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Analytic and geometric aspects of Laplace operator on Riemannian manifold

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Abstract
In the past decade there has been a flurry of work at intersection of spectral theory and Riemannian geometry. In this paper we present some of recent results on abstract spectral theory depending on Laplace-Beltrami operator on compact Riemannian manifold. Also, we will emphasize the interplay between spectrum of operator and geometry of manifolds by discussing two main problems (direct and inverse problems) with an eye towards recent developments.

Keywords
Spectrum, eigenvalue, Laplacian, spectral geometry, isospectral manifolds.

AMS Subject Classification
47A10, 58C40, 53C20, 58J50, 58J53.

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Contents

1. Introduction

Let M be a compact, connected Riemannian manifold. Let $\varphi \in L^2(M)$ space of all square integrable real value on M .We define Laplace-Beltrami operator $\triangle\varphi = -divgrad\varphi$ where div is divergence , grad is the gradient or simply we write $\triangle\varphi = -\bigtriangledown(\bigtriangledown\varphi)$ which is differential unbounded self-adjoint operator.
The inner product is defined by $\langle\varphi,\psi\rangle = \int_M \varphi.\psi dV$,where V is volume form of M. In local coordinates $\{x_i\}$, the Laplace-Beltrami is defined by
$\triangle_g f = \frac{-1}{\sqrt{g}} \sum_{j,k} \frac{\partial}{\partial x_j}\left(g^{jk}\sqrt{g}\frac{\partial}{\partial x_k} f\right)$ where $g = |g_{jk}|$, $g^{jk} = (g_{jk})^{-1}$,
f is smooth function on M. We will discern $\triangle$ with metric when Laplace operator is associated by metric, we write $\triangle_g$. Particulary in Euclidean case the form is written as $\triangle f = -\sum_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} f$, f is a smooth function on R^n.
Suppose that M is compact Riemannian manifold, we will deal with a class of eigen value problems as follows
Closed problem $\triangle\varphi = \lambda\varphi$ in M $\quad \partial M = \phi$
Dirchlet problem $\triangle\varphi = \lambda\varphi$ in M $\quad \varphi|_{\partial M} = 0$
Neumann problem $\triangle\phi = \lambda\phi$ in M $\quad \frac{\partial\varphi}{\partial N}|_{\partial M} = 0$
Where N is outward oriented unit vector field normal to boundary. The discrete set of all eigenvalue λ_j with multiplicity m_j; $j = 1,2,3,\dots$ is spectrum of $\triangle_g$ and its denoted by $spec(M)$ or $spec(\triangle_g)$.
$spec(M) = \{\lambda_j(M)\}$ such that $\triangle_g(\varphi_j) = \lambda_j\varphi_j$, φ_j is called eigen function.
The relationship between geometric structure of manifolds and spectrum of differential operators created a new concept which is spectral geometry. In the case of Laplace -Beltrami operator on closed Riemannian manifold this field sets two questions.
(1) Direct problem
(2) Inverse problem

Direct problem discusses how spectrum can be determined from Riemannian manifold from this point on many inequalities have been established like Cheeger and Cheng inequality see [2].
Inverse problem seeks to identify features of geometry from information about Laplace's spectrum, some results are appeared in inverse problem when Milnor [13] gave answer of the question that Kac posted see [10] , the analogy of this question is " Is the spectrum of associated on smooth function Laplacian determine the shape of manifold? "In general, Sunada rise to give examples which clarifies iso-Spectral manifolds see [16]. This paper is covered by good references for Riemannian geometry see [5]. For spectral geometry see [6]. We refer to [12] for bounds of eigenvalues on Riemannian manifolds and [4] for general review in isospectral manifolds.

2. Preliminaries

Definition 2.1. *An n-dim manifold M is second countable, Hausdroff space for which every point $p \in M$ has a neighbourhood U_p homemorphic to an open subset of R^+, the complement of int(M) is boundary of M and denoted by ∂M . It should be noted that the term "compact manifold" often implies "manifold without boundary," which is the sense in which it is used here. When there is need for a separate term, a compact boundaryless manifold is called a closed manifold. also, it can be superimposed by local charts.*
The mapping $\phi : U \to M$ is a local chart if it is bijective and smooth. In addition to that, the Jacobian matrix of ϕ has to have full rank. Furthermore, the point $x \in \phi^{-1}(p)$ is the local coordinate of $p \in M$. Now we assume that the function $\phi : M \to R$ takes points from M and maps them to R . One way to apply that function to the parameter space is to use the local chart to convert it into local coordinates, so that $\tilde{f} = fo\phi$.

