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Skew-constacyclic codes over $\mathscr{R}$

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Abstract

Let p be an odd prime and $q = p^m$, where m is a positive integer. We study the Θ_t-cyclic and (Θ_t, λ)-cyclic code over a finite commutative non-chain ring $\mathscr{R} = \mathbb{F}_q[u,v,w]/\langle u^2 = u, v^2 = v, w^2 = 1, uv = vu = 0, uw = wu, wv = vw\rangle$, where λ is a unit in $\mathscr{R}$.

Keywords

Skew cyclic codes; quasi-cyclic codes; equivalent codes; linear codes; arbitrary lengths.

AMS Subject Classification

94B05, 94B15.

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1. Introduction

Constacyclic codes constitute a remarkable generalization of cyclic codes and form an important class of linear codes in the coding theory. Constacyclic codes also have some practical applications as they can be encoded with shift registers.

In 2007, Boucher et al. [6] firstly studied cyclic codes in a non commutative ring called the skew polynomials ring $\mathbb{F}[x;\theta]$, where θ denotes the automorphism of the finite field $\mathbb{F}$, and they produced many good linear codes which better than existing ones. One advantage of skew polynomials ring is that the polynomial $x^n - 1$ has more factors in skew polynomials ring than commutative rings. Later, in [7] Boucher et al. generalised this idea to skew constacyclic codes in the skew polynomials ring. In 2011, Siap et al. [21] studied the skew cyclic codes of arbitrary length and established a strong connection with well known codes. In 2012, Abualrub et al. [1] studied θ-cyclic codes over the non-chain ring $\mathbb{F}_2 + v\mathbb{F}_2, v^2 = v$ with respect to Euclidean and Hermitian inner products. Jitman et al. [17] studied skew constacyclic codes over finite chain rings and gave the generators of Euclidean and Hermitian dual codes. Later, these codes over non-chain rings are extensively studied. For instance, the rings $\mathbb{F}_3 + v\mathbb{F}_3$ in [2], $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$ in [12], $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu = 0$ in [3] are considered to study skew cyclic codes. Also, Yao et al. [20] and Dertli & Cengellenmis [9] studied these codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$. In 2017, Gao et al. [11] obtained the structure of skew constacyclic codes over non-chain ring $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$ and they obtained skew $(-1+2v)$-constacyclic codes. Islam and Prakash have determined the structural properties of skew constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$ in [15] and $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu = 0$ in [16]. In 2019, Bhardwaj and Raka [4] studied skew constacyclic codes over the ring $\mathbb{F}_q[u,v]\langle f(u), g(v), uv - vu\rangle$ by using two non trivial automorphisms.

Motivated by above studies, in this paper, we consider a Commutative ring $\mathscr{R} = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + w\mathbb{F}_q + uv\mathbb{F}_q$ where $u^2 = u, v^2 = v, w^2 = w, uv = vu, uw = 0 = wv$ and study Θ_t-cyclic and (Θ_t, λ)-cyclic codes over it.

2. Gray Map

Let $q = p^m$, where p is an odd prime and $\mathscr{R} = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + w\mathbb{F}_q + uv\mathbb{F}_q$ where $u^2 = u, v^2 = v, w^2 = w, uv = vu, uw = 0 = wv$ Note that $\mathscr{R}$ is a finite commutative non chain extension of $\mathbb{F}_q$ and isomorphic to the ring $\frac{\mathbb{F}_q[u,v,w]}{\langle u^2-u, v^2-v, w^2-w, uv-vu, uw, wv\rangle}$.

Also, $\mathscr{R}$ is local with unique maximal ideal $\langle u, v, w\rangle$ and quotient ring $\frac{\mathscr{R}}{\langle u,v,w\rangle}$ is isomorphic to $\mathbb{F}_q$. A non-empty subset $\mathscr{C}$ of $\mathscr{R}^n$ is said to be a linear code of length n if $\mathscr{C}$ is an $\mathscr{R}$-

submodule of $\mathscr{R}^n$. The elements of $\mathscr{C}$ are called codewords.

Since $\mathscr{R} = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + w\mathbb{F}_q + uv\mathbb{F}_q$ where $u^2 = u, v^2 = v, w^2 = w, uv = vu, uw = 0 = wv$. Any element in $\mathscr{R}$ can be represented uniquely as $r_1 + ur_2 + vr_3 + wr_4 + uvr_5$ where $r_i \in \mathbb{F}_q$. Let $\zeta_1 = (u-uv), \zeta_2 = (1-u-v-w+uv), \zeta_3 = (v-uv), \zeta_4 = uv$ and $\zeta_5 = w$ are elements in $\mathscr{R}$. Then it satisfies $(\zeta_i)^2 = \zeta_i$ for $1 \le i \le 5, \zeta_i\zeta_j = 0$ for $i \neq j$ and $\zeta_1+\zeta_2+\zeta_3+\zeta_4+\zeta_5 = 1$.

Therefore, the ring $\mathscr{R}$ can be decomposed to $\mathscr{R} = \zeta_1\mathscr{R} \oplus \zeta_2\mathscr{R} \oplus \zeta_3\mathscr{R} \oplus \zeta_4\mathscr{R} \oplus \zeta_5$. As, $\zeta_i\mathscr{R} \cong \zeta_i\mathbb{F}_q$, for $1 \le i \le 5$, then $\mathscr{R} \cong \zeta_1\mathbb{F}_q \oplus \zeta_2\mathbb{F}_q \oplus \zeta_3\mathbb{F}_q \oplus \zeta_4\mathbb{F}_q \oplus \zeta_5\mathbb{F}_q$. Thus we can represent the element of $\mathscr{R}$ as $a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3 + a_4\zeta_4 + a_5\zeta_5$ where $a_i \in \mathbb{F}_q$.

We recall the Frobenius automorphism Θ_t on $\mathbb{F}_q$ defined by $\Theta_t(a) = a^{p^t}$, where $t|m$. The extension of the automorphism on $\mathscr{R}$ is define as

$$\Theta_t(r) = \sum_{i=1}^{5} \zeta_i\Theta_t(r_i) = \sum_{i=1}^{5} \zeta_i r_i^{p^t},$$

where $r = \oplus_{i=1}^{5} e_i r_i$ and $r_i \in \mathbb{F}_q$ for $1 \le i \le 5$. The multiplication of skew polynomials is defined as $(ax^i)(bx^j) = a\Theta_t(b)^i x^{i+j}$. Therefore, the skew polynomial ring $\mathscr{R}[x;\Theta_t] = \{f(x) \in \mathscr{R}[x]\}$ is a non-commutative ring under the above multiplication and standard addition of polynomials. Whenever we considered Θ_t identity automorphism it becomes commutative ring. It is evident to see that the invariant ring under the automorphism is $\mathscr{R}_{inv} = \mathscr{R}/_{m=t}$ and hence $\mathscr{R}_{inv}[x;\Theta_t] = \mathscr{R}_{inv}[x]$ is a commutative ring. Further, the polynomial $x^n - 1$ is a central element in $\mathscr{R}[x;\Theta_t]$ if $nt \mid m$. In that case $\langle x^n - 1\rangle$ is a two sided ideal and hence, $\mathscr{R}_n = \mathscr{R}[x;\Theta_t]/\langle x^n - 1\rangle$ is a ring.

