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Geometrical approach on set theoretical solutions of Yang-Baxter equation in Lie algebras

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Abstract. In this paper, we handle set-theoretical solutions of Yang-Baxter equation and Lyubashenko set theoretical solutions in Lie algebras. We present a new commutative binary operation on these structures, and we obtain new set theoretical solutions including this operation by using property of commutativity of it. Also, we show that some set theoretical solutions of Yang-Baxter equation corresponds to the Lyubashenko set theoretical solutions on these structures. Additionally, we give some relations to verify set theoretical solution of Yang-Baxter equation. Moreover, we put an interpretation for these solutions from the point of geometrical view in Euclidean space, Minkowski space and differentiable manifolds by using Lie algebras.

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Keywords: Yang-Baxter equation, Lyubashenko solution, Lie algebras, Set theoretical solutions, Differentiable manifolds, Minkowski space.

Contents

1. Introduction

Yang-Baxter equation introduced by the Nobel laureate C.N. Yang in theoretical physics [18] and by R.J. Baxter in statistical mechanics [1, 2]. Yang-Baxter equation has been recently attracted more attention among researchers in a wide range of disciplines such as knot theory, link invariants, quantum computing, braided categories, geometrical structures, quantum groups, the analysis of integrable systems, quantum mechanics, physics and etc. For example, Boucetta and Medina worked on solutions of the Yang-Baxter equations on quadratic Lie group by

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using the case of oscillator groups [25]. Berceanu et al. [4] examined algebraic structures arising from Yang-Baxter Systems. Oner, Senturk et. al constructed new set theoretical solutions of Yang-Baxter equation in MV-algebras [14]. Massuyeau and Nichita considered the problem of constructing knot invariants from Yang-Baxter operators associated to (unitary associative) algebra structures [15]. Abedin and Maximov classified the classical twists of standard Lie bialgebra structure on a loop algebra. Besides all these works, Gateva-Ivanova examined set theoretical solutions of the Yang-Baxter equation on braces and symmetric groups [7]. Wang and Ma provided a new framework of obtaining singular solutions of the quantum Yang-Baxter equation by constructing weak quasi-triangular structures [8]. Belavin and Drinfeld worked on solutions of the classical Yang-Baxter equation for simple Lie algebras [5]. Burban and Henrich handled semi-stable vector bundles on elliptic curves and their relation with associative Yang-Baxter equation [26]. Nichita and Parashar studied Spectral-parameter dependent Yang-Baxter operators and Yang-Baxter systems from algebraic structures [12], and etc. Moreover, Lyubashenko set theoretical solution of Yang-Baxter equation was given in [20]. This solution method have been used in many works such as [21–24].

In accordance with these works, we consider a geometrical approach of set theoretical solutions of Yang-Baxter equation in Lie algebras by defining a new operator. On the other hand, Lie algebra were introduced for the first time by Marius Sophus Lie in the 1870s to study the concept of infinitesimal transformations [13]. Moreover, this algebraic structure has widely served for many areas in science especially physics and geometry, such as [6, 9, 16].

In this study, we handle set theoretical solutions of Yang-Baxter equation in Lie algebras. For this aim, we define a new operator to find new set theoretical solutions of Yang-Baxter equation in this structure. Moreover, we reach that some set-theoretical solutions of Yang-Baxter equation corresponds to the Lyubashenko set theoretical solutions on these structures. We verify that some solutions are preserved under Lie homomorphism, additional homomorphism and $\circledast$-homomorphism. Besides, we deal with geometrical interpretation of set theoretical solution in Lie algebras which are defined in Euclidean space, Minkowski Space and differentiable manifolds. This paper is organized as follows: In Section 2, we recall basic notions which are going to be needed. In Section 3, we give set theoretical solutions of Yang-Baxter equation in Lie algebras. Moreover, we define a new operator to get new solutions which verifies Yang-Baxter condition. Moreover, we examine geometrical interpretation of some solutions in 3-dimensional Euclidean space. In Section 4, we interpret of geometrical meaning of set theoretical solutions of Yang-Baxter equation in Minkowski Space. Finally in Section 5, we construct a bridge between differentiable manifolds and set theoretical solutions of Yang-Baxter equation.

2. Preliminaries

In this section, we recall some definitions which are used during this work.

Definition 2.1. *[17] A Lie algebra over $\mathbb{R}$ is a real vector space U with a bilinear map*

$$[,]: U \times U \to U$$

such that:
- $[X, Y] = -[Y, X]$ *and*
- $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacoby identity) for all* $X, Y, Z \in U$.

Definition 2.2. *[10] A Lie homomorphism is a linear map from a Lie algebra ϱ_1 to a Lie algebra ϱ_2 such that it is compatible with the Lie bracket*

$$\Psi : \varrho_1 \to \varrho_2, \quad \Psi([l, m]) = [\Psi(l), \Psi(m)]$$

where $l, m \in \varrho_1$.

Definition 2.3. *[19] Let M be a Hausdorff space. A differentiable structure on M of dimension n is a collection of open charts $(U_i, \phi_i)_{i\varepsilon\Lambda}$ on M where $\phi_i(U_i)$ is an open subset of $\mathbb{R}^n$ such that the following conditions are satisfied:*

(a) $M = \cup_{i\varepsilon\Lambda} U_i$,

(b) For each pair $i, j\varepsilon\Lambda$ the mapping $\phi_i \cdot \phi_j^{-1}$ is a differentiable mapping of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$,

(c) The collection $(U_i, \phi_i)_{i\varepsilon\Lambda}$ is a maximal family of open charts for which (a) and (b) hold.

Definition 2.4. *[19] A differentiable manifold of dimension n is a Hausdorff space with differentiable structure of dimension n.*

Definition 2.5. *[17] Let M be a real $n-$dimensional differentiable manifold and $\chi(M)$ the module of differentiable vector fields of M and $f \in C^\infty(M, \mathbb{R})$. If X and Y are in $\chi(M)$, then the Lie bracket $[X, Y]$ is defined as a mapping from the ring of functions on M into itself by*

$$[X, Y]f = X(Y(f)) - Y(X(f))$$

where $X(f)$ is the directional derivative of f function in direction X.

Definition 2.6. *[11] The Minkowski Space is the metric space $E_1^3 = (\mathbb{R}^3, \langle,\rangle)$, where the metric is given by*

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3$$

which is called the Minkowski metric for $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$.

