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On vertex integer-magic spectra of Caterpillar graphs

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Abstract

Consider any graph $G = (V(G), E(G))$ and k be any positive integer. Then a graph G is said to be $\mathbb{Z}_k$-vertex magic graph if there exist a map $l : V(G) \longrightarrow \mathbb{Z}_k - \{0\}$ such that for any vertex $v \in V(G)$, sum of the labels of vertices in the open neighborhood of v is a constant. ie, $\omega(v) = \sum_{u\in N(v)} l(u) = \mu$, $\forall v \in V(G)$. The set $VIM(G) = \{k \in \mathbb{Z}^+ | \text{G is } \mathbb{Z}_k - \text{vertex magic}\}$ is called vertex integer magic spectrum. In this paper, we determine VIM of caterpillar, super caterpillar and extended super caterpillar graphs.

Keywords

Super Caterpillar, Extended Super Caterpillar, vertex integer magic spectrum.

AMS Subject Classification

05C30.

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Contents

1. Introduction

Consider $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$ be a finite, simple, connected, undirected graph. We refer Hernstein [2] for group theoretic concepts and Bondy & Murthy [1] for graph-theoretic terminology. Throughout this paper $\mathbb{Z}_k$ denote the group of integer modulo k, where $k \geq 3$ unless otherwise mentioned and degree of a vertex v by $deg(v)$. we denote $[a,b]$ for set of all integer x such that $a \leq x \leq b$.

Any labeling $f : E(G) \longrightarrow A - \{0\}$, where A be an abelian group is said to be A–magic labeling of G if the induced map $f^+ : V(G) \longrightarrow A$ defined by $f^+(u) = \sum_{(u,v)\in E(G)} f(u,v)$ is a constant map and the graph G is called A–magic graph[10]. If $A = \mathbb{Z}_k$, then G is $\mathbb{Z}_k$-magic graph and the set $\{k \in \mathbb{Z}^+ | \text{G is } \mathbb{Z}_k - \text{magic}\}$ is called integer magic spectrum.

M.Doob studied A-magic graphs in ([3],[4]). In 2002, S-M Lee, Alexander Nien-Tsu Lee, Hugo Sun, and Ixin Wen [5] introduced integer magic spectrum of graphs. Several other researchers focused on studying $\mathbb{Z}_k$- magic graphs and integer magic spectra ([6],[7],[8], [9],[10], [11]). N.kamatchi, K.paramasivam, A.V Prajesh, K. Muhammed Sabeel, S. Arumugam [12] introduced group vertex magic labeling and obtained many necessary conditions for group vertex magic graphs. In 2014, Md.Forhad Hossain, Md. Momin AlAziz, and M.Kaykobad [13] studied graceful labelings of caterpillar, super caterpillar and extended super caterpillar graphs. In this paper, we introduced vertex integer magic spectrum of graphs and determine vertex integer magic spectrum of caterpillar, super caterpillar and extended super caterpillar graph.

Definition 1.1. *[12] For any abelian group A, a mapping $l : V(G) \longrightarrow A - \{0\}$ is said to be A-vertex magic labeling of G if there exists an element μ of A such that $\omega(v) = \sum_{u\in N(v)} l(u) = \mu$ for any vertex v of G,where $N(v)$ is the open neighborhood of v. A graph G that admits such a labeling is called A-vertex magic graph. If A=$\mathbb{Z}_k$, then G is called $\mathbb{Z}_k$ vertex magic graph and set of all positives integers for which G is $\mathbb{Z}_k$ vertex magic is called vertex integer magic spectrum.*

Definition 1.2. *[13] Supercaterpillar graph is a graph which formed by several arbitrary caterpillar graph $Ti, i = 0,1,...,m$, and one of the end backbone vertex in each caterpillar Ti be* *joined with root vertex v_{c_1} by an edge.*

Definition 1.3. *[13] Let there be any numbers of mp caterpillars. These caterpillars are grouped in p groups each having m caterpillars. Let backbones of each group i of caterpillars be connected to vertex v_{c_i} that is connected to root vertex v_r. Then the resulting tree is called an extended super-caterpillar.*