Example 2.2. $S^m = \{x \in R^{m+1} \big| \parallel x \parallel = r\}$ *m− sphere is smooth manifold.*
$T^m = S^{11} \times \ldots \times S^{11}$ *m− dim tours (closed surface defined as product of m circles).*

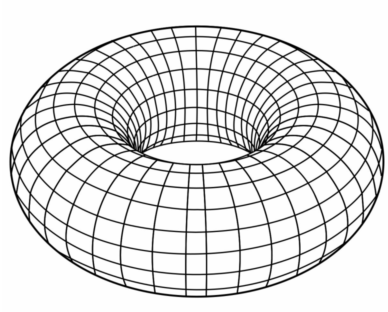


Figure (1)

Definition 2.3. *Tangent vector is the derivation on $C^\infty(M)$ $X : C^\infty(M) \to R$ such satisfies Leibnitz rule $X(fg) = X(f).g(x) + f(x).X(g)$ for $f, g \in C^\infty(M), x \in M$ the set of all derivations is n- dimensional tangent denoted by $T_x(M)$ the disjoint union of tangent spaces is tangent bundle $TM = \bigcup_{x \in M} T_xM$.*

Definition 2.4. *Riemannian metric is a smooth map $g : T_x(M) \times T_x(M) \to R$ which associates each point $x \in M$ by scalar product $g(x)(,)$. Riemannian manifold is smooth and is denoted by (M, g).*

Example 2.5. ***Example of Riemannian Metric Space***
The simple example is canonical metric when $M = R^n$ for $x \in R^n$, $X, Y \in T_x(R^n)$
$X = (x_1, x_2, \ldots, x_n)$ *and* $Y = (y_1, y_2, \ldots, y_n)$
$< X, Y >= \sum_{i=1}^{n} x_i y_i$

Definition 2.6. *Vector field of manifold is a smooth map $M \to TM$(i.e. $x \to T_x(M)$).*
Definitions of some spaces that we will use later
$C^\infty(M) = \{f : M \to R | f \;\; is \;\; smooth\}$
$C_0^{\inf}(M) = \{f \in C_\infty(M) | supp(f) \;\; is \;\; compact\}$
A space of all square integrable real valued functions on M is $L^2(M)$ such that inner product on $L^2(M)$ is defined as $< f_1, f_2 >= \int_M f_1(x) f_2(x) v_g$ *for* $f_1, f_2 \in L^2(M)$

Definition 2.7. *Laplace-Beltrami $\triangle_g$ is unbounded, self-adjoint operator on $C^\infty(M)$ takes the formula $\triangle_g f = -divgrad f$ where div is divergence of vector field V of M given by $divV = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(\sqrt{g}V^i)$, grad is the gradient such that*
$gradf = \sum_i \sum_j g^{ij} \frac{\partial \tilde{f}}{\partial x^j}$ *where* $V = \sum_i V^i \frac{\partial}{\partial x_i} \quad g = detg_{ij}$
Laplace Beltrami is denoted by $\triangle_g$ with the form
$\triangle_g f = -divgrad \; f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\sqrt{g}.g^{ij}.\frac{\partial}{\partial x^j} f)$, $f : M \to R$, $g^{ij} = (g_{ij})^{-1}$.

Definition 2.8. *The discrete set of eigenvalues (λ_j) which satisfies the equation $\triangle_g(\phi_j) = \lambda_j(\phi_j)$ where $\phi_j \in C^\infty(M)$, $j = 1, 2, \ldots$ is called spectrum of $\triangle_g$ and denoted by $spec(M) = \{\lambda_j(M)\}, \phi_j$ is called eigen function.*

We need some definitions of sobolev spaces
$H^1M = \{f \in L^2(M) \big| |df| \in L^2(M)\}$
for $f, g \in H^1(M)$ $\langle f, g \rangle = \int_M fgdV + \int_M \langle df, dg \rangle dV$ *where* $dV = \sqrt{g} dx^1 dx^2 \ldots dx^n$ *is canonical measure of* (M, g), $dx^1 dx^2 \ldots dx^n$ *is standard Lebesgue measure of* R^n.
$H_0^1(M) = \{f \in H^1(M) \;\; \exists \;\; f_n \in C_0^\infty(M) \| f_n - f \|_1 \to 0$ *as* $n \to \infty\}$ *i.e.* $H_0^1(M)$ *is closure of* C_0^∞ *in* $H^1(M)$.