However, under the left multiplication define by $c(x)(a(x) + \langle x^n - 1\rangle) = c(x)a(x) + \langle x^n - 1\rangle$, where $c(x), b(x) \in \mathscr{R}[x;\Theta_t]$, $\mathscr{R}_n$ is a left $\mathscr{R}[x;\Theta_t]$-module.

A non-empty subset $\mathscr{C}$ of $\mathscr{R}^n$ is called a linear code of length n over $\mathscr{R}$ if $\mathscr{C}$ is an $\mathscr{R}$-submodule of $\mathscr{R}^n$ and members of $\mathscr{C}$ are codewords.

Definition 2.1. *[6] A non-trivial $\mathscr{R}$-submodule $\mathscr{C}$ of $\mathscr{R}^n$ is called a Θ_t-cyclic code if for any $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathscr{C}$, $\sigma(c) = (\Theta_t(c_{n-1}), \Theta_t(c_0), \ldots, \Theta_t(c_{n-2})) \in \mathscr{C}$. The operator σ is called as Θ_t-cyclic shift operator on $\mathscr{R}^n$.*

Theorem 2.2. *Let $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in R_n$ be the polynomial represenation of the codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathscr{C}$. Then the linear code $\mathscr{C}$ is a Θ_t-cyclic code iff $\mathscr{C}$ is a left $\mathscr{R}[x;\Theta_t]$-submodule of $\mathscr{R}_n$.*

Proof. Let $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ then the corresponding polynomial representation of c is $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$. Since $\mathscr{C}$ is a Θ_t-cyclic code, we have $\sigma(c) = (\Theta_t(c_{n-1}), \Theta_t(c_0), \Theta_t(c_1), \ldots, \Theta_t(c_{n-2})) \in \mathscr{C}$ then the polynomial representation can be seen as follows $x(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) = \Theta_t(c_0)x + \Theta_t(c_1)x^2 + \cdots + \Theta_t(c_{n-2})x^{n-1} + \Theta_t(c_{n-1})x^n$
$= \Theta_t(c_{n-1}) + \Theta_t(c_0)x + \Theta_t(c_1)x^2 + \cdots + \Theta_t(c_{n-2})x^{n-1} \in \mathscr{C}$. Implies that $xc(x) \in \mathscr{C}$ then by induction on $i \ge 1$, we have $x^ic(x) \in \mathscr{C}$. Using linearity in $\mathscr{C}$, we have $r(x)c(x) \in C$ for $r(x) \in \mathscr{R}[x,\Theta_t]$. Therefore, $\mathscr{C}$ is a left $\mathscr{R}[x;\Theta_t]$-submodule of R_n.
On the other hand, let $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathscr{C}$ whose corresponding polynomial representation is $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$. Let $\mathscr{C}$ be a left $\mathscr{R}[x;\Theta_t]$-submodule of R_n. Then $\sigma(c) = xc(x) \in \mathscr{C}$, represents the Θ_t-cyclic shift of $c(x)$. Therefore, $\mathscr{C}$ is a Θ_t-cyclic code of length n over $\mathscr{R}$. □

We define a new gray map from $\mathscr{R}$ to $\mathbb{F}_q$,

$$\begin{aligned} \delta : \mathscr{R} &\mapsto \mathbb{F}_q^5 \\ \delta(a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3 + a_4\zeta_4 + a_5\zeta_5) &= (a_1, a_2, a_3, a_4, a_5) \end{aligned}$$

and it can be extended to n length by $\delta((a_{1,0}, a_{1,1}, \cdots, a_{1,n-1})\zeta_1 + (a_{2,0}, a_{2,1}, \cdots, a_{2,n-1})\zeta_2 + \cdots + (a_{5,0}, a_{5,1}, \cdots, a_{5,n-1})\zeta_5) = (a_{1,0}, a_{1,1}, \cdots, a_{1,n-1}, a_{2,0}, a_{2,1}, \cdots, a_{5,n-1})$

The Hamming weight of $a \in \mathbb{F}_q^n$ is defined as number of non zero entries in a and is denoted as $w_H(a)$ For any element $r = (a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3 + a_4\zeta_4 + a_5\zeta_5) \in \mathscr{R}$ we define the Gray weight of a code as $w_G(r) = w_H(\delta(r))$. Then Gray distance of code $\mathscr{C}$ is $d_G(C) = min(w_G(c_i - c_j))$ where $c_i, c_j \in \mathscr{C}$.

Theorem 2.3. *The gray map δ is a $\mathbb{F}_q$-linear map and distance preserving from $\mathscr{R}^n$(Gray Distance) to $\mathbb{F}_q^{5n}$(Hamming distance).*

Proof. Let $a = a_0, a_1, \cdots, a_{n-1}$ and $b = b_0, b_1, \cdots, b_{n-1}$ be an element in $\mathscr{R}^n$ where $a_j = \sum_{i=1}^{5} a_{i,j}\zeta_i, b_j = \sum_{i=1}^{5} b_{i,j}\zeta_i$ for $0 \le j \le n-1$ then for any $m_1, m_2 \in \mathbb{F}_q$ we have

$$\begin{aligned} \delta(m_1a + m_2b) &= \delta(m_1(\sum_{i=1}^{5} a_{i,0}\zeta_i, \sum_{i=1}^{5} a_{i,1}\zeta_i, \cdots, \sum_{i=1}^{5} a_{i,n-1}\zeta_i) \\ &\quad + m_2(\sum_{i=1}^{5} b_{i,0}\zeta_i, \sum_{i=1}^{5} b_{i,1}\zeta_i, \cdots, \sum_{i=1}^{5} b_{i,n-1}\zeta_i)) \end{aligned}$$

$= m_1(a_{1,0}, a_{1,1}, \cdots, a_{1,n-1}, a_{2,0}, a_{2,1}, \cdots, a_{2,n-1}, a_{5,0}, a_{5,1}, \cdots, a_{5,n-1}) + m_2(b_{1,0}, b_{1,1}, \cdots, b_{1,n-1}, b_{2,0}, b_{2,1}, \cdots, b_{2,n-1}, b_{5,0}, b_{5,1}, \cdots, b_{5,n-1}) = m_1\delta(a) + m_2\delta(b)$ Which implies δ is a $\mathbb{F}_q$-linear map. Now using linear property we have $d_G(a,b) = w_G(a-b) = w_H(\delta(a) - \delta(b)) = d_H(\delta(a), \delta(b))$. Thus δ is a weight preserving map. □

Theorem 2.4. *Let $\mathscr{C}$ be a linear code of length n over $\mathscr{R}$. Then $\delta(\mathscr{C}^\perp) = (\delta(\mathscr{C}))^\perp$.*

Proof. Let $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathscr{C}$ and $d = (d_0, d_1, \ldots, d_{n-1}) \in \mathscr{C}^\perp$ where $c_j = \sum_{i=1}^{6} e_i s_j^i$ and $d_j = \sum_{i=1}^{6} e_i t_j^i$ for $s_j^i, t_j^i \in \mathbb{F}_q$, $0 \le j \le n-1$. Now, we have $c \cdot d = 0$ implies $\sum_{j=0}^{n-1} c_j d_j = 0$, i.e., $\sum_{j=0}^{n-1}\sum_{i=1}^{6} s_j^i t_j^i = 0$. Again, we have

$$\delta(c) \cdot \delta(d) = \sum_{j=0}^{n-1}\sum_{i=1}^{6} s_j^i t_j^i = 0.$$

Therefore, $\delta(\mathscr{C}^\perp) \subseteq (\delta(\mathscr{C}))^\perp$. As δ is bijective linear map, $\mid \delta(\mathscr{C}^\perp) \mid = \mid (\delta(\mathscr{C}))^\perp \mid$. Therefore $\delta(\mathscr{C}^\perp) = (\delta(\mathscr{C}))^\perp$. □