Definition 2.7. *[11] A vector $v \in E_1^3$ is called*

(1) spacelike if $\langle v, v\rangle > 0$ or $v = 0$,

(2) timelike if $\langle v, v\rangle < 0$,

(3) null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$.

3. Set Theoretical Solutions of Yang-Baxter Equation in Lie Algebras and Geometrical View in Euclidean Space

In this part of the paper, we give some set theoretical solutions of Yang-Baxter equation in Lie algebras. And also, we determine which of them are corresponding to Lyubashenko set theoretical solutions of Yang-Baxter equation on these structures. Along with these, we discuss the geometrical interpretations of some set theoretical solutions of Yang-Baxter equation in Euclidean space.

Let F be a field where tensor products are defined and W be a F-space. The mapping over $W \otimes W$ is denoted by μ. The twist map on this structure is given by $\mu(w_1 \otimes w_2) = w_2 \otimes w_1$ and the identity map on F is defined by $I : W \to W$; for a $F-$linear map $S : W \otimes W \to W \otimes W$, let $S^{12} = S \otimes I$, $S^{13} = (I \otimes \mu)(S \otimes I)(\mu \otimes I)$ and $S^{23} = I \otimes S$.

Definition 3.1. *[3] A Yang-Baxter operator is an invertible $F-$linear map $S : W \otimes W \to W \otimes W$ that verifies the braid condition (called the Yang-Baxter equation):*

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}. \tag{3.1}$$

If S verifies Equation (3.1), then both $S \circ \mu$ and $\mu \circ S$ verify the quantum Yang–Baxter equation (QYBE):

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}. \tag{3.2}$$

Lemma 3.2. *[3] Equations (3.1) and (3.2) are equivalent to each other.*

To construct set theoretical solutions of Yang-Baxter equation in Lie algebras, we need the following definition.

Definition 3.3. *[3] Let L be a set and $S : L \times L \to L \times L$, $S(l,m) = (f(l), g(m))$ be a map. The map S is set theoretical solution of Yang-Baxter equation if it verifies the following equality for $l,m,n \in L$:*

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \tag{3.3}$$

which is also equivalent to

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}, \tag{3.4}$$

where

$$\begin{aligned} S^{12} &: L^3 \to L^3, \quad S^{12}(l,m,n) = (f(l), g(m), n), \\ S^{23} &: L^3 \to L^3, \quad S^{23}(l,m,n) = (l, f(m), g(n)), \\ S^{13} &: L^3 \to L^3, \quad S^{13}(l,m,n) = (f(l), m, g(n)). \end{aligned}$$

First of all, we handle some fundamental set theoretical solutions of Yang-Baxter equation in Lie algebras.

Lemma 3.4. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = ([c,m], 0)$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l,m \in L$ on this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([c,m], 0, n), \\ S^{23}(l,m,n) &= (l, [c,n], 0). \end{aligned}$$

We need to satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(l,m,n) \in L^3$ as below:

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(l,m,n) &= S^{12}(S^{23}(S^{12}(l,m,n))) \\ &= S^{12}(S^{23}([c,m], 0, n)) \\ &= S^{12}([c,m], [c,n], 0) \\ &= ([c,[c,n]], 0, 0) \\ &= S^{23}([c,[c,n]], 0, 0) \\ &= S^{23}(S^{12}(l, [c,n], 0)) \\ &= S^{23}(S^{12}(S^{23}(l,m,n))) = (S^{23} \circ S^{12} \circ S^{23})(l,m,n). \end{aligned}$$

Therefore, the mapping $S(l,m) = ([c,m], 0)$ is a set theoretical solution of Yang-Baxter equation in Lie algebras. ■

Lemma 3.5. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = (0, [l,c])$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l,m \in L$ on this structure.

Proof. It follows from similar procedure in the proof of Lemma 3.4. ■

Lemma 3.6. *Let $(L,[,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = ([c,m],[l,c])$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l,m \in L$ on this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([c,m],[l,c],n), \\ S^{23}(l,m,n) &= (l,[c,n],[m,c]). \end{aligned}$$

We need to satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(l,m,n) \in L^3$.

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(l,m,n) &= S^{12}(S^{23}(S^{12}(l,m,n))) \\ &= S^{12}(S^{23}([c,m],[l,c],n)) \\ &= S^{12}([c,m],[c,n],[[l,c],c]) \\ &= ([c,[c,n]],[[c,m],c],[[l,c],c]) \\ &= ([c,[c,n]],[c,[m,c]],[[l,c],c]). \end{aligned}$$

By the using the property $[c,[m,c]] = [[c,m],c]$ of Lie brackets, then we obtain

$$\begin{aligned} &= S^{23}([c,[c,n]],[l,c],[m,c]) \\ &= S^{23}(S^{12}(l,[c,n],[m,c])) \\ &= S^{23}(S^{12}(S^{23}(l,m,n))) = (S^{23} \circ S^{12} \circ S^{23})(l,m,n). \end{aligned}$$

Therefore, the mapping $S(l,m) = ([c,m],[l,c])$ is a set theoretical solution of Yang-Baxter equation in Lie algebras. ■

Lemma 3.7. *Let $(L,[,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = ([c,m]+c,c)$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l,m \in L$ on this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([c,m]+c,c,n), \\ S^{23}(l,m,n) &= (l,[c,n]+c,c). \end{aligned}$$

We have to verify the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(l, m, n) \in L^3$.

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(l, m, n) &= S^{12}(S^{23}(S^{12}(l, m, n))) \\
&= S^{12}(S^{23}([c, m] + c, c, n)) \\
&= S^{12}([c, m] + c, [c, n] + c, c) \\
&= ([c, [c, n] + c], c, c) \\
&= ([c, [c, n]], c, c) \\
&= ([c, [c, n]], [c, c] + c, c) \\
&= S^{23}([c, [c, n]], c, c) \\
&= S^{23}([c, [c, n] + c], c, c) \\
&= S^{23}(S^{12}(l, [c, n] + c, c)) \\
&= S^{23}(S^{12}(S^{23}(l, m, n))) = (S^{23} \circ S^{12} \circ S^{23})(l, m, n).
\end{aligned}$$

Then, the mapping $S(l, m) = ([c, m] + c, c)$ is a set theoretical solution of Yang–Baxter equation for any $c \in L$ in Lie algebras. ■

Lemma 3.8. *Let $(L, [,])$ be a Lie algebra. Then, the mapping*

$$S(l, m) = (c, [l, c] + c)$$

is a set theoretical solution of Yang-Baxter equation for any constant element $c \in L$ and $l, m \in L$ on this structure.