Definition 2.9. *Let M be a smooth Manifold and $\Gamma(TM) =$ is the space of all vector fields of M .The connection ∇ on*

M is bilinear map $\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ defined by $(X,Y)\to\nabla_{XY}$, ∇ is said to be Levi- Civita if
(1) For all smooth vector fields X,Y of M the next relation is hold $\nabla_X Y-\nabla_Y X=[X,Y]$
(2) ∇ is compatible with metric g. i.e. for all smooth vector fields X,Y,Z.
(3) $Xg(Y,Z)=g(\nabla_{X,Y},Z)+g(Y,\nabla_{X,Z})$
Where $[X,Y]$ is Lie bracket given by $[X,Y](f)=X(Y(f))-Y(X(f))$, for all $f\in C^\infty(M)$. We can regrade each smooth vector field X on manifold M as differential operator on $C^\infty(M)$.

Definition 2.10. ***Curvature on Riemannian manifold***
Let ∇ be Levi – Civita connection associated to the metric g , let X,Y be smooth vector fields on M. the Curvature $R(X,Y)$ is a map from the set of all vector fields on M into itself, so for U vector field.
$R(X,Y)U=\nabla_X(\nabla_Y U)-\nabla_Y(\nabla_X U)-\nabla_{[X,Y]}U$
For four vector fields we have the formula $R(X,Y,Z,W)=g(R(X,Y),W,Z)$

In this paper we will use Ricci curvature by the equality $Ricci_x(V,V)=\sum\limits_{j=1}^{n}R(V,e_i,V,e_i)$ where
$V\in T_x(M), x\in M, (e_i)_{i=1,2,\ldots n}$
orthogonal basis of vector space $T_x(M)$, also we refer to scalar curvature $R(x)$ of M by $R(x)=\sum\limits_{i=1}^{n}Ricci_x(e_i,e_i)\in R$.

3. Standard Result About Spectrum

Let M be a compact Riemannian manifold with boundary ∂M (possibly empty) suppose one of mentioned eigenvalue problems, then
(1) Spectrum consists of infinite sequence of eigenvalues $0\prec\lambda_1\le\lambda_2\le\ldots\le\ldots$
Where 0 is not an eigenvalue of Dirichlet problem. Each eigenvalue has finite multiplicity and eigenvectors corresponding to distinct eigenvalues are L^2 orthogonal.
(2) Each eigenvalue is C^∞ smooth analytic.

4. Properties and Estimates

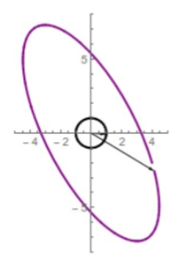
Given a compact Riemannian Manifold (M,g) can we find the spectrum $\{\lambda_k(M)\}_{\{k\ge 0\}}$ of M , this question comes under **Direct problem**.
In fact, we can discern that the explicit computation of spectrum is not easy task, there are few examples where the spectrum of manifold is known, like (sphere, flat tori, balls), for this reason will describe some of estimates of spectrum:
The aim is to find a_k and b_k depending on geometrical invariants $a_k\le\lambda_k\le b_k$ we will focus on the boundary of λ_1 for that we will introduce Min-Max principle
$R(f)=\frac{\int_M|\nabla f|^2dV}{\int_M f^2dV}=\frac{\int_M\triangle f.fdV}{\int_M f^2dV}$ is called Rayleigh form on (M,g)

If f is eigenfunction associated to eigenvalue then it takes the formula
$R(f)=\frac{\int_M\triangle f.fdV}{\int_M f^2dV}=\lambda$
$f\in H^1(M)$ in case of Neumann problem, $f\in H_0^1(M)$ in case of Dirichlet problem.

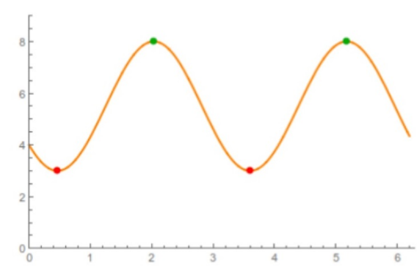
Definition 4.1. *minimax principle*
For each spectral problem and for $k\ge 1$ we have
$\lambda_k(M)=\min\limits_{\substack{E\subset H_0^1(M)\\ dim(E)=k}}\max\limits_{f\in E}\{R(f)\}$*for Dirichlet problem*

and $\lambda_k(M)=\min\limits_{\substack{E\subset H^1(M)\\ dim(E)=k}}\max\limits_{f\in E}\{R(f)\}$ *for Neumann problem.*

For ease, we will serve the explanation of min max principle for another operator
For example, let take matrix of dim (2), $A=\begin{pmatrix}4&-2\\-2&7\end{pmatrix}$
both operators $\triangle$ and A are self-adjoint.
The eigenvalues are $\lambda_1=3,\lambda_2=8$ with corresponding normalized eigenvector $f_1=\frac{1}{\sqrt5}(2,1)$, $f_2=\frac{1}{\sqrt5}(-1,2)$
$R(f)=\frac{\langle Af,f\rangle}{\langle f,f\rangle}=\langle Af,f\rangle$, the Rayleigh quotient formula is constant in any subspace E of dimension 1 and 2.
In the figure (2),(3) the Rayleigh quotient has its minimum value when f_1 equals the value will be at 3 .we find $maxR(f)$ is 8 this exactly when $f=f_2$, $\lambda=8$.