Theorem 1.4. *[8] A Graph G is $\mathbb{Z}_2$- magic if and only if every vertex of G has same parity.*

Theorem 1.5. *[14] The system of congruences $x \equiv b_i \ (mod\, m_i)$, $m_i \neq 0$, $i \in [1,p]$ has solution iff $gcd(m_i,m_j) \mid (b_i - b_j)$, $\forall i,j \in [1,p]$.*

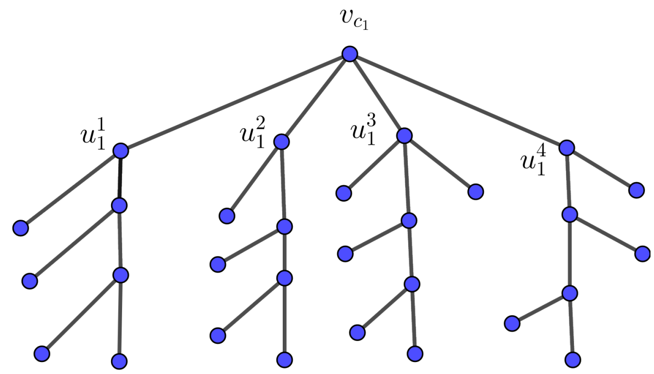

Figure 1. *A super caterpillar graph.*

2. Main Results

A Graph G is $\mathbb{Z}_2$- vertex magic iff each vertex $v \in V(G)$ has same parity. This result is straight forward and obtaining similar results for $\mathbb{Z}_k, k > 2$ seems to be difficult. Caterpillar graph is a graph in which elimination of all end vertices produces a path graph. In this present work, we investigate the integer vertex magic spectrum of special class of trees such as caterpillar, super caterpillar, extended super caterpillar

Lemma 2.1. *Let a be an element in Z_k. Then for any $n \geq 2$, $a = \sum_{i=1}^{n} g_i$ for some $g_i \in Z_k - \{0\}$.*

Theorem 2.2. *Any caterpillar graph G is $\mathbb{Z}_k$ vertex magic graph iff G has no vertex of degree two.*

Proof. Let $v \in V(G)$ such that $deg(v) = 2$. Then $\exists u_i, u_{i+1} \in V(G)$ such that $u_i \in N(u_{i+1})$ with $deg(u_i) = 2$ and $deg(u_{i+1}) > 2$. If not G must be a path graph. Let u_{i-1} be the other vertex adjcent to u_i, then $\omega(u_i) = l(u_{i+1}) + l(u_{i-1}) = \mu + l(u_{i-1}) = \mu$. Hence $l(u_{i-1}) = 0$, which is a contradiction.
Conversly, let $w_1^i, w_2^i, \ldots w_{n_i}^i$ are the pendant vertices of u_i. It follows from lemma 2.1, $k - g = \sum_{r=1}^{n_i} g_r^i$, $0 = \sum_{r=1}^{n_i} h_r^i$ for $i \in [1,n]$.
Now, define $l : V(T) \longrightarrow \mathbb{Z}_k - \{0\}$ by
$l(u_i) = g$ for $i \in [1,n]$
$l(w_r^i) = g_r^i$ if $r \in [1,n_i]$, $i \in [2,n-1]$ and $deg(u_i) \geq 4$
$l(w_1^i) = k - g$ if $i \in [2,n-1]$ and $deg(u_i) = 3$
$l(w_r^1) = h_r^1$, for $r \in [1,n_1]$
$l(w_r^n) = h_r^n$, for $r \in [1,n_2]$
Then G is $\mathbb{Z}_k$ vertex magic graph with magic constant g. □

Clearly the above theorem holds for $k = 1$, since any integer can be written as sum of 2 or more nonzero integers. Thus any caterpillar graph G is $\mathbb{Z}$-vertex magic graph iff G has no vertex with degree exactly 2.