Proof. It is clearly obtained by using similar method in the proof of Lemma 3.7. ■

Definition 3.9. *[20] The mappings*

$$\begin{aligned}
S : L \times L &\to L \times L \\
(l, m) &\to S(l, m) = (f(l), g(m))
\end{aligned}$$

or

$$\begin{aligned}
S : L \times L &\to L \times L \\
(l, m) &\to S(l, m) = (f(m), g(l))
\end{aligned}$$

are called Lyubashenko set theoretical solutions for $l, m \in L$ where $f : L \to L$ and $g : L \to L$ are functions such that $f(g(x)) = g(f(x))$ for each $x \in L$.

Corollary 3.10. *Using Definition 3.9, we obtain the following results:*

- *In Lemma 3.4, we show that the mapping $S(l, m) = ([c, m], 0)$ is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take $f(m) = [c, m]$ and $g(l) = 0$, then we obtain*

  $$f(g(x)) = f(0) = [c, 0] = 0 = g([c, x]) = g(f(x))$$

  for each $x \in L$. So, we conclude that the mapping $S(l, m) = ([c, m], 0)$ is also a Lyubashenko set theoretical solution.

- *In Lemma 3.5, we show that the mapping $S(l, m) = (0, [l, c])$ is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take $f(m) = 0$ and $g(l) = [l, c]$, then we obtain*

  $$f(g(x)) = f([x, c]) = 0 = [0, c] = g(0) = g(f(x))$$

  for each $x \in L$. So, we conclude that the mapping $S(l, m) = (0, [l, c])$ is also a Lyubashenko set theoretical solution.

- *In Lemma 3.6, we show that the mapping $S(l,m) = ([c,m],[l,c])$ is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take $f(m) = [c,m]$ and $g(l) = [l,c]$, then we have $f(x) = [c,x] = -[x,c] = -g(x)$ for each $x \in L$. By using this relation and since $f(x)$ is an odd function, we obtain*
$$f(g(x)) = f(-f(x)) = -f(f(x)) = g(f(x))$$
for each $x \in L$. So, we conclude that the mapping $S(l,m) = ([c,m],[l,c])$ is also a Lyubashenko set theoretic solution.

- *In Lemma 3.7, we show that the mapping $S(l,m) = ([c,m]+c, c)$ is a set theoretical solution of Yang-Baxter equation for Lie algebras. Besides, if we take $f(m) = [c,m]+c$ and $g(l) = c$ where c is any constant element in L, then we obtain*
$$f(g(x)) = f(c) = [c,c]+c = c = g(f(x))$$
for each $x \in L$. So, we conclude that the mapping $S(l,m) = ([c,m]+c, c)$ is also a Lyubashenko set theoretical solution.

- *In Lemma 3.8, we obtain that the mapping $S(l,m) = (c,[l,c]+c)$ is also a Lyubashenko set theoretical solution on L by using similar procedure as above.*

Now, we give a theorem which gives us a general set theoretical solution of Yang-Baxter equation for Lie Algebras in 3-dimensional Euclidean space (E^3).

Theorem 3.11. *Let E be a Euclidean space and let $(E^3,[,])$ be a Lie algebra. Then, the mapping*
$$S(l,m) = ([l,m],0)$$
is a set theoretical solution of Yang-Baxter equation for $l,m \in E^3$.

Proof. Let S^{12} and S^{23} be defined as follows:
$$S^{12}(l,m,n) = ([l,m],0,n),$$
$$S^{23}(l,m,n) = (l,[m,n],0)$$
where l, m and n are linearly dependent with Euclidean bases e_1, e_2 and e_3, respectively.

We satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l,m,n \in E^3$.
$$\begin{aligned} & (S^{12} \circ S^{23} \circ S^{12})(l,m,n) \\ =\ & S^{12}(S^{23}(S^{12}(l,m,n))) \\ =\ & S^{12}(S^{23}([l,m],0,n)) \\ =\ & S^{12}([l,m],0,0) \\ =\ & (0,0,0) \end{aligned} \tag{3.5}$$
and
$$\begin{aligned} & (S^{23} \circ S^{12} \circ S^{23})(l,m,n) \\ =\ & S^{23}(S^{12}(S^{23}(l,m,n))) \\ =\ & S^{23}(S^{12}(l,[m,n],0)) \\ =\ & S^{23}([l,[m,n]],0,0) \\ =\ & ([l,[m,n]],0,0). \end{aligned} \tag{3.6}$$

From the Equations (3.5) and (3.6), we have $([l,[m,n]],0,0) = (0,0,0)$. As it seen, the condition is satisfied when $[l,[m,n]] = 0$. In 3-dimensional Euclidean space, Lie bracket corresponds to cross product. Therefore, the vector l is parallel to $[m,n]$. So, the equality $[l,[m,n]] = 0$ is verified in 3-dimensional Euclidean space. Hence, the mapping $S(l,m) = ([l,m],0)$ is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space. ■

By Theorem 3.11, we attain a more general case as follows.

Corollary 3.12. *The mapping*

$$S(l,m) = (f(l,m),0)$$

is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space where $f(l,0) = f(0,m) = 0$ *for each* $l,m \in E^3$ *since the condition* $f(l,f(m,n)) = 0$ *is verified for each* $l,m,n \in E^3$ *in 3-dimensional Euclidean space.*

Using similar method in Theorem 3.11, we obtain the following theorem.

Theorem 3.13. *Let* $(E^3,[,])$ *be a Lie algebra. The mapping*

$$S(l,m) = (0,[m,l])$$

is a set theoretical solution of Yang-Baxter equation for $l,m \in E^3$ *in 3-dimensional Euclidean space.*

By the help of Theorem 3.13, we also get a more general case as follows.

Corollary 3.14. *The mapping*

$$S(l,m) = (0,g(l,m))$$

is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space where $g(l,0) = g(0,m) = 0$ *for each* $l,m \in E^3$ *since the condition* $g(g(l,m),n)) = 0$ *is verified for each* $l,m,n \in E^3$ *in 3-dimensional Euclidean space.*

Now, we introduce a new binary operation in Lie algebras. This operation has advantages to find set theoretical solutions of Yang-Baxter equation on these structures.

Definition 3.15. *Let* $(L,[,])$ *be a Lie algebra. The binary operation* $\circledast$*-operation defined as*

$$l \circledast m := [l,m] + l + m$$

for each $l,m \in L$.