Corollary 2.5. *Let $\mathscr{C}$ be a linear code of length n over $\mathscr{R}$. Then $\mathscr{C}$ is self-dual if and only if $\delta(C)$ is self-dual. Further $\delta(\mathscr{C})$ is a self-orthogonal over $\mathbb{F}_q$ if $\mathscr{C}$ is self-orthogonal.*

Proof. Let $\mathscr{C}$ be a self-dual linear code of length n over R. That is $\mathscr{C} = \mathscr{C}^{\perp}$. Since δ is a bijective map we hve $\delta(\mathscr{C}) = \delta(\mathscr{C}^{\perp})$, and hence by Theorem 2.4, we have $\delta(\mathscr{C}) = (\delta(\mathscr{C}))^{\perp}$. Thus $\delta(\mathscr{C})$ is a self-dual linear code of length $5n$ over $\mathbb{F}_q$. Conversely, let $\delta(\mathscr{C})$ be a self-dual linear code of length $5n$ over $\mathbb{F}_q$ that is $\delta(\mathscr{C}) = (\delta(\mathscr{C}))^{\perp}$. Since δ is onto the inverse image is unique, thus $\mathscr{C} = \mathscr{C}^{\perp}$. Therefore, $\mathscr{C}$ is a self-dual linear code of length n over R.
Moreover, let $\mathscr{C} \subseteq \mathscr{C}^{\perp}$ by using Theorem 2.4 and above we have $\delta(\mathscr{C}) \subseteq (\delta(\mathscr{C}))^{\perp}$. Hence $\delta(\mathscr{C})$ is self-orthogonal linear code of length $5n$ over $\mathbb{F}_q$. □

Let $A_i \subseteq \mathbb{F}_q$, then we define the following
$A_1 \oplus A_2 = \{a_1 + a_2 \mid a_i \in A_i\}$ and $A_1 \otimes A_2 = \{(a_1, a_2) \mid a_i \in A_i\}$ where $A_i \subseteq \mathscr{R}$
Define $\mathscr{C}_i \subseteq \mathbb{F}_q$ as follows,
$\mathscr{C}_1 = \{a_1 \in \mathbb{F}_q^n \mid \sum_{i=1}^{5} a_i\zeta_i \in \mathscr{C},$ for some $a_j \in \mathbb{F}_q^n (j \neq 1)\}$
$\mathscr{C}_2 = \{a_2 \in \mathbb{F}_q^n \mid \sum_{i=1}^{5} a_i\zeta_i \in \mathscr{C},$ for some $a_j \in \mathbb{F}_q^n (j \neq 2)\}$
$\mathscr{C}_3 = \{a_3 \in \mathbb{F}_q^n \mid \sum_{i=1}^{5} a_i\zeta_i \in \mathscr{C},$ for some $a_j \in \mathbb{F}_q^n (j \neq 3)\}$
$\mathscr{C}_4 = \{a_4 \in \mathbb{F}_q^n \mid \sum_{i=1}^{5} a_i\zeta_i \in \mathscr{C},$ for some $a_j \in \mathbb{F}_q^n (j \neq 4)\}$
$\mathscr{C}_5 = \{a_5 \in \mathbb{F}_q^n \mid \sum_{i=1}^{5} a_i\zeta_i \in \mathscr{C},$ for some $a_j \in \mathbb{F}_q^n (j \neq 5)\}$
Then cyclic code $\mathscr{C}$ over $\mathscr{R}^n$ can be represented as $\mathscr{C} = \zeta_1\mathscr{C}_1 \oplus \zeta_2\mathscr{C}_2 \oplus \zeta_3\mathscr{C}_3 \oplus \zeta_4\mathscr{C}_4 \oplus \zeta_5\mathscr{C}_5$.

Theorem 2.6. *Let $\mathscr{C}$ be a linear code of length n over $\mathscr{R}$. Then $\delta(\mathscr{C}) = \otimes_{i=1}^{5}\mathscr{C}_i$ and $\mid \mathscr{C} \mid = \prod_{i=1}^{6} \mid \mathscr{C}_i \mid$.*

Proof. Let $z = (a_{0,1}, a_{1,1}, \ldots, a_{n-1,1}, a_{0,2}, a_{1,2}, \ldots, a_{n-1,2}, \cdots, a_{0,5}, a_{1,5}, \ldots, a_{n-1,5}) \in \delta(\mathscr{C})$ and $r_i = \sum_{j=1}^{5} \zeta_j a_i^j$, for $0 \leq i \leq n-1$. The map δ being bijective, so $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathscr{C}$. Therefore, by the definition of $\mathscr{C}_i$, we have $(a_0^i, a_1^i, \ldots, a_{n-1}^i) \in \mathscr{C}_i$ for $1 \leq i \leq 5$. Therefore, $z \in \otimes_{i=1}^{5}\mathscr{C}_i$ and hence $\delta(\mathscr{C}) \subseteq \otimes_{i=1}^{5}\mathscr{C}_i$.
Conversely, let $z = (a_{0,1}, a_{1,1}, \ldots, a_{n-1,1}, a_{0,2}, a_{1,2}, \ldots, a_{n-1,2}, \cdots, a_{0,5}, a_{1,5}, \ldots, a_{n-1,5}) \in \otimes_{i=1}^{5}\mathscr{C}_i$. Then $a^i = (a_{0,i}, a_{1,i}, \ldots, a_{n-1,i}) \in \mathscr{C}_i$ for $1 \leq i \leq 5$. To show $z \in \delta(\mathscr{C})$, we have to find $z' = \sum_{i=1}^{5} e_i s_i \in \mathscr{C}$ such that $\delta(z') = z$. Consider $s_j = \sum_{i=1}^{5} e_i t_{i,j}$ where $t_{i,i} = a^i$ for $1 \leq i \leq 6$. Thus $z' = \sum_{i=1}^{5} e_i a^i$ and $\delta(z') = z$. Consequently, $\bigotimes_{i=1}^{5}\mathscr{C}_i \subseteq \delta(\mathscr{C})$. Combining both sides, we have $\delta(\mathscr{C}) = \bigotimes_{i=1}^{5}\mathscr{C}_i$.
Moreover, δ being bijection, $\mid \mathscr{C} \mid = \mid \delta(\mathscr{C}) \mid$. Consequently, $\mid \mathscr{C} \mid = \mid \bigotimes_{i=1}^{5}\mathscr{C}_i \mid = \prod_{i=1}^{5} \mid \mathscr{C}_i \mid$. □

Corollary 2.7. *Let $\mathscr{C} = \oplus_{i=1}^{5} e_i\mathscr{C}_i$ be a linear code of length n over $\mathscr{R}$. Then $\delta(\mathscr{C})$ is a $[6n, \sum_{i=1}^{6} k_i, d_H(\mathscr{C})]$, where $\mathscr{C}_i$ is a $[n, k_i, d_H(\mathscr{C}_i)]$ linear code over $\mathbb{F}_q$ for $1 \leq i \leq 5$ and $d_H(\mathscr{C}) = min\{d_H(\mathscr{C}_i) \mid i = 1, 2, \ldots, 5\}$.*

