2} \rfloor$ empty diagonals (i.e., without bishops or bishops removed) from the bottom left corner and $\lfloor \frac{k}{2} \rfloor$ empty diagonals from the top right corner of the main diagonal. Now fill these empty cells with pawns. Thus, $k-1$ diagonals will be filled with $2x$ pawns where $1 \leq x \leq \lfloor \frac{k}{2} \rfloor$ and the remaining one diagonal below the main diagonal is filled with $k - \lfloor \frac{k}{2} \rfloor$ pawns.

(ii) When k is even we have $\lfloor \frac{k}{2} \rfloor$ empty diagonals on either side of the main sum diagonals. The cells in these sum diagonals are replaced with $\sum_{x=1}^{\lfloor \frac{k}{2} \rfloor} 2x$ pawns.

Here in Fig. 2(b). $\sum_{x=1}^{3} 2x + k - \lfloor \frac{k}{2} \rfloor = 2(1) + 3 - 1 = 4$, where 1, 2, and 3 denotes bishops to be removed.   □

Theorem 3.2. *For* $n = 1, 2$, $s_B(\gamma_{pf}, \gamma_{pf} - k, n) = nk$

Proof. For the case $n = 1$, it is trivial as $\gamma_{pf} = 1$. For $n = 2$, we have $\gamma_{pf} = 2$ where the two bishops are placed either in a single row or a column. Here the two bishops dominates the cells in which they are placed and the remaining two cells of the board are dominated by one bishop each. Thus, removal of a bishop separates the board with 2 pawns reducing the domination number by one (i.e., $s_B(\gamma_{pf}, \gamma_{pf} - 1, 2) = 2$).   □

Theorem 3.3. *For* $n = 3$, $s_B(\gamma_{pf}, \gamma_{pf} - k, n) = k^2$

Proof. We prove this by considering 3 bishops on a 3×3 board as $\gamma_{pf}(B_3) = 3$. Since we know that there are $2n - 1$ sum diagonals on a square board as mentioned in Theorem 3.1 we need a maximum of $2n - 1$ bishops. But in this case since the board is of order 3×3, placing a bishop in the center of the board covers three sum diagonals and placing one bishop each on the remaining two diagonals on either side of main sum diagonal covers the entire board satisfying the property of perfect domination.

To reduce the perfect domination number we start removing the bishops one after the other from either side of the sum diagonal. Here, each bishop on either side of the board dominates one empty cell in the sum diagonal and one cell occupied by bishop in the difference diagonal. Now reducing γ_{pf} by one results in separation with one pawn, since the cell from which the bishop removed is also dominated by another bishop. Thus, removing the second bishop leaves 3 non-dominating cells. Therefore, a minimum of k^2 pawns are required to separate a minimum of $\gamma_{pf} - k$ bishops on a 3×3 board. □

4. Kings perfect domination separation

Lemma 4.1. *For* $n \geq 3$ *and* $n = 3a$,
$s_K(\gamma_{pf}, \lfloor \frac{n+2}{3} \rfloor^2 - k, n) = 9k$

Proof. When the board is of order $3a$, we divide the board into sub-boards of order 3×3 and place $\lfloor \frac{n+2}{3} \rfloor^2$ kings which is the perfect domination number as mentioned in [6]. We prove that $\lfloor \frac{n+2}{3} \rfloor^2 - k$ kings can be separated using $9k$ pawns according to the moves of the king that dominates at most nine cells. Since the board is divided into sub-boards of order 3×3, each sub-board has a king that dominates exactly nine cells which in turn covers the entire board. Thus, removal of a king on a sub-board would result in nine non-dominating cells including the cell from which the king is removed and therefore, these cells will be filled with nine pawns. Similarly when k kings are removed we place $9k$ pawns in the respective non-dominating cells resulting in a separation board. See Fig.3̃. □

Lemma 4.2. *For the case* $n = 3a + 1$ *and* $n \geq 3$, *kings separation with perfect domination is as follows:*