Lemma 3.16. *Let* $(L,[,])$ *be a Lie algebra. Then, the identities*

(i) $l \circledast l = 2l$,

(ii) $l \circledast 0_L = l$,

(iii) $l \circledast (-l) = 0_L$,

(iv) $(l \circledast m) \circledast (m \circledast l) = 2([[l,m],l] + [[l,m],m] + l + m)$,

(v) $l \circledast (m - l) = (l \circledast m) - l$,

(vi) $(l \circledast m) + (m \circledast l) = 2(l + m)$

are verified for each $l,m \in L$.

Theorem 3.17. *Let* $(E^3, [,])$ *be a Lie algebra. Then, the mapping*

$$S(l,m) = (l \circledast m, 0)$$

is a set theoretical solution of Yang-Baxter equation for $l, m \in E^3$ *in 3-dimensional Euclidean space.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$S^{12}(l,m,n) = (l \circledast m, 0, n) = ([l,m] + l + m, 0, n),$$
$$S^{23}(l,m,n) = (l, m \circledast n, 0) = (l, [m,n] + m + n, 0)$$

where l, m and n are linearly dependent to Euclidean bases e_1, e_2 and e_3, respectively.

We have to satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l, m, n \in E^3$.

$$\begin{aligned} & (S^{12} \circ S^{23} \circ S^{12})(l,m,n) \\ =\ & S^{12}(S^{23}(S^{12}(l,m,n))) \\ =\ & S^{12}(S^{23}([l,m] + l + m, 0, n)) \\ =\ & S^{12}([l,m] + l + m, [0,n] + 0 + n, 0) \\ =\ & S^{12}([l,m] + l + m, n, 0) \\ =\ & ([[l,m] + l + m, n] + [l,m] + l + m + n, 0, 0) \\ =\ & ([[l,m],n] + [l,n] + [m,n] + [l,m] + l + m + n, 0, 0) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & (S^{23} \circ S^{12} \circ S^{23})(l,m,n) \\ =\ & S^{23}(S^{12}(S^{23}(l,m,n))) \\ =\ & S^{23}(S^{12}(l, [m,n] + m + n, 0)) \\ =\ & S^{23}([l, [m,n] + m + n] + l + [m,n] + m + n, 0, 0) \\ =\ & ([l, [m,n] + m + n] + l + [m,n] + m + n, 0, 0) \\ =\ & ([l,[m,n]] + [l,m] + [l,n] + [m,n] + l + m + n, 0, 0). \end{aligned} \tag{3.8}$$

Using Equation (3.7) and (3.8), we have

$$\begin{aligned} & ([[l,m],n] + [l,n] + [m,n] + [l,m] + l + m + n, 0, 0) \\ =\ & ([l,[m,n]] + [l,m] + [l,n] + [m,n] + l + m + n, 0, 0). \end{aligned}$$

By the help of above equality, we need to satisfy the following condition

$$[[l,m],n] = [l,[m,n]].$$

From the point of geometrical view, since we can correspond cross product to Lie bracket, we obtain

$$l \wedge m = [l,m].$$

Hence, we get

$$[[l,m],n] = ((l \wedge m) \wedge n) = 0. \quad \text{(Since } (l \wedge m) \text{ is parallel to } n\text{)}$$
$$[l,[m,n]] = (l \wedge (m \wedge n)) = 0. \quad \text{(Since } (m \wedge n) \text{ is parallel to } l\text{)}$$

So, the mapping $S(l,m) = ([l,m] + l + m, 0)$ is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space. ■

After these theorems and lemmas, we can give the following examples in E^3.

Example 3.18. *Let* $(E^3, [,])$ *be a Lie algebra. Then, the mappings*
(i) $S(l,m) = (l \cdot (l \wedge m), 0)$,
(ii) $S(l,m) = (l \cdot (l \wedge m), m)$,
(ii) $S(l,m) = (l \cdot (l \wedge m), m \cdot (l \wedge m))$
are set theoretical solutions of Yang-Baxter equation in 3-dimensional Euclidean space where the operation " $\cdot$"corresponds dot product such that $l \cdot (m \wedge n) = (l \wedge m) \cdot n = det(l,m,n)$ *where* $l,m,n \in E^3$.

Theorem 3.19. *Let* $(L, [,])$ *be a Lie algebra. Then, the mapping*

$$S(l,m) = (\frac{1}{2}((l \circledast m) + (m \circledast l)), 0)$$

is a set theoretical solution of Yang-Baxter equation for $l,m \in L$ *in Lie algebras.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$S^{12}(l,m,n) = (\tfrac{1}{2}((l \circledast m) + (m \circledast l)), 0, n),$$
$$S^{23}(l,m,n) = (l, \tfrac{1}{2}((m \circledast n) + (n \circledast m)), 0).$$

We need to verify the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l,m,n \in L$.

$$\begin{aligned}
&(S^{12} \circ S^{23} \circ S^{12})(l,m,n) \\
&= S^{12}(S^{23}(S^{12}(l,m,n))) \\
&= S^{12}(S^{23}(\frac{1}{2}((l \circledast m) + (m \circledast l)), 0, n)) \\
&= S^{12}(S^{23}(\frac{1}{2}(2(l+m), 0, n)), && \text{(By Lemma 3.16 } (vi)) \\
&= S^{12}(S^{23}(l+m, 0, n)) \\
&= S^{12}(l+m, \frac{1}{2}((0 \circledast n) + (n \circledast 0)), 0) \\
&= S^{12}(l+m, \frac{1}{2}(2n), 0) && \text{(By Lemma 3.16 } (ii)) \\
&= S^{12}(l+m, n, 0) \\
&= (\frac{1}{2}(((l+m) \circledast n) + (n \circledast (l+m))), 0, 0) \\
&= (\frac{1}{2}(2(l+m+n), 0, 0)) && \text{(By Lemma 3.16 } (vi)) \\
&= (l+m+n, 0, 0)
\end{aligned} \tag{3.9}$$

and we have

$$
\begin{aligned}
&(S^{23} \circ S^{12} \circ S^{23})(l,m,n) \\
=\ & S^{23}(S^{12}(S^{23}(l,m,n))) \\
=\ & S^{23}(S^{12}(l, \frac{1}{2}((m \circledast n) + (n \circledast m)), 0)) \\
=\ & S^{23}(S^{12}(l, \frac{1}{2}(2(m+n), 0)) && \text{(By Lemma 3.16 } (vi)) \\
=\ & S^{23}(S^{12}(l, m+n, 0)) \\
=\ & S^{23}(\frac{1}{2}((l \circledast (m+n)) + ((m+n) \circledast l)), 0, 0) \\
=\ & S^{23}(\frac{1}{2}(2(l+m+n)), 0, 0) && \text{(By Lemma 3.16 } (vi)) \\
=\ & S^{23}(l+m+n, 0, 0) \\
=\ & (l+m+n, 0, 0). && (3.10)
\end{aligned}
$$

From the equality of (3.9) and (3.10), the mapping $S(l,m) = (\frac{1}{2}((l \circledast m) + (m \circledast l)), 0)$ is satisfied Yang-Baxter equation in Lie algebras. ■

By Theorem 3.17 and Theorem 3.19, we attain a more general case as follows.