Theorem 2.8. *Let $\mathscr{C} = \oplus_{i=1}^{6} e_i\mathscr{C}_i$ be a linear code of length n over $\mathscr{R}$. Then $\mathscr{C}^{\perp} = \oplus_{i=1}^{5} e_i\mathscr{C}_i^{\perp}$. Moreover, $\mathscr{C}$ is self-dual code if and only if $\mathscr{C}_i's$ $(i = 1, 2, \ldots, 5)$ are self-dual codes over $\mathbb{F}_q$.*

Proof. Let $\overline{\mathscr{C}}_i = \{r_i \in \mathbb{F}_q^n \mid$ there exists $r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_5 \in \mathbb{F}_q^n$ such that $\sum_{i=1}^{6} e_i r_i \in \mathscr{C}^{\perp}\}$. Then $\mathscr{C}^{\perp}$ has the unique expression $C^{\perp} = \oplus_{i=1}^{5} e_i\overline{\mathscr{C}}_i$. It is easy to see $\overline{\mathscr{C}}_1 \subseteq \mathscr{C}_1^{\perp}$. If $z \in \mathscr{C}_1^{\perp}$, then $z \cdot x_1 = 0$ for all $x_1 \in \mathscr{C}_1$. Let $s = \sum_{i=1}^{5} e_i x_i \in \mathscr{C}$. Then $e_1 z s = e_1 x_1 z = 0$, and which implies $e_1 z \in \mathscr{C}^{\perp}$. From the construction of $\mathscr{C}^{\perp}$, we have $z \in \overline{\mathscr{C}}_1$. Therefore, $\mathscr{C}_1^{\perp} \subseteq \overline{\mathscr{C}}_1$. Hence, $\overline{\mathscr{C}}_1 = \mathscr{C}_1^{\perp}$. By similar process we have $\mathscr{C}_i^{\perp} = \overline{\mathscr{C}}_i$ for $i = 2, 3, 4, 5$. Consequently, $\mathscr{C}^{\perp} = \oplus_{i=1}^{5} e_i\mathscr{C}_i^{\perp}$.
Moreover, let $\mathscr{C}$ be a self-dual linear code. Then $\mathscr{C} = \mathscr{C}^{\perp}$, i.e., $\oplus_{i=1}^{5} e_i\mathscr{C}_i = \oplus_{i=1}^{5} e_i\mathscr{C}_i^{\perp}$, Hence $\mathscr{C}_i^{\perp} = \mathscr{C}_i$ for $1 \leq i \leq 5$. Conversely, let $\mathscr{C}_i's$ $(i = 1, 2 \ldots, 5)$ be self-dual linear codes. Then $\mathscr{C}_i^{\perp} = \mathscr{C}_i$ for $1 \leq i \leq 5$. Thus, $\mathscr{C}^{\perp} = \oplus_{i=1}^{5} e_i\mathscr{C}_i^{\perp} = \oplus_{i=1}^{5} e_i\mathscr{C}_i = \mathscr{C}$. Hence $\mathscr{C}$ is a self-dual linear code over $\mathscr{R}$. □

Definition 2.9. *A linear code $\mathscr{C}$ of length nm over $\mathbb{F}_q$ is said to be a Θ_t-quasi-cyclic code of index m if $\rho_m(\mathscr{C}) = \mathscr{C}$, where ρ_m is the Θ_t-quasi-cyclic shift on $(\mathbb{F}_q^n)^m$ define by*

$$\rho_m(a^1 \mid a^2 \mid \cdots \mid a^m) = (\sigma(a^1) \mid \sigma(a^2) \mid \cdots \mid \sigma(a^m)), \tag{2.1}$$

and σ is the Θ_t-cyclic shift operator.

Lemma 2.10. *Let σ be the Θ_t-cyclic shift and ρ_5 be the Θ_t-quasi-cyclic shift defined in equation (2.1) and δ be the Gray map from $\mathscr{R}^n$ to $\mathbb{F}_q^{5n}$ defined in equation. Then $\delta\sigma = \rho_5\delta$.*

Proof. Let $r_j = \sum_{i=1}^{5} e_i a_{j,i} \in R$ for $0 \leq j \leq n-1$ where $a_j^i \in \mathbb{F}_q$ for $1 \leq i \leq 5$. Then $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$. Now,

$$\begin{aligned}\delta\sigma(r) &= \delta(\Theta_t(r_{n-1}), \Theta_t(r_0), \ldots, \Theta_t(r_{n-2})) \\ &= (\Theta_t(a_{n-1,1}), \Theta_t(a_{0,1}), \ldots, \Theta_t(a_{n-2,1}), \ldots, \\ &\quad \Theta_t(a_{n-1,5}), \Theta_t(a_{0,5}), \ldots, \Theta_t(a_{n-2,5})).\end{aligned}$$

On the other hand,

$$\begin{aligned}\rho_5\delta(r) &= \rho_5(a_{n-1,1}, a_{0,1}, \ldots, a_{n-2,1}, \ldots, a_{n-1,5}, \\ &\quad a_{0,5}, \ldots, a_{n-2,5}) \\ &= (\Theta_t(a_{n-1,1}), \Theta_t(a_{0,1}), \ldots, \Theta_t(a_{n-2,1}), \ldots, \\ &\quad \Theta_t(a_{n-1,5}), \Theta_t(a_{0,5}), \ldots, \Theta_t(a_{n-2,5})).\end{aligned}$$

Therefore, $\delta\sigma = \rho_5\delta$. □

Theorem 2.11. *The code $\mathscr{C} = \zeta_1\mathscr{C}_1 \oplus \zeta_2\mathscr{C}_2 \oplus \zeta_3\mathscr{C}_3 \oplus \zeta_4\mathscr{C}_4 \oplus \zeta_5\mathscr{C}_5$ over $\mathscr{R}$ is cyclic code of length n iff $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4$ and $\mathscr{C}_5$ are cyclic codes over $\mathbb{F}_q$ of length n.*

Proof. Assume that $\mathscr{C}$ is a cyclic code over $\mathscr{R}$. Let $l_i = (l_{0,i}, l_{1,i}, \cdots, l_{n-1,i}) \in \mathscr{C}_i$ for $1 \leq i \leq 5$ where $l_{i,j} \in \mathbb{F}_q$. Then $m_i = \zeta_1 l_{i,1} + \zeta_2 l_{i,2} + \zeta_3 l_{i,3} + \zeta_4 l_{i,4} + \zeta_5 l_{i,5} \in \mathscr{R}$ and hence $(m_1, m_2, \cdots, m_n) = l_1\zeta_1 + l_2\zeta_2 + l_3\zeta_3 + l_4\zeta_4 + l_5\zeta_5 \in \mathscr{C}$ is cyclic, $\varphi_1(m_1, m_2, \cdots, m_n) = \zeta_1\varphi_1(l_1) + \zeta_2\varphi_1(l_2) + \zeta_3\varphi_1(l_3) + u_4\varphi_1(l_4) + \zeta_5\varphi_1(l_5) \in \mathscr{C}$. Therefore, $\varphi_1(l_1) \in \mathscr{C}_1, \varphi_1(l_2) \in \mathscr{C}_2, \varphi_1(l_3) \in \mathscr{C}_3, \varphi_1(l_4) \in \mathscr{C}_4, \varphi_1(l_5) \in \mathscr{C}_5$. Hence $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5$ are cyclic codes.