Corollary 2.3. *The vertex integer magic spectrum of a caterpillar graph G,*

$$VIM(G) = \begin{cases} \phi & \text{if } G \text{ has vertex of degree } 2 \\ \mathbb{N} - \{2\} & \text{if } G \text{ has no vertex of degree } 2 \end{cases}$$

Theorem 2.4. *Super caterpillar graph G is $\mathbb{Z}_k$ vertex magic graph iff G has no vertex of degree 2 and $gcd(deg(v_{c_1}) - 1, k) > 1$.*

Proof. Let super caterpillar graph G is $\mathbb{Z}_k$ vertex magic graph with magic constant $\mu = g$. Suppose G has atleast a vertex with degree two. Let $v \neq v_{c_1}$ such that $deg(v) = 2$. Then $\exists u_i \in V(G)$ such that $u_i \in N(u_{i+1})$ with $deg(u_i) = 2$ and $deg(u_{i+1}) > 2$. suppose not, then u_i be vertex of caterpillar T_i which is a path. If $deg(u_{i+1}) > 2$ then $l(u_{i+1}) = g$,

$$\begin{aligned} \omega(u_i) &= l(u_{i+1}) + l(u_{i-1}), \text{ where } u_{i-1} \in N(u_i) \\ &= g + l(u_{i-1}) = g. \end{aligned}$$

Thus $l(u_{i-1})$=0 , which contradicts the fact that G is group vertex magic.
If $deg(v_{c_1}) = 2$ then $\omega(v_{c_1}) = l(u_1^1) + l(u_1^2) = 2g \neq g$.
Now, all vertices u_1^i of T_i for $i \in [1,m]$ must have pendant vertex. Otherwise u_1^i is of degree 2. Then $l(u_1^i) = g$ for all $i \in [1,m]$ and $\omega(v_{c_1}) = deg(v_{c_1})g = mg = g$. Therefore $o(g)$ divides both $m-1$ and k. Hence $gcd(m-1,k) > 1$, where $deg(v_{c_1}) = m$.
Conversly, we label all vertices of T_i for $i \in [1,m]$ same as in above theorem 2.2 except the pendant vertices of u_1^i for $i \in [1,m]$. Label all pendant vertices of u_1^i such that $\omega(u_j^i) = g$ and $l(v_{c_1}) = g$. Now, if we choose the nonzero element g in such a way that $o(g) = d$, where $d = gcd(deg(v_{c_1}) - 1, k)$. Then $\omega(v_{c_1}) = deg(v_{c_1})g = g$. Thus $\omega(v) = g$ for all $v \in V(G)$.

□

Corollary 2.5. *Let G be a super caterpillar graph with $deg(v_{c_1}) = m$ and $m - 1 = p_1^{r_1}.p_2^{r_2} \ldots p_s^{r_s}$, where p_i for $i \in [1,s]$ be prime. Then,*

$$VIM(G) = \begin{cases} \phi & \text{if } G \text{ has vertex of degree } 2 \\ \bigcup_{i=1}^{s} p_i\mathbb{N} - \{2\} & \text{if } G \text{ has no vertex of degree } 2 \end{cases}$$

Proof. We have $deg(v_{c_1}) = m$, and for any $n \in \mathbb{N}$, $gcd(m-1, np_i) > 1$ for $i \in [1,s]$.

□

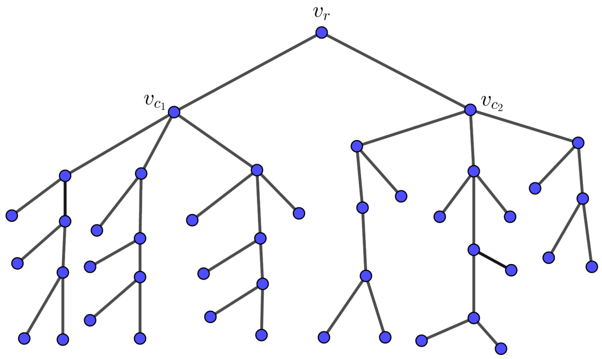

Figure 2. *An extended super caterpillar graph.*

Theorem 2.6. *Consider an extended super caterpillar graph G without a vertex $v \neq v_r$ with $deg(v) = 2$ is $\mathbb{Z}_k$-vertex magic graph if*

(i). $deg(v_{c_i})$ is same for all $i \in [1, p]$.