Corollary 3.20. *The mapping*

$$S(l,m) = (f(l,m), 0)$$

is a set theoretical solution of Yang-Baxter equation in 3-dimensional Euclidean space where $f(l,0) = f(0,m) = 0$ *for each* $l, m \in E^3$ *since the condition* $f(f(l,m),n) = f(l, f(m,n))$ *is verified for each* $l, m, n \in E^3$ *in 3-dimensional Euclidean space.*

Definition 3.21. *Let L be a Lie algebra.Then, the mapping Ψ is called a $\circledast$-homomorphism if the equality*

$$\Psi(l \circledast m) = \Psi(l) \circledast \Psi(m)$$

is satisfied for each $l, m \in L$.

Lemma 3.22. *Let Ψ be a Lie homomorphism and additional homomorphism, then Ψ is also a $\circledast-$homomorphism in Lie algebras.*

Proof. Let L be a Lie algebra and $l, m \in L$. Then, we obtain

$$
\begin{aligned}
\Psi(l \circledast m) &= \Psi([l,m] + l + m) \\
&= \Psi([l,m]) + \Psi(l) + \Psi(m) \\
&= ([\Psi(l), \Psi(m)]) + \Psi(l) + \Psi(m) \\
&= \Psi(l) \circledast \Psi(m).
\end{aligned}
$$

■

Lemma 3.23. *Let L be a Lie algebra and the mapping $f : L^2 \to L$ only consist of the combinations of binary operations "$[,]$", "$+$" and "$\circledast$". Besides, the mapping Ψ be a Lie homomorphism and additional homomorphism. Then, the mapping*

$$\Psi(f(l,m)) = \begin{cases} f(\Psi(l), \Psi(m)), & \text{if } f(l,m) \text{ does not contain any constant} \\ f(\Psi(l), \Psi(m)) + \Psi(c) - c, & \text{if } f(l,m) \text{ contains any constant element } c \end{cases}$$

is verified for each $l, m \in L$.

Proof. We make induction on the number of the operations of the f mapping.
• Assume that the mapping $f(l,m)$ consists of only the binary operation " + " and it does not contain any constant element. Then, we use induction on the number of " + " operators as logical induction on the number of connectives:

- If $f(l,m) = p_{i_1} + p_{i_2}$ such that $p_{i_1}, p_{i_2} \in \{l,m\}$. $\Psi(f(l,m)) = \Psi(p_{i_1} + p_{i_2}) = \Psi(p_{i_1}) + \Psi(p_{i_2}) = f(\Psi(l), \Psi(m))$.

- Let $f(l,m) = p_{i_1} + p_{i_2} + ... + p_{i_n}$ such that $p_{i_1}, p_{i_2}, ..., p_{i_n} \in \{l,m\}$. Assume that $\Psi(f(l,m)) = \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n}) = \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) = f(\Psi(l), \Psi(m))$ is verified for each $p_{i_1}, p_{i_2}, ..., p_{i_n} \in \{l,m\}$. Let $g(l,m) = p_{i_1} + p_{i_2} + ... + p_{i_n} + p_{i_{n+1}}$ such that $p_{i_{n+1}} \in \{l,m\}$. Then, we show that $\Psi(g(l,m)) = g(\Psi(l), \Psi(m))$ as follows:

$$\begin{aligned}\Psi(g(l,m)) &= \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n} + p_{i_{n+1}})\\ &= \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n}) + \Psi(p_{i_{n+1}})\\ &= \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + \Psi(p_{i_{n+1}})\\ &= g(\Psi(l), \Psi(m)).\end{aligned}$$

• We assume that the mapping $f(l,m)$ consists of only the binary operation " + " and it contains any constant element such as c. Then we have the following conditions:

- If $f(l,m) = c$ is any constant funtion, then we obtain clearly $\Psi(f(l,m)) = \Psi(c) = \Psi(c) + c - c = f(\Psi(l), \Psi(m)) + \Psi(c) - c$.

- Let $f(l,m) = p_{i_1} + p_{i_2} + ... + p_{i_n} + c$ such that $p_{i_1}, p_{i_2}, ..., p_{i_n} \in \{l,m\}$ and c be any constant element. Then we obtain

$$\begin{aligned}\Psi(f(l,m)) &= \Psi(p_{i_1} + p_{i_2} + ... + p_{i_n} + c)\\ &= \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + \Psi(c)\\ &= \Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + \Psi(c) + c - c\\ &= (\Psi(p_{i_1}) + \Psi(p_{i_2}) + ... + \Psi(p_{i_n}) + c) + \Psi(c) - c\\ &= f(\Psi(l), \Psi(m)) + \Psi(c) - c\end{aligned}$$

So, the equality $f(l,m) = f(\Psi(l), \Psi(m))$ is satisfied when the mapping consists of only "+"operation.
By using similar procedure as the "+"operation, we verify this equality for another operations and their combinations on Lie algebras. ■

Theorem 3.24. *Let Ψ be a Lie homomorphism and an additional homomorphism. If $S(l,m) = (f(l,m), g(l,m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras where $f(l,m)$ and $g(l,m)$ do not contain any constant element and consist of only " $[,]$", " $+$"and " $\circledast$"operations, then $\Psi(S(l,m))$ is also a set theoretical solution of Yang-Baxter equation in Lie algebras.*