Conversely, we assume that $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5$ are cyclic codes. Let $(m_1, m_2, \cdots, m_n) \in \mathscr{C}$ where $m_i = u_1 l_{i,1} + u_2 l_{i,2} +$

$u_3 l_{i,3} + u_4 l_{i,4} + u_5 l_{i,5}$. Let $l_i = (l_{1_i}, l_{2_i}, \cdots, l_{n-1_i}) \in \mathscr{C}_i$. Since $\mathscr{C}_i$ is cyclic, $\varphi_1(l_i) \in \mathscr{C}_i$. Therefore, $\zeta_1\varphi_1(l_1) + \zeta_2\varphi_1(l_2) + \zeta_3\varphi_1(l_3) + \zeta_4\varphi_1(l_4) + \zeta_5\varphi_1(l_5) \in \mathscr{C}$. That is, $\varphi_1(m_1, m_2, \cdots, m_n) \in \mathscr{C}$. Hence, $\mathscr{C}$ is cyclic. □

Theorem 2.12. *Let $\mathscr{C}$ be a cyclic code over $\mathscr{R}$ then it is generated by a unique polynomial $\langle f(x)\rangle = \langle \zeta_1 f_1(x), \zeta_2 f_2(x), \zeta_3 f_3(x), \zeta_4 f_4(x), \zeta_5 f_6(x)\rangle$ and divides $x^n - 1$.*

Proof. Assume that $\mathscr{C}$ is a cyclic code then Theorem 2.11 implies that each $\mathscr{C}_i$ is cyclic code over $\mathbb{F}_q$ and each $\mathscr{C}_i$ is generated by a polynomial $f_i(x)$ then $\langle \zeta_i f_i(x)\rangle \subseteq \langle f(x)\rangle$ for each i. An element in $\langle f(x)\rangle$ is of the form $\sum_{i=1}^{5} \zeta_i f_i(x) g_i(x)$ then multiply by ζ_i we get $\zeta_i f_i(x) g_i(x)$ which is an element in $\zeta_i f_i(x)$ which implies that $\langle f(x)\rangle \subseteq \langle \zeta_1 f_1(x), \zeta_2 f_2(x), \zeta_3 f_3(x), \zeta_4 f_4(x), \zeta_5 f_6(x)\rangle$. Hence L.H.S=R.H.S

Let $f(x) = \sum_{i=1}^{5} \zeta_i f_i(x)$. There exists a unique polynomial $h_i(x)$ in $\mathbb{F}_q[x]$ such that $f_i(x)h_i(x) = x^n - 1$ then multiply both sides by ζ_i Further $\sum_{i=1}^{5} \zeta_i f_i(x) h_i(x) = x^n - 1 = (\sum_{i=1}^{5} \zeta_i h_i(x)) f(x)$. Hence $f(x) | (x^n - 1)$ □

3. Structure of Θ_t-cyclic codes

The Θ_t-cyclic codes of length n over finite field are determined by Siap et al. [21]. Using the structure on finite field, here we obtain their properties on the ring $\mathscr{R}$.

Lemma 3.1. *[21] Let $\mathscr{C}$ be a Θ_t-cyclic code of length n over $\mathbb{F}_q$. Then there exists a polynomial $f(x) \in \mathbb{F}_q[x;\Theta_t]$ such that $\mathscr{C} = \langle f(x)\rangle$ and $x^n - 1 = g(x)f(x)$ in $\mathbb{F}_q[x;\Theta_t]$.*

We recall from [6] that for a Θ_t-cyclic code $\mathscr{C} = \langle f(x)\rangle$ of length n over $\mathbb{F}_q$ such that $x^n - 1 = g(x)f(x)$, its dual $\mathscr{C}^\perp = \langle g^*(x)\rangle$ is also a Θ_t-cyclic code where $g^*(x)$(called it Θ_t-reciprocal polynomial) is given by $g^*(x) = g_{n-r} + \Theta_t(g_{n-r-1})x + \cdots + \Theta_t^{n-r-1}(g_1)x^{n-r-1} + \Theta_t^{n-r}(g_0)x^{n-r}$, for the polynomial $g(x) = g_0 + g_1 x + \cdots + g_{n-r}x^{n-r}$.

Theorem 3.2. *Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a linear code of length n over $\mathscr{R}$. Then $\mathscr{C}$ is a Θ_t-cyclic code if and only if $\mathscr{C}_i' s$ $(i = 1, 2, \ldots, 5)$ are Θ_t-cyclic codes of length n over $\mathbb{F}_q$.*

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$. Let

$$z^i = (z_{0,i}, z_{1,i}, \ldots, z_{n-1,i}) \in \mathscr{C}_i \text{ for } 1 \le i \le 5, \text{ and}$$

$$y_j = \sum_{i=1}^{5} \zeta_i z_{j,i} \text{ for } 0 \le j \le n-1.$$

Then $y = (y_0, y_1, \ldots, y_{n-1}) \in \mathscr{C}$ and hence $\sigma(y) = (\Theta_t(y_{n-1}), \Theta_t(y_0), \ldots, \Theta_t(y_{n-2})) \in \mathscr{C}$. Now,

$$\sigma(y) = \sum_{i=1}^{5} \zeta_i \sigma(z^i) \in C = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i.$$

Therefore, $\sigma(z^i) \in \mathscr{C}_i$ for $1 \le i \le 5$. Hence $\mathscr{C}_i' s$ $(i = 1, 2, \ldots, 5)$ are Θ_t-cyclic code of length n over $\mathbb{F}_q$.

For the converse part, let $\mathscr{C}_i' s$ $(i = 1, 2, \ldots, 5)$ be Θ_t-cyclic codes of length n over $\mathbb{F}_q$. Let $y = (y_0, y_1, \ldots, y_{n-1}) \in \mathscr{C}$ where $y_j = \sum_{i=1}^{5} \zeta_i z_{j,i}$ for $0 \le j \le n-1$. Then $z^i = (z_{0,i}, z_{1,i}, \ldots, z_{n-1,i}) \in \mathscr{C}_i$ for $1 \le i \le 5$ and hence $\sigma(z^i) \in \mathscr{C}_i$ for $1 \le i \le 5$. Again $\sigma(y) = \sum_{i=1}^{5} \zeta_i \sigma(z^i) \in \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i = \mathscr{C}$. Consequently, $\mathscr{C}$ is a Θ_t-cyclic code of length n over $\mathscr{R}$. □

Theorem 3.3. *Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$. Then*

$$\mathscr{C} = \langle \zeta_1 f_1(x), \zeta_2 f_2(x), \cdots, \zeta_5 f_5(x)\rangle$$

and $|\mathscr{C}| = q^{5n - \sum_{i=1}^{5} \varepsilon_i}$, where $\mathscr{C}_i = \langle f_i(x)\rangle$ and $x^n - 1 = g_i(x)f_i(x)$ in $\mathbb{F}_q[x;\Theta_t]$ and $deg(f_i(x)) = \varepsilon_i$, for $1 \le i \le 5$.

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$. By Thereom 3.2, $\mathscr{C}_i' s$ $(i = 1, 2, \ldots, 5)$ are Θ_t-cyclic codes of length n over $\mathbb{F}_q$. Now, by Lemma 3.1, we have $\mathscr{C}_i = \langle f_i(x)\rangle$ and $x^n - 1 = g_i(x)f_i(x)$ in $\mathbb{F}_q[x;\Theta_t]$ for $1 \le i \le 5$. Then $\zeta_i f_i(x) \in \mathscr{C}$ for $1 \le i \le 5$. Since for any $f(x) \in \mathscr{C}$, we have $f(x) = \sum_{i=1}^{5} \zeta_i h_i(x) f_i(x)$ where $h_i(x) \in \mathbb{F}_q[x;\Theta_t]$ for $1 \le i \le 5$. Thus $f(x) \in \langle \zeta_1 f_1(x), \zeta_2 f_2(x), \cdots, \zeta_5 f_5(x)\rangle$. Therefore, $\mathscr{C} = \langle \zeta_1 f_1(x), \zeta_2 f_2(x), \cdots, \zeta_5 f_5(x)\rangle$.