(ii). $gcd(deg(v_{c_1}) - 1, deg(v_{c_2}) - 1, .., deg(v_{c_p}) - 1, k) > 1$.

Proof. (i). Suppose $deg(v_{c_i}) = s$ for all $i \in [1, p]$. Define $l : V(T) \longrightarrow \mathbb{Z}_k - \{0\}$ by $l(v_r) = k - sg + g$ and It follows from lemma 2.2 that v_{c_i} for all $i \in [1, p]$ label in such a way that $\sum_{i=1}^{p} l(v_{c_i}) = g$ and label $v \neq v_r \in N(v_{c_i})$ such that the sum of the labels on the pendant vertices equals to $k - l(v_{c_i})$ for all $i \in [1, p]$. Thus $\omega(v_r) = g$ and $\omega(v_{c_i}) = g$ for all $i \in [1, p]$. Now label all other vertices of each super caterpillar as same as theorem 2.4 such that $\omega(v) = g$ for all $v \in V(G)$. Thus G is $\mathbb{Z}_k$-vertex magic graph with magic constant $\mu = g$.

(ii). Let $d > 1$ be $gcd(deg(v_{c_1}) - 1, deg(v_{c_2}) - 1, .., deg(v_{c_p}) - 1, k)$. Then $\exists g \in \mathbb{Z}_k$ such that $o(g) = d$. Now label all vertices as in (i) except root vertex v_r. let $l(v_r) = g$ and $N(v_r) = \{v_{c_i} : i \in [1, p]\}$, thus $\omega(v) = g$ for all v except $v \in N(v_r)$. Since, $d \mid (deg(v_{c_i}) - 1)$ which implies $\omega(v_{c_i}) = deg(v_{c_i})g = g$.

□

Theorem 2.7. *Consider an extended super caterpillar graph G without a vertex $v \neq v_r$ with $deg(v) = 2$ is $\mathbb{Z}_k$-vertex magic graph iff there exist some prime q such that $q \mid k$ and $q \mid (deg(v_{c_i}) - deg(v_{c_j}))$ for all $i, j \in [1, p]$.*

Proof. Label all vertices except v_r as same as in theorem 2.6. Then $\omega(v) = g\ \forall v$ except $v \in N(v_r)$. Let $l(v_r) = x$, then G is $\mathbb{Z}_k$-vertex magic graph iff the system of linear congruences $(deg(v_{c_i}) - 1)g + x \equiv g \pmod{k}$, ie. $x \equiv -(deg(v_{c_i}) - 2)g \pmod{k}$ for $i \in [1, p]$ has solution. Now by theorem 1.6, G is $\mathbb{Z}_k$-vertex magic graph iff $(deg(v_{c_i}) - deg(v_{c_j}))g \equiv 0 \pmod{k}\ \forall i, j \in [1, p]$. Therefore $o(g) \mid (deg(v_{c_i}) - deg(v_{c_j}))$ for all $i, j \in [1, p]$. Thus $\exists$ prime q such that $q \mid k$ and $q \mid (deg(v_{c_i}) - deg(v_{c_j}))$ for all $i, j \in [1, p]$, since $o(g) > 1$.

□

Corollary 2.8. *Consider an extended Super caterpillar graph with no vertex $v \neq v_r$ such that $deg(v) = 2$. Then,*

$$VIM(G) = \begin{cases} \mathbb{N} - \{2\} & \text{if } deg(v_{c_i}) \text{is same for all } i \in [1, p] \\ p_i\mathbb{N} - \{2\} & \text{if } p_i \mid (deg(v_{c_i}) - 1) \text{for all } i \in [1, p] \\ q\mathbb{N} - \{2\} & \text{if } q \mid (deg(v_{c_i}) - deg(v_{c_j})) \\ & \text{for all } i, j \in [1, p] \end{cases}$$

3. Conclusion

In this paper, we give a charecterization of $\mathbb{Z}_k$-vertex magic labeling of caterpillar, super Caterpillar and extended super caterpillars graphs. We also determined vertex integer magic spectrum of these graphs.

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