(i) $0 < k \leq 2$, $s_K(\gamma_{pf}, \lfloor \frac{n+2}{3} \rfloor^2 - k, n) = 2k$

(ii) $2 < k \leq \frac{6n+3}{9}$, $s_K(\gamma_{pf}, \lfloor \frac{n+2}{3} \rfloor^2 - k, n) = 3k - 2$

(iii) $\frac{6n+3}{9} < k \leq \lfloor \frac{n+2}{3} \rfloor^2$,
$s_K(\gamma_{pf}, \lfloor \frac{n+2}{3} \rfloor^2 - k, n) = 2n - 1 + 9(k - \gamma + \lfloor \frac{n+2}{3} \rfloor^2)$

Proof. We first divide the board into boards of height 2 from top and bottom and the remaining with height three. Then divide the board at width 2 from the first and the last column and the remaining center columns with width 3 as mentioned
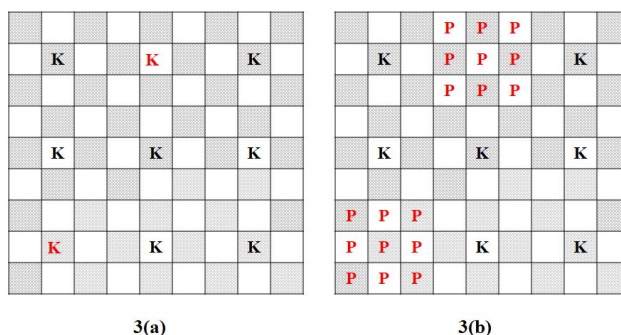

Figure 3. (a)Red K's denote kings to be removed,
(b) $s_k(\gamma_{pf}, \lfloor \frac{n+2}{3} \rfloor^2 - 2, 9) = 9(2) = 18$
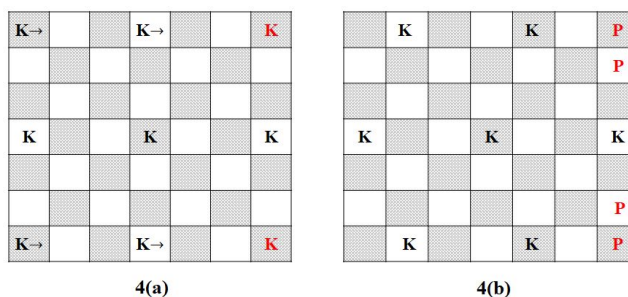

Figure 4. (a)Red K's denote kings to be removed,
(b) $s_k(\gamma_{pf}, \lfloor \frac{n+2}{3} \rfloor^2 - 2, 7) = 2(2) = 4$

in [7]. Now place a king in each sub-board obtained as mentioned in [6]. From this we can say that a king can dominate exactly 4 cells when placed in the corner, exactly 6 cells when placed in the first or last, row or column leaving the corners and exactly 9 cells when placed in any of the cells leaving the outer.

(i) For $0 < k \leq 2$ remove the kings from the corners of the same column on after the other and shift the kings to the next immediate cell in the same row from which the king is removed. Thus by removing a king, each king present in the row after shifting dominates a 2×3 sub-board leaving a 2×1 sub-board not-dominated by any king for which will be occupied by pawns as the board is of the form $3a + 1$. Therefore to reduce the perfect domination number of king by one we need 2 pawns and hence, $s_K(\gamma_{pf}, \gamma_{pf} - k, n) = 2k$ in this case. See Fig 4(a) and 4(b).

(ii) For $2 < k \leq \frac{6n+3}{9}$, We first remove the left over kings present in the column from which the previous kings are removed in case(i) and simultaneously shift the kings as done in case(i). Here note that whenever kings are removed from a column(row) we shift the kings present in the respective row(column) from which it is removed to the next immediate cell (cell just below it). Fig. 4(b) shows the kings removed from a column.