Further, we have $|\mathscr{C}_i| = q^{n - \varepsilon_i}$ and $|\mathscr{C}| = \prod_{i=1}^{5} |\mathscr{C}_i| = q^{5n - \sum_{i=1}^{5} \varepsilon_i}$, where $deg(f_i(x)) = \varepsilon_i$, for $1 \le i \le 5$. □

Corollary 3.4. *If $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ is a Θ_t-cyclic code of length n over $\mathscr{R}$, there exists a polynomial $f(x) \in R[x;\Theta_t]$ such that $\mathscr{C} = \langle f(x)\rangle$ and $x^n - 1 = g(x)f(x)$ in $\mathscr{R}[x;\Theta_t]$.*

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$. Then by Theorem 3.3, we have $\mathscr{C} = \langle \zeta_1 f_1(x), \zeta_2 f_2(x), \cdots, \zeta_5 f_5(x)\rangle$ where $\mathscr{C}_i = \langle f_i(x)\rangle$ and $x^n - 1 = g_i(x)f_i(x)$ in $\mathbb{F}_q[x;\Theta_t]$ for $1 \le i \le 5$. Let $f(x) = \sum_{i=1}^{5} \zeta_i f_i(x) \in \mathscr{R}[x;\Theta_t]$. Then $f(x) \in \mathscr{C}$. On the other hand $\zeta_i f_i(x) = \zeta_i f(x) \in \langle f(x)\rangle$ for $i = 1, 2, \ldots, 5$. Consequently, $\mathscr{C} = \langle f(x)\rangle$. Further, $[\sum_{i=1}^{5} \zeta_i g_i(x)] f(x) = \sum_{i=1}^{5} \zeta_i g_i(x) f_i(x) = \sum_{i=1}^{5} \zeta_i (x^n - 1) = x^n - 1$. Then $x^n - 1 = g(x)f(x)$ in $\mathscr{R}[x;\Theta_t]$, where $g(x) = \sum_{i=1}^{5} \zeta_i g_i(x)$. □

Theorem 3.5. *Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$. Then $\mathscr{C}^\perp = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i^\perp$ is also a Θ_t-cyclic code of length n over $\mathscr{R}$ and $|\mathscr{C}^\perp| = q^{\sum_{i=1}^{5} \varepsilon_i}$ where $\mathscr{C}_i = \langle f_i(x)\rangle$ and $deg(f_i(x)) = \varepsilon_i$ for $1 \le i \le 5$.*

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$. Then by Theorem 3.2, $\mathscr{C}_i' s$ $(i = 1, 2, \ldots, 5)$ are Θ_t-cyclic codes of length n over $\mathbb{F}_q$. Then $\mathscr{C}_i^\perp$'s $(i = 1, 2, \ldots, 5)$ are also Θ_t-cyclic codes of length n over $\mathbb{F}_q$. Again by Theorem 3.2, $\mathscr{C}^\perp = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i^\perp$ is also a Θ_t-cyclic code of length n over $\mathscr{R}$. Further, as $|\mathscr{C}_i^\perp| = q^{\varepsilon_i}$ for $i = 1, 2, \ldots, 5$, therefore $|\mathscr{C}^\perp| = \prod_{i=1}^{5} |\mathscr{C}_i^\perp| = q^{\sum_{i=1}^{5} \varepsilon_i}$. where $deg(f_i(x)) = \varepsilon_i$ for $1 \le i \le 5$. □

Corollary 3.6. *Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$ where $\mathscr{C}_i = \langle f_i(x)\rangle$ and $x^n - 1 = g_i(x)f_i(x)$ for $1 \le i \le 5$. Then there exists a polynomial $G(x)$ such that $\mathscr{C}^\perp = \langle G(x)\rangle$, where $G(x) = \sum_{i=1}^{5} \zeta_i g_i^*(x)$.*

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$ be a Θ_t-cyclic code of length n over $\mathscr{R}$ and $\mathscr{C}_i = \langle f_i(x)\rangle$ where $x^n - 1 = g_i(x)f_i(x)$ for $i = 1,2\ldots,5$. Then by Theorem 3.5, we have $\mathscr{C}^{\perp} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i^{\perp}$ is a Θ_t-cyclic code of length n over $\mathscr{R}$ where $\mathscr{C}_i^{\perp}$'s $(i = 1,2,\ldots,5)$ are Θ_t-cyclic codes over $\mathbb{F}_q$. Therefore, $\mathscr{C}_i^{\perp} = \langle g^*(x)\rangle$ where $g_i^*(x)$ is the Θ_t-reciprocal polynomial of $g_i(x)$ for $1 \le i \le 5$. Take $G(x) = \sum_{i=1}^{5}\zeta_i g_i^*(x)$, then we checked that $\mathscr{C}^{\perp} = \langle G(x)\rangle$. □

4. Structure of (Θ_t, λ)-cyclic codes

In the present section, we extend our study from Θ_t-cyclic to (Θ_t, λ)-cyclic codes of length n over R. The complete structure of these codes are obtained by decomposition method.

Definition 4.1. *[17] Let $\mathscr{C}$ be a linear code of length n over $\mathscr{R}$ and $\lambda \in R_{inv}$ be a unit in $\mathscr{R}$. Then $\mathscr{C}$ is said to be a (Θ_t, λ)-cyclic code if for $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathscr{C}$, we have $\sigma_\lambda(c) = (\lambda\Theta_t(c_{n-1}), \Theta_t(c_0), \ldots, \Theta_t(c_{n-2})) \in \mathscr{C}$. Note that for $\lambda = 1$, $\mathscr{C}$ is a Θ_t-cyclic code and σ_λ is the (Θ_t, λ)-cyclic shift operator.*

Theorem 4.2. *Let $\mathscr{R}_{n,\lambda} = \mathscr{R}[x;\Theta_t]/\langle x^n - \lambda\rangle$. A linear code $\mathscr{C}$ of length n over $\mathscr{R}$ is (Θ_t, λ)-cyclic code if and only if $\mathscr{C}$ is a left $\mathscr{R}[x;\Theta_t]$-submodule of $\mathscr{R}_{n,\lambda}$.*

Proof. Same as the proof of Theorem 2.2. □

Theorem 4.3. *Let n be an integer such that $\lambda^{n+1} = 1$. Then the map $\Gamma : \mathscr{R}_n \longrightarrow \mathscr{R}_{n,\lambda}$ define by $\Gamma(a(x)) = a(\lambda x)$, is a left $\mathscr{R}[x;\Theta_t]$-module isomorphism.*

Proof. Let $a(x), b(x) \in \mathscr{R}_n$ such that $a(x) = b(x)$. Thus $a(x) - b(x) \equiv 0\ (mod\ x^n - 1)$. Now replacing both side x by λx, we have $a(\lambda x) - b(\lambda x) \equiv 0\ (mod\ x^n\lambda^n - 1)$, i.e., $a(\lambda x) - b(\lambda x) \equiv 0,\ \lambda^n(mod\ x^n - \lambda)$. Thus $a(\lambda x) = b(\lambda x)$ in $\mathscr{R}_{n,\lambda}$. Therefore $\Gamma(a(x)) = \Gamma(b(x))$. Consequently, Γ is an injective and well-defined map. Further, Γ is surjection and left $\mathscr{R}[x;\Theta_t]$-module homomorphism. Hence the results. □