Proof. Let L be a Lie algebra. Assume that $S(l,m) = (f(l,m), g(l,m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras where $f(l,m)$ and $g(l,m)$ do not contain a constant element and consist of only " $[,]$", "$+$"and " $\circledast$"operations. Then, we have the following equality:

$$(S^{12} \circ S^{23} \circ S^{12})(l,m,n) = (S^{23} \circ S^{12} \circ S^{23})(l,m,n) \tag{3.11}$$

for each l, m and $n \in L$. So, we obtain the following conclusions:

$$\begin{aligned}
& (S^{12} \circ S^{23} \circ S^{12})(l, m, n) \\
&= S^{12}(S^{23}(f(l, m), g(l, m), n)) \\
&= S^{12}(f(l, m), f(g(l, m), n), g(g(l, m), n)) \\
&= (f(f(l, m), f(g(l, m), n)), g(f(l, m), f(g(l, m), n)), g(g(l, m), n))
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
& (S^{23} \circ S^{12} \circ S^{23})(l, m, n) \\
&= S^{23}(S^{12}(l, f(m, n), g(m, n))) \\
&= S^{23}(f(l, f(m, n)), g(l, f(m, n)), g(m, n)) \\
&= (f(l, f(m, n)), f(g(l, f(m, n)), g(m, n)), g(g(l, f(m, n)), g(m, n))).
\end{aligned} \tag{3.13}$$

From the Equality (3.11), we achieve the Equation (3.12) and the Equation (3.13). Moreover, we get

$$\Psi(S(l, m)) = \Psi((f(l, m), g(l, m))) = (f(\Psi(l), \Psi(m)), g(\Psi(l), \Psi(m))).$$

If we show the accuracy of the below equation

$$(S^{12} \circ S^{23} \circ S^{12})(\Psi(l), \Psi(m), \Psi(n)) = (S^{23} \circ S^{12} \circ S^{23})(\Psi(l), \Psi(m), \Psi(n)) \tag{3.14}$$

then we prove that $\Psi(S(l, m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras.
Now, we verify the Equality (3.14) by the help of Lemma 3.23,the Equations (3.12) and (3.13), respectively.

$$\begin{aligned}
& (S^{12} \circ S^{23} \circ S^{12})(\Psi(l), \Psi(m), \Psi(n)) \\
&= (f(f(\Psi(l), \Psi(m)), f(g(\Psi(l), \Psi(m)), \Psi(n))), g(f(\Psi(l), \Psi(m)), \\
&\quad f(g(\Psi(l), \Psi(m)), \Psi(n))), g(g(\Psi(l), \Psi(m)), \Psi(n))) \\
&= \Psi((f(f(l, m), f(g(l, m), n)), g(f(l, m), f(g(l, m), n)), g(g(l, m), n))) \\
&= \Psi(f(f(l, m), f(g(l, m), n)), g(f(l, m), f(g(l, m), n)), g(g(l, m), n)) \\
&= \Psi(f(l, f(m, n)), f(g(l, f(m, n)), g(m, n)), g(g(l, f(m, n)), g(m, n))) \\
&= f(\Psi(l), f(\Psi(m), \Psi(n))), f(g(\Psi(l), f(\Psi(m), \Psi(n))), g(\Psi(m), \Psi(n))), \\
&\quad g(g(\Psi(l), f(\Psi(m), \Psi(n))), g(\Psi(m), \Psi(n))).
\end{aligned}$$

Then, we reach

$$(S^{12} \circ S^{23} \circ S^{12})(\Psi(l), \Psi(m), \Psi(n)) = (S^{23} \circ S^{12} \circ S^{23})(\Psi(l), \Psi(m), \Psi(n))$$

for each $l, m, n \in L$. So, $\Psi(S(l, m))$ is a set theoretical solution of Yang-Baxter equation in Lie algebras. ■

Theorem 3.25. *Let Ψ be a Lie homomorphism and an additional homomorphism. If $S(l, m) = (f(l), g(m))$ is a Lyubashenko set theoretical solution of Yang-Baxter equation in Lie algebras where the funtions f and g consist of only "$[,]$", "$+$"and "$\circledast$"operations, then $\Psi(S(l, m))$ is also a Lyubashenko set theoretical solution of Yang-Baxter equation in Lie algebras.*

Proof. It follows from Lemma 3.23 and Theorem 3.24. ■

4. Set Theoretical Solutions of Yang-Baxter Equation in Minkowski Space

In the previous section, we study in Euclidean space and realize that we need more flexible space to obtain new solutions. For this reason, we decide to study in Minkowski space. The following theorems correspond to geometrical interpretations of set theoretical solutions of Yang-Baxter equation in Minkowski space via Lie algebras.

Theorem 4.1. *Let $(E_1^3, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = ([m,l]-l,0)$$

is a set theoretical solution of Yang-Baxter equation for $l,m \in E_1^3$ in Minkowski space where l is a null vector and m is a spacelike, timelike or null vector.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([l,m],[m,l],n), \\ S^{23}(l,m,n) &= (l,[m,n],[n,m]). \end{aligned}$$

We need to reach the equation $S^{12}\circ S^{23}\circ S^{12} = S^{23}\circ S^{12}\circ S^{23}$ for all $l,m,n \in E_1^3$. Then, we get

$$\begin{aligned} (S^{12}\circ S^{23}\circ S^{12})(l,m,n) &= S^{12}(S^{23}(S^{12}(l,m,n))) \\ &= S^{12}(S^{23}([m,l]-l,0,n)) \\ &= S^{12}([m,l]-l,0,0) \\ &= (-[m,l]+l,0,0) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} (S^{23}\circ S^{12}\circ S^{23})(l,m,n) &= S^{23}(S^{12}(S^{23}(l,m,n))) \\ &= S^{23}(S^{12}(l,[n,m]-m,0)) \\ &= S^{23}([[n,m]-m,l]-l,0,0) \\ &= ([[n,m]-m,l]-l,0,0) \\ &= ([[n,m],l]-[m,l]-l,0,0). \end{aligned} \tag{4.2}$$

From the Equations (4.1) and (4.2), we have

$$(-[m,l]+l,0,0) = ([[n,m],l]-[m,l]-l,0,0).$$

Thus, we need to satisfy the following condition

$$[[n,m],l] = 2l. \tag{4.3}$$

If we use cross product instead of Lie bracket, then we get

$$(n\wedge m)\wedge l = 2l. \tag{4.4}$$

From the geometrical meaning of cross product, we know that if $a\wedge b = c$ then c is both orthogonal to a and b where $a,b,c \in E_1^3$. So, the Equation (4.4) is not achieved because a vector can not orthogonal to itself in Euclidean space. However in Minkowski Space a null vector is orthogonal to itself so if l is a null vector and $(n\wedge m)$ is a spacelike vector, then the Equation (4.3) is satisfied. ■