Since the sub- boards present in the center are of order 3×3 and the board is of the form $3a + 1$, there will be a
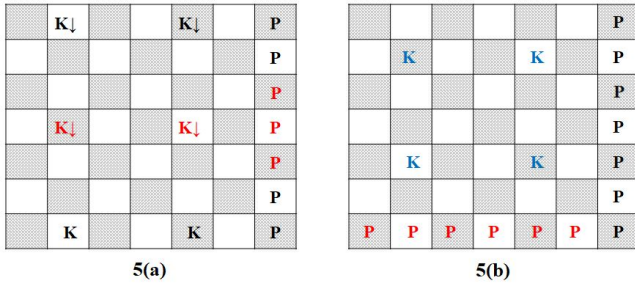
Figure 5. (a) Red K's denote kings to be removed and red K's with arrow denotes kings to be shifted,

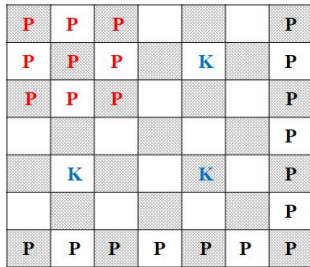(b) $s_k(\gamma_{pf}, \lfloor\frac{n+2}{3}\rfloor^2 - 5, 7) = 3(5) - 2 = 13$

Figure 6. $s_k(\gamma_{pf}, \lfloor\frac{n+2}{3}\rfloor^2 - 6, 7) = 22$

sub-board of order 3×1 whose cells are not dominated by any king. These cells are now occupied by pawns.

Now, start removing the kings from the first row of the board one after the other and start shifting the kings as mentioned earlier. This results in a sub-board of order 3×1 with 3 non-dominant cells which are now occupied by pawns. Continue in the same way until the required k kings are removed as shown in Fig. 5(a) and 5(b).

Fig. 5(a) is obtained from Fig. 4(b). The pawns in red in Fig. 5(a) shows the sub-board from which the king present in Fig. 4(a) is removed and the red kings represent the kings shifted in that respective row from which it is removed. The arrows in Fig. 5(a) are the kings to be shifted to obtain Fig.5(b) by removing kings one after the other from the first row as mentioned in the proof.

(iii) In this case for $\frac{6n+3}{9} < k \leq \lfloor\frac{n+2}{3}\rfloor^2$, we are left with a row and a column of the board of order $3a + 1$ filled with $2n - 1$ pawns, and the remaining board of order $3a$ is covered by $\lfloor\frac{n}{3}\rfloor^2$ kings. Now by using Lemma 1 we remove these $\lfloor\frac{n}{3}\rfloor^2$ kings one after the other till the desired k is obtained and replace each king with 9 pawns. This separates $\lfloor\frac{n+2}{3}\rfloor^2 - k$ kings with $2n - 1 + 9(k - \gamma + \lfloor\frac{n}{3}\rfloor^2)$ pawns. See Fig. 6
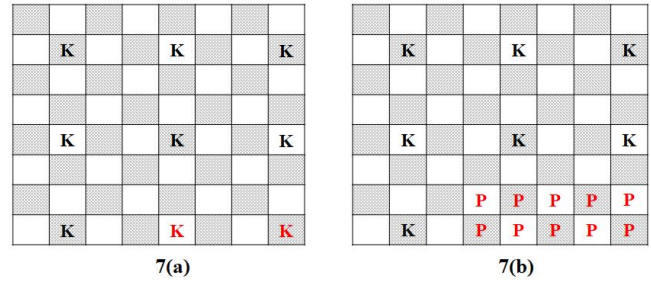
$\square$

Figure 7. (a) Red K's denote kings to be removed,

(b) $s_k(\gamma_{pf}, \lfloor\frac{n+2}{3}\rfloor^2 - 2, 8) = 10$

Lemma 4.3. *For $n = 3a + 2$ and $n \geq 3$, kings separation with perfect domination is as follows:*