Corollary 4.4. *Let n be an integer such that $\lambda^{n+1} = 1$. If $\mathscr{C}$ is a Θ_t-cyclic code of length n over R, then $\Gamma(\mathscr{C})$ is a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$.*

Proof. Let $\mathscr{C}$ be a Θ_t-cyclic code of length n over $\mathscr{R}$. In other words, $\mathscr{C}$ is a left $\mathscr{R}[x;\Theta_t]$-submodule of $\mathscr{R}_n$. By Theorem 4.3, $\Gamma(\mathscr{C})$ is a left $\mathscr{R}[x;\Theta_t]$-submodule of $\mathscr{R}_{n,\lambda}$. Thus $\Gamma(\mathscr{C})$ is a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. □

Lemma 4.5. *[11] Let $\mathscr{C}$ be a (Θ_t, α)-cyclic code of length n over $\mathbb{F}_q$. Then there exists a polynomial $f(x) \in \mathbb{F}_q[x;\Theta_t]$ such that $\mathscr{C} = \langle f(x)\rangle$ and $x^n - \alpha = g(x)f(x)$ in $\mathbb{F}_q[x;\Theta_t]$.*

Lemma 4.6. *Let $\lambda \in \mathscr{R}$ be a non-zero element such that $\lambda = \sum_{i=1}^{5}\zeta_i\lambda_i$, where $\lambda_i \in \mathbb{F}_q$ for $1 \le i \le 5$. Then λ is a unit in $\mathscr{R}$ if and only if $\lambda_i's$ $(i = 1,2,\ldots,5)$ are units in $\mathbb{F}_q$.*

Proof. Let $\lambda = \sum_{i=1}^{5}\zeta_i\lambda_i$ be a unit in $\mathscr{R}$ where $\lambda_i \in \mathbb{F}_q$ for $1 \le i \le 5$. Then there exists a unit $\lambda' = \sum_{i=1}^{5}\zeta_i\lambda_i'$ where $\lambda_i' \in \mathbb{F}_q^*$ for $1 \le i \le 5$. Now $\lambda\lambda' = 1$ implies $\sum_{i=1}^{5}\zeta_i\lambda\lambda' = 1$, i.e., $\zeta_i\lambda\lambda' = \zeta_i$, and hence $\lambda\lambda' = 1$ for $1 \le i \le 5$. Therefore, $\lambda_i's$ $(i = 1,2,\ldots,5)$ are units in $\mathbb{F}_q$.
For converse part, let $\lambda_i's$ $(i = 1,2,\ldots,5)$ be units in $\mathbb{F}_q$. Then $\lambda\lambda' = 1$ where $\lambda' = \sum_{i=1}^{5}\zeta_i\lambda_i^{-1} \in \mathscr{R}$. Therefore, λ is a unit in $\mathscr{R}$. □

Here, we study the (Θ_t, λ)-cyclic codes of length n over $\mathscr{R}$ where $\lambda = a_1 + a_2u + a_3v + a_4w + a_5uw + a_6vw \in R$ is a unit and $a_i \in \mathbb{F}_q$ for $1 \le i \le 6$. By calculation the units, $\lambda_i's$ are given by

$$\begin{aligned} &\lambda_1 = a_1 + a_2, \lambda_2 = a_1, \lambda_3 = a_1 + a_3, \\ \lambda_4 =& a_1 + a_2 + a_3 + a_4, \lambda_5 = a_5. \end{aligned} \tag{4.1}$$

Theorem 4.7. *Let $C = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$ be a linear code of length n over $\mathscr{R}$. Then $\mathscr{C}$ is a (Θ_t, λ)-cyclic code if and only if $\mathscr{C}_i's$ are (Θ_t, λ_i)-cyclic codes over $\mathbb{F}_q$, respectively for $1 \le i \le 5$, where $\lambda_i's$ are given by equation (4.1).*

Proof. Let $\mathscr{C}$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Let $z^i = (z_{0,i}, z_{1,i}, \ldots, z_{n-1,i}) \in \mathscr{C}_i$ for $1 \le i \le 5$ and $y_j = \sum_{i=1}^{5}\zeta_i z_j^i$ for $0 \le j \le n-1$. Now $y = (y_0, y_1, \ldots, y_{n-1}) \in \mathscr{C}$ and hence $\sigma_\lambda(y) = (\lambda\Theta_t(y_{n-1}), \Theta_t(y_0), \ldots, \Theta_t(y_{n-2})) \in \mathscr{C}$. Again we have $\sigma_\lambda(y) = \sum_{i=1}^{5}\zeta_i\sigma_{\lambda_i}(z^i) \in \mathscr{C} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$. Therefore, $\sigma_{\lambda_i}(z^i) \in \mathscr{C}_i$ for $1 \le i \le 5$. Hence $\mathscr{C}_i's$ are (Θ_t, λ_i)-cyclic codes over $\mathbb{F}_q$, respectively for $1 \le i \le 5$.
On the other side, let $\mathscr{C}_i's$ be (Θ_t, λ_i)-cyclic codes over $\mathbb{F}_q$, respectively for $1 \le i \le 5$. Let $y = (y_0, y_1, \ldots, y_{n-1}) \in \mathscr{C}$ where $y_j = \sum_{i=1}^{5}\zeta_i z_j^i$ for $0 \le j \le n-1$. Then $z^i = (z_{0,i}, z_{1,i}, \ldots, z_{n-1,i}) \in \mathscr{C}_i$ for $1 \le i \le 5$ and hence $\sigma_{\lambda_i}(z^i) \in \mathscr{C}_i$ for $1 \le i \le 5$. Now $\sigma_\lambda(y) = \sum_{i=1}^{5}\zeta_i\sigma_{\lambda_i}(z^i) \in \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i = \mathscr{C}$. Therefore, $\mathscr{C} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$ is (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. □

Theorem 4.8. *Let $\mathscr{C} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Then*

$$\mathscr{C} = \langle \zeta_1 f_2(x), \zeta_2 f_2(x), \ldots, \zeta_6 f_6(x)\rangle$$

where $\mathscr{C}_i = \langle f_i(x)\rangle$ and $x^n - \lambda_i = g_i(x)f_i(x)$ in $\mathbb{F}_q[x;\Theta_t]$ for $1 \le i \le 5$.