$figure(2)$
Unitvectors



$figure(3)$
Rayleighquotient

For example, of how geometry control the spectrum, Cheeger inequality appears the lower bound of first eigenvalue if (M,g) a compact ,connected Riemannian Manifold , and consider one of eigenvalue problem, Ω is bounded regular subset of M ,Cheeger constant is defined as $h(M,g)=infh(\Omega,g)$ where $h(\Omega,g)=\frac{vol(\partial\Omega,g)}{vol(\Omega,g)}$, $\partial\Omega$ the boundary of Ω with vol(n-1), such that $vol(\Omega)\le\frac{vol(M,g)}{2}$ we have $\lambda_1(M,g)\ge\frac{h^2(M,g)}{4}$.
For the upper there are classical results on eigenvalues on surfaces. [17] gave an upper bound for the first Neumann eigenvalue λ of the Laplacian on a simply connected bounded Ω domain . Where $\varepsilon=1.8412$ is the constant related to the first zero of some Bessel function. The equality holds if and only if Ω is a disk.

$\lambda_1 \leq \frac{\varepsilon^2\pi}{S(\Omega)}.S(\Omega)$ is area of Ω.
There are generations of the Szego inequality. H. Weinberger [18] generalized Szego result to higher dimensions, that is, for a simply connected bounded domain $\Omega \in R^n$ the first Neumann eigenvalue λ_1 satisfies $\lambda_1 \leq \frac{c}{vol(\Omega)^{\frac{2}{n}}}$ where c is a constant related to the volume of unit ball, and the equality holds if and only if Ω is a ball. The proof of the this Szego-Weinberger inequality is based on the minimax principle and classical analysis.

For **Inverse problem** the data of spectrum doesn't determine the shape of manifold however, we will interduce some of positive results to show the geometric effect that we can take out from spectral invariant.

5. Application of heat kernel in Riemannian geometry

Definition 5.1. ***Spectral Partition Function***
Let (M,g) be a compact Riemannian manifold ,consider we have unbounded operator (in our case) Laplace operator on $C^\infty(M)$, let $\{\lambda_k(M)\}$ the spectrum of Laplace i.e. $spec(\triangle_g) = \{\lambda_M\}_{\{k\geq 1\}}$
One of the important tools in study of spectrum of Laplace operator is heat kernel.
The map $H(x,y,t) \to M \times M \times R^+ \to R$ is heat kernel if
(1) H is C^0 in x,y,t variables, C^2 in variable y, C^1 in variable t.
(2) $lim_{t\to 0^+} H(x,y,t) = \partial_x(y)$, $lim_{t\to 0^+}\int_M H(x,y,t)f(y)dy = f(x)$
(3) $\frac{\partial H(x,y,t)}{\partial t} = \triangle_{g,y} H(x,y,t)$ where f is compactly supported function, $\triangle_{g,y}$ is Laplace operator for second variable y.
Mathematical relationship between heat kernel and spectrum of Laplace is regarded by $H(x,y,t) = \sum_k e^{-\lambda_k t} e_k(y).e_k(x)$ this formula helps to describe
The spectral partition function $Z(t) = \sum_k e^{-\lambda_k t}$ as spectral invariant.

$$\begin{aligned}
\int_M H(x,y,t)dx &= \int_M \sum_k e^{-\lambda_k t} e_k(x).e_k(x)dx \\
&= \int_M \sum_k e^{-\lambda_k t} e_k^2(x)dx \\
&= \sum_k e^{-\lambda_k t}\int_M e_k^2(x)dx \\
\sum_k e^{-\lambda_k t}\|e_k\|_{L^2}^2 &= \sum_k e^{-\lambda_k t}
\end{aligned}$$