Theorem 4.2. *Let $(E_1^3, [,])$ be a Lie algebra. Then, the mapping*

$$S(l,m) = (0, [l,m] + l)$$

is a set theoretical solution of Yang-Baxter equation for $l, m \in E_1^3$ in Minkowski space where l is a null or spacelike vector and m is a spacelike vector.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(l,m,n) &= ([l,m],[m,l],n), \\ S^{23}(l,m,n) &= (l,[m,n],[n,m]). \end{aligned}$$

We have to get the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $l, m, n \in E_1^3$. Then

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(l,m,n) &= S^{12}(S^{23}(S^{12}(l,m,n))) \\ &= S^{12}(S^{23}([l,m],[m,l],n)) \\ &= S^{12}(S^{23}(0,[l,m]+l,n)) \\ &= S^{12}(0,0,[[l,m]+l,n]+[l,m]+l) \\ &= (0,0,[[l,m]+l,n]+[l,m]+l) \\ &= (0,0,[[l,m],n]+[l,n]+[l,m]+l) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(l,m,n) &= S^{23}(S^{12}(S^{23}(l,m,n))) \\ &= S^{23}(S^{12}(l,[m,n],[n,m])) \\ &= S^{23}(S^{12}(l,0,[m,n]+m)) \\ &= S^{23}(0,l,[m,n]+m) \\ &= (0,0,[l,[m,n]+m]+l) \\ &= (0,0,[l,[m,n]]+[l,m]+l). \end{aligned} \tag{4.6}$$

From the Equations (4.5) and (4.6), we have

$$(0,0,[[l,m],n]+[l,n]+[l,m]+l) = (0,0,[l,[m,n]]+[l,m]+l).$$

Thus, we need to satisfy the following condition

$$[[l,m],n]+[l,n] = [l,[m,n]]. \tag{4.7}$$

With the help of Jacobi identity of Lie bracket definition, we know that

$$[l,[m,n]] = -[m,[n,l]] - [n,[l,m]] \tag{4.8}$$

and on the other hand from anti-symmetry identity of Lie bracket definition, we have

$$-[n,[l,m]] = [[l,m],n]. \tag{4.9}$$

Finally, from the Equations (4.7), (4.8) and (4.9) , we attain

$$[l,n] = [m,[l,n]]. \tag{4.10}$$

When we examine the Equation (4.10) in geometric terms via cross product, we get

$$(l \wedge n) = m \wedge (l \wedge n).$$

So, if $(l \wedge n)$ is a null vector and m is a spacelike vector, then the Equation (4.7) is verified. ■

Theorem 4.3. *The mappings in Theorem 4.1 and 4.2 preserve the Yang-Baxter condition in Minkowski space under the Ψ mapping where it is a Lie and additional homomorphism.*

Proof. It follows from Theorem 3.24. ■

5. Set Theoretical Solutions of Yang-Baxter Equation in Differentiable Manifolds via Lie Algebras

Until this section, we have discussed some set theoretical solutions of Yang-Baxter equations in Lie algebras, 3-dimensional Euclidean space and Minkowski space. Finally, we give a definition of Yang-Baxter equation in differentiable manifolds and we attain some solutions for this equation on this structure. Additionally, we present the definition of manifold quantum Yang-Baxter equation.

Definition 5.1. *Let M be a differentiable manifold, U, V and W be vector fields on M and f be a smooth function. Then, the mapping*

$$((S^{12} \circ S^{23} \circ S^{12})(U,V,W))_f = ((S^{23} \circ S^{12} \circ S^{23})(U,V,W))_f$$

is called manifold quantum Yang-Baxter equation.

Theorem 5.2. *Let M be an n-dimensional differentiable manifold and its local coordinate system denoted by $(x_1, x_2, ..., x_n)$. Assume that $U = g_1 \dfrac{\partial}{\partial x_1}$ and $V = g_2 \dfrac{\partial}{\partial x_2}$ are vector fields on M where f, g_1 and g_2 are smooth functions such that the functions g_1 and g_2 depend on the variables x_1 or x_3 and x_2 or x_3, respectively. Then, the mapping*

$$\begin{aligned} S(U,V) &= ([U,V]f, 0), \\ &= ([g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}]f, 0) \end{aligned}$$

is a set theoretical solution of Yang-Baxter equation on manifold M.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(U,V,W) &= ([U,V]f, 0, W), \\ S^{23}(U,V,W) &= (U, [V,W]f, 0) \end{aligned}$$

where $U = g_1 \dfrac{\partial}{\partial x_1}, V = g_2 \dfrac{\partial}{\partial x_2}$ and $W = g_3 \dfrac{\partial}{\partial x_3}$ are vector fields on M such that the functions g_1, g_2 and g_3 depend on the variables x_1 or x_3; x_2 or x_3 and x_1 or x_2 or x_3, respectively.

We satisfy the equation

$$((S^{12} \circ S^{23} \circ S^{12})(g_1 \frac{\partial}{\partial x_1}, g_2) \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f = ((S^{23} \circ S^{12} \circ S^{23})(g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f$$

for each g_1, g_2 and g_3 functions with the help of the Definition 2.5.

$$
\begin{aligned}
&((S^{12} \circ S^{23} \circ S^{12})(g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f \\
&= (S^{12}(S^{23}(S^{12}(g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f)_f)_f \\
&= (S^{12}(S^{23}(([g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}]f, 0, g_3 \frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}((g_1 \frac{\partial}{\partial x_1}(g_2 \frac{\partial}{\partial x_2}(f)) - g_2 \frac{\partial}{\partial x_2}(g_1 \frac{\partial}{\partial x_1}(f)), 0, g_3 \frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}((g_1 \frac{\partial g_2}{\partial x_1} \frac{\partial f}{\partial x_2} + g_1 g_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} - g_2 \frac{\partial g_1}{\partial x_2} \frac{\partial f}{\partial x_1} - g_2 g_1 \frac{\partial^2 f}{\partial x_2 \partial x_1}, 0, g_3 \frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}(g_1 g_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} - g_2 g_1 \frac{\partial^2 f}{\partial x_2 \partial x_1}, 0, g_3 \frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(S^{23}(0, 0, g_3 \frac{\partial}{\partial x_3}))_f)_f \\
&= (S^{12}(0, 0, 0))_f \\
&= (0, 0, 0)
\end{aligned}
$$