(i) $0 < k \leq 2\lfloor\frac{n}{3}\rfloor + 1$, $s_K(\gamma_{pf}, \lfloor\frac{n+2}{3}\rfloor^2 - k, n) = 4 + 6(k - 1)$

(ii) $2\lfloor\frac{n}{3}\rfloor + 1 < k \leq 2\lfloor\frac{n+2}{3}\rfloor^2$, $s_K(\gamma_{pf}, 2\lfloor\frac{n+2}{3}\rfloor^2 - k, n) = 4(n - 1) + 9[k - (2\lfloor\frac{n}{3}\rfloor + 1)]$

Proof. We prove this lemma by first dividing the board as in [1] and from Fig.7(a), which shows the board partitioned at height 2 from bottom with width 3 till the $(n - 1)^{th}$ column and the remaining board at height 3. Now place a king in each of the sub-boards satisfying the concept of perfect domination. Since the board is of the form $3a + 2$ the board of order $3a$ will have $\lfloor\frac{n}{3}\rfloor^2$ kings one each in the 3×3 sub-boards. The remaining 2 extra rows and columns which intersect at a corner is placed with a king which dominates a 2×2 sub-board, and the other $2 \times n - 2$ board and $n - 2 \times 2$ board is placed with $2\lfloor\frac{n-2}{3}\rfloor$ kings one in each sub-board of order 2×3. Thus we have placed $\lfloor\frac{n+2}{3}\rfloor^2$ kings in total.

(i) For $0 < k \leq 2\lfloor\frac{n}{3}\rfloor + 1$, first remove the king from the only corner with 2×2 sub-board which dominates less number of cells and then fill these 4 cells with pawns. Now start removing the kings from the sub-boards of order 2×3 and fill the 6 cells dominated by the respective kings removed with pawns until k reaches $2\lfloor\frac{n}{3}\rfloor + 1$. Thus in this case $\gamma_{pf} - k$ pawns can be separated by $4 + 6(k - 1)$ pawns. See Fig.7(b)

(ii) For $2\lfloor\frac{n}{3}\rfloor + 1 < k \leq \lfloor\frac{n+2}{3}\rfloor^2$, since we have already replaced the kings with $4(n - 1)$ pawns in the sub-boards of order less than 3×3, we start removing the $\lfloor\frac{n}{3}\rfloor^2$ kings one after the other from the sub-board of order $3a$ as proved earlier in Lemma 1. Thus removing the kings inductively till the k reaches $\lfloor\frac{n+2}{3}\rfloor^2$ gives separated board with $\lfloor\frac{n+2}{3}\rfloor^2 - k$ kings and $4(n - 1) + 9[k - (2\lfloor\frac{n}{3}\rfloor + 1)]$. See Fig. 8(a) and 8(b)
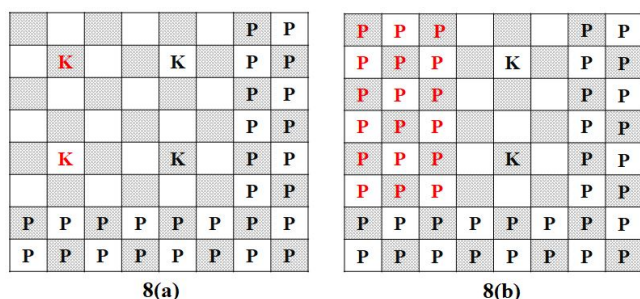
$\square$

Figure 8. (a) Red K's denote kings to be removed,
(b) $s_k(\gamma_{pf}, \lfloor \frac{n+2}{3} \rfloor^2 - 7, 8) = 46$

5. conclusion

The chessboard separation problem which is an extension of chessboard domination problem has been discussed in detailed. The separation number for the rooks, bishops and kings on a square chessboard were obtained by decreasing the perfect domination number mentioned in [6] using the pawns. We are presently working on separation problems on rectangular and hexagonal board with respect to various domination parameters.

Acknowledgment

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