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Then by Theorem 4.7, $\mathscr{C}_i's$ are (Θ_t, λ_i)-cyclic codes of length n over $\mathbb{F}_q$, respectively for $1 \le i \le 5$. Again by Lemma 4.5, we have $\mathscr{C}_i = \langle f_i(x)\rangle$ and $x^n - \lambda_i = g_i(x)f_i(x)$ in $\mathbb{F}_q[x;\Theta_t]$ for $1 \le i \le 5$. Therefore, $\mathscr{C} = \langle \zeta_1 f_2(x), \zeta_2 f_2(x), \ldots, \zeta_5 f_5(x)\rangle$. □

Corollary 4.9. *Let $\mathscr{C} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Then there exists a polynomial $f(x) \in \mathscr{R}[x;\Theta_t]$ such that $\mathscr{C} = \langle f(x)\rangle$ and $x^n - \lambda = g(x)f(x)$ in $\mathscr{R}[x;\Theta_t]$.*

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5}\zeta_i\mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Then by Theorem 4.8, we have $\mathscr{C} = \langle \zeta_1 f_2(x), \zeta_2 f_2(x), \ldots, \zeta_5 f_5(x)\rangle$, where $\mathscr{C}_i = \langle f_i(x)\rangle$ and $x^n - \lambda_i = g_i(x)f_i(x)$ in $\mathbb{F}_q[x;\Theta_t]$ for $1 \le i \le 5$. Take $f(x) = \sum_{i=1}^{5}\zeta_i f_i(x)$. Then $\langle f(x)\rangle \subseteq \mathscr{C}$. On the other hand, $\zeta_i f_i(x) = \zeta_i f(x) \in \langle f(x)\rangle$ for

$1 \leq i \leq 5$. Thus $\mathscr{C} \subseteq \langle f(x) \rangle$. Combining both sides, we conclude $\mathscr{C} = \langle f(x) \rangle$. Further, $[\sum_{i=1}^{5} \zeta_i g_i(x)] f(x) = \sum_{i=1}^{5} \zeta_i g_i(x) f_i(x) = \sum_{i=1}^{5}(x^n - \lambda_i) = x^n - \lambda$. Thus $x^n - \lambda = g(x)f(x)$ in $\mathscr{R}[x;\Theta_t]$ where $g(x) = \sum_{i=1}^{5} \zeta_i g_i(x)$. □

Theorem 4.10. *Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. If $gcd(n, \frac{m}{t}) = 1 = gcd(n,q)$, then there exists an idempotent polynomial $i(x) \in \mathscr{R}[x;\Theta_t]$ such that $C = \langle i(x) \rangle$ and $i(x)$ is a right divisor of $x^n - \lambda$.*

Proof. Since $gcd(n, \frac{m}{t}) = 1 = gcd(n,q)$, there exists idempotent polynomials $i_j(x)'s$ such that $\mathscr{C} = \langle i_j(x) \rangle$ and $i_j(x)$ is a right divisor of $x^n - \lambda_j$ in $\mathbb{F}_q[x]$ for $1 \leq j \leq 5$. Then by similar argument as of Corollary 4.9, we have $\mathscr{C} = \langle i(x) \rangle$ and $i(x)$ is a right divisor of $x^n - \lambda$. □

Theorem 4.11. *Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Then $\mathscr{C}^{\perp} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i^{\perp}$ be a (Θ_t, λ^{-1})-cyclic code of length n over $\mathscr{R}$.*

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Then by Theorem 4.7, $\mathscr{C}_i' s$ are (Θ_t, λ_i)-cyclic code of length n over $\mathbb{F}_q$, respectively for $1 \leq i \leq 5$. Since $\Theta_t(\lambda) = \lambda$, then $\Theta_t(\lambda_i) = \lambda_i$ for $1 \leq i \leq 5$. Therefore, $\mathscr{C}_i^{\perp}$ is a $(\Theta_t, \lambda_i^{-1})$-cyclic code of length n over $\mathbb{F}_q$, respectively for $1 \leq i \leq 5$. As $\lambda^{-1} = \sum_{i=1}^{5} \zeta_i \lambda_i^{-1}$, then by Theorem 4.7, $\mathscr{C}^{\perp} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i^{\perp}$ is a (Θ_t, λ^{-1})-cyclic code of length n over $\mathscr{R}$. □

Corollary 4.12. *Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. Then $\mathscr{C}^{\perp} = \langle G(x) \rangle$ where $G(x) = \sum_{i=1}^{5} \zeta_i g_i^*(x)$ and $x^n - \lambda_i^{-1} = g_i(x) f_i(x)$ in $\mathbb{F}_q[x;\Theta_t]$ for $1 \leq i \leq 5$.*

Proof. Let $\mathscr{C} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i$ be a (Θ_t, λ)-cyclic code of length n over $\mathscr{R}$. The by Theorem 4.11, $\mathscr{C}^{\perp} = \oplus_{i=1}^{5} \zeta_i \mathscr{C}_i^{\perp}$ is a (Θ_t, λ^{-1})-cyclic code of length n over $\mathscr{R}$ where $\mathscr{C}_i^{\perp}$ is a $(\Theta_t, \lambda_i^{-1})$-cyclic code of length n over $\mathbb{F}_q$, respectively for $1 \leq i \leq 5$. Let $\mathscr{C}_i^{\perp} = \langle g^*(x) \rangle$ where $x^n - \lambda_i^{-1} = g_i(x) f_i(x)$ and $g_i^*(x)$ is the Θ_t-reciprocal polynomial of $g_i(x)$ for $1 \leq i \leq 5$. Therefore, by similar argument as of (Θ_t, λ)-cyclic code, we conclude that $\mathscr{C}^{\perp} = \langle G(x) \rangle$ where $G(x) = \sum_{i=1}^{5} \zeta_i g_i^*(x)$. □

Corollary 4.13. *Let $C = \oplus_{i=1}^{6} e_i C_i$ be a linear code of length n over R. Then C is a self-dual (Θ_t, λ)-cyclic code if and only if C_i is a self-dual (Θ_t, λ_i)-cyclic code respectively for $1 \leq i \leq 6$. Moreover, C is a sef-dual (Θ_t, λ)-cyclic code if and only if $\lambda_i^2 = 1$, for $1 \leq i \leq 6$.*

Example 4.14. *Let $q = 25$ and $R = \mathbb{F}_{25}[u,v,w]/\langle u^2 = u, v^2 = v, w^2 = w, uv = vu, uw = vw = 0, \rangle$. Now the automorphism $\Theta_1(a) = a^5$ for all $a \in \mathbb{F}_{25}$ and hence for $r \in R$, we have $\Theta_1(r) = \sum_{i=1}^{5} \zeta_i \Theta_1(r_i) = \sum_{i=1}^{5} \zeta_i r_i^5$ where $r_i \in \mathbb{F}_{25}$ and $1 \leq i \leq 5$. Let $\lambda_1 = \lambda_2 = -1, \lambda_3 = \lambda_4 = \lambda_5 = 1$. Now,*

$$x^6 - 1 = (x^2 + w^4)(x+1)(x+4)(x^2 + w^{20}) \in \mathbb{F}_{25}[x]$$
$$= (x^2 + w^{20})(x+1)(x+4)(x^2 + w^4) \in \mathbb{F}_{25}[x]$$

and,

$$x^6 + 1 = (x^2 + w^8)(x+2)(x+3)(x^2 + w^{16}) \in \mathbb{F}_{25}[x].$$

Let $f_1(x) = f_2(x) = (x+3)(x^2 + w^{16}), f_3(x) = f_4(x) = (x+4)(x^2 + w^{20})$ and $f_5(x) = (x+4)(x^2 + w^4)$. Then $\mathscr{C} = \langle \sum_{i=1}^{5} \zeta_i f_i(x) \rangle$ is a (Θ_1, λ)-cyclic code of length 6 over R. Since $\lambda_i^2 = 1$ for $i = 1, 2, \ldots, 6$, then $\lambda^2 = 1$. Therefore, by Corollary 4.13, $\mathscr{C}$ is a self-dual (Θ_1, λ)-cyclic code over $\mathscr{R}$. Hence, by Corollary 2.5, $\phi(C)$ is a self-dual $[30, 21, 3]$ linear code over $\mathbb{F}_{25}$.

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