and

$$
\begin{aligned}
&((S^{23} \circ S^{12} \circ S^{23})(g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f \\
&= (S^{23}(S^{12}(S^{23}(g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}))_f)_f)_f \\
&= (S^{23}(S^{12}(g_1 \frac{\partial}{\partial x_1}, [g_2 \frac{\partial}{\partial x_2}, g_3 \frac{\partial}{\partial x_3}]f, 0))_f)_f \\
&= (S^{23}(S^{12}(g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial}{\partial x_2}(g_3 \frac{\partial}{\partial x_3}(f)) - g_3 \frac{\partial}{\partial x_3}(g_2 \frac{\partial}{\partial x_2}(f)), 0))_f)_f \\
&= (S^{23}(S^{12}((g_1 \frac{\partial}{\partial x_1}, g_2 \frac{\partial g_3}{\partial x_2} \frac{\partial f}{\partial x_3} + g_2 g_3 \frac{\partial^2 f}{\partial x_2 \partial x_3} - g_3 \frac{\partial g_2}{\partial x_3} \frac{\partial f}{\partial x_2} - g_3 g_2 \frac{\partial^2 f}{\partial x_3 \partial x_2}, 0))_f)_f \\
&= (S^{23}(S^{12}(g_1 \frac{\partial}{\partial x_1}, g_2 g_3 \frac{\partial^2 f}{\partial x_2 \partial x_3} - g_3 g_2 \frac{\partial^2 f}{\partial x_3 \partial x_2}, 0))_f)_f \\
&= (S^{23}(S^{12}(g_1 \frac{\partial}{\partial x_1}, 0, 0))_f)_f \\
&= (S^{23}(0, 0, 0))_f \\
&= (0, 0, 0).
\end{aligned}
$$

■

Example 5.3. *Let M be an n-dimensional differentiable manifold and two vector fields are defined as $U = x_1^2 x_3 \frac{\partial}{\partial x_1}$ and $V = tan(x_3) \frac{\partial}{\partial x_2}$ on M. Then, the mapping*

$$
\begin{aligned}
S(U, V) &= ([U, V]f, 0), \\
&= ([x_1^2 x_3 \frac{\partial}{\partial x_1}, tan(x_3) \frac{\partial}{\partial x_2}]f, 0)
\end{aligned}
$$

is a set theoretical solution of Yang-Baxter equation on manifold M.

Each mapping must not satisfy manifold quantum Yang-Baxter condition. For example, we can define a mapping which does not verify this condition as follows.

Example 5.4. *Let M be a differentiable manifold, U and V be vector fields on M and f be smooth function. Then the mapping*

$$\begin{aligned} S(U,V) &= ([U,V]f,[V,U]f), \\ &= (-V_f,V_f) \end{aligned}$$

is not a set theoretical solution of Yang–Baxter equation on manifold M.
Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(U,V,W) &= ([U,V]f,[V,U]f,W), \\ S^{23}(U,V,W) &= (U,[V,W]f,[W,V]f) \end{aligned}$$

where W is a vector field on M. We need to satisfy the equation

$$((S^{12}\circ S^{23}\circ S^{12})(U,V,W)_f = ((S^{23}\circ S^{12}\circ S^{23})(U,V,W)_f$$

for each U,V,W vector fields. Then, we obtain

$$\begin{aligned} &(S^{12}\circ S^{23}\circ S^{12})(U,V,W)_f \\ &= (S^{12}(S^{23}(S^{12}(U,V,W)_f)_f)_f \\ &= (S^{12}(S^{23}([U,V]f,[V,U]f,W)_f)_f \\ &= (S^{12}(S^{23}(-V,V,W)_f)_f \\ &= (S^{12}(-V,[V,W]f,[W,V]f))_f \\ &= (S^{12}(-V,-W,W))_f \\ &= (S^{12}([-V,-W]f,[-W,-V]f,W))_f \\ &= (W,-W,W) \end{aligned}$$

and

$$\begin{aligned} &((S^{23}\circ S^{12}\circ S^{23})(U,V,W))_f \\ &= (S^{23}(S^{12}(S^{23}(U,V,W)_f)_f)_f \\ &= (S^{23}(S^{12}(U,[V,W]f,[W,V]f))_f)_f \\ &= (S^{23}(S^{12}(U,-W,W))_f)_f \\ &= (S^{23}(([U,-W]f,[-W,U]f,W))_f)_f \\ &= (S^{23}(W,-W,W))_f)_f \\ &= (S^{23}(W,[-W,W]f,[W,-W]f))_f \\ &= (W,0,0). \end{aligned}$$

As we can see

$$(W,-W,W)\neq(W,0,0)$$

by this way the mapping does not verify manifold quantum Yang-Baxter condition.

Furthermore we have the following theorem for preserving of set theoretical solution of Yang-Baxter equation under the $\Psi-$homomorphism.

Theorem 5.5. *The mappings in Theorem 5.2 preserve the Yang-Baxter condition in differentiable manifolds under the Ψ mapping where it is a Lie and additional homomorphism.*

Proof. It follows from Theorem 3.24. ■

6. Conclusion

Lie algebra is one of the important algebraic structure which has been extensively investigated by many researchers. This structure has an important role for different areas such as physics and geometry. In this study, we examine fundamental set theoretical solutions of Yang-Baxter equation in Lie algebras. We indicate that some set theoretical solutions of Yang-Baxter equation corresponds to the Lyubashenko set theoretical solutions on these structures. Then, we define $\circledast$-operation on this structure. In accordance with this, we prove that all set theoretical solutions which do not contain any constant element are preserved under the homomorphism. Moreover, we give an interpretation for these solutions from the point of geometrical view in Euclidean space, Minkowski space and differentiable manifolds by constructing a bridge among Lie algebras and these all geometrical structures. As a result of this study, we think that further researchers should focus on new set theoretical solutions of Yang-Baxter equation in many algebraic structures with geometrical approach such as MV-algebras, C-algebras, Jordan algebras and etc. Furthermore, they should use applications of these solutions in different areas such as physics, statistical mechanics, quantum groups, quantum mechanics, knot theory and etc.

Compliance with ethical standarts

Conflict of interest The author declares that he has no conflict of interests.
Human and animal participants This article does not contain any studies with human participants performed by any of the authors.

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