Definition 5.2. *The Minakshisundaram-pleijel expansion*
Let (M,g) be closed Riemannian Manifold of dim n, asymptotic expansion of heat trace is $H(x,x,t) = (4\pi t)^{(\frac{n}{2})}(\alpha_1 t + \alpha_2 t^2 + \ldots)$ where α_j integral over M depend on curvature and covariant derivative , it's difficult to compute all formulas α_j but we get some of t. Then $\alpha_0 = vol(M)$, $\alpha_1 = \frac{1}{6}\int_M S$
Where S is scalar curvature.
$\alpha_2 = \frac{1}{360}\int_M 5S^2 - 2(Ric)^2 - 10|K|^2$ where Ric is Ricci curvature.
K is main curvature. By Minakshisundaram-pleijel expansion we can see that dim, vol, scalar curvature is known by spectrum.
***Note**: If is two dim then by Gauss –Bonnet theorem we get Euler characteristic of M is also spectral invariant.*
$\alpha_0 = vol(M), \alpha_1 = \frac{\pi}{3}\chi(M)$

Definition 5.3. *Vardhan's formula*
Vardhan's formula is used to be another application of heat kernel in Riemannian geometry.
$lim_{t\to 0^+} tlogE(x,y,t) = \frac{-d_M^2(x,y)}{4}$ where $d_M(x,y)$ is Riemannian distance between x and y. We can see from above examples that the Spectrum invariant is defined in terms of Riemannian manifold (M,g).

For negative results in inverse problem we will extend some of iso-spectral non-isometric Manifolds, first let's define isometric and isospectral manifolds

Definition 5.4. *Two manifolds M,N are isometric if there is a diffeomorphism (diffeomorphism is a map between manifolds which is differentiable and has differentiable inverse.) such that Riemannian metric from M pull back to metric on N .*

Definition 5.5. *Two closed Riemannian manifolds are said to be iso-spectral if the eigenvalues 0f their (Laplace-Beltrami operator) counted multiplicities coincide.*

Note: Isometric manifolds are Isospectral manifolds.

6. Isospectral manifolds

We will refer to outline of constructing isospectral manifolds
(*) Direct computation
(*) Representation theorem
(*) Riemannian submersion

6.1 First Method : (Direct computation)

There are a few examples of manifolds which we can compute the spectrum by direct computation one of them is Rectangle with Dirichlet conditions and $[0,1]\times[0,1]$ is the domain in R^2, $sin(n_1\pi x)sin(n_2\pi y)$ $n_1,n_2 \in N^*$ form a Hilbert basis of $\{f \in L^2([0,1]\times[0,1]); f(1) = 0\}$
$spec(M,g) = \{\pi^2(n_1^2+n_2^2)\}$ eigenvectors are $sin(n_1^2\pi)sin(n_2^2\pi)$ $n_1,n_2 \in N^*$ where $M = [0,1]\times[0,1]$ is Riemannian manifold with canonical metric.

Example 6.1. ***Milnor's counter***
There are two lattices Γ,Γ' of R^{16} such that the associated tori $T^{16}(\Gamma)$ and $T^{16}(\Gamma')$ are isospectral,but not isometric.

$T^{16}(\Gamma)$ is flat tori-quotients of Euclidean space R^{16} by lattice Γ of full rank. $T^{16}(\Gamma^{'})$ is flat tori-quotients of Euclidean space R^{16} by dual lattice $\Gamma^{'}$.
Milnor's work [13] is not a direct answer to the original version of Kac's [10] question because it is not concerned with the planar domains. But from Milnor's example one learned that there really exist non-congruent domains with the same eigenvalue spectrum.

6.2 Second method : (Representation method)

Let G be locally compact group acts on Riemannian manifold. If Γ_1, Γ_2 are cocompact subgroups of group G . g be a metric on G acts on G if Γ_1, Γ_2 are representation equivalent then $spec(\Gamma_1|G) = spec(\Gamma_2|G)$.
Two co-compact subgroup of G is said to be representation equivalent if there exists a unitary isomorphism $T : L^2(\Gamma_1|G) \to L^2(\Gamma_2|G)$ and $T(\rho\Gamma_1(x))T^{-1} = \rho\Gamma_2(x), x \in G$
$L^2(\Gamma_1|G)$ is the space of measurable square integrable functions with respect to haar measure on $(\Gamma_1|M)$. We refer to $\rho\Gamma_1(x)f = f \circ R_{\Gamma,x}$ where $R_{\Gamma,x}$ right translation operator on $\Gamma_1|G$ i.e. $R_{\Gamma,x}(\Gamma x) = (\Gamma xa)$ for all $a \in G$ and $f \in L^2(\Gamma_1|G)$.

Theorem 6.2. *If $\Gamma_1\Gamma_2$ are representation equivalent subgroups of group G, acts on G compact Riemannian Manifold then $spec(\Gamma_1|M) = spec(\Gamma_2|M)$.*

Note: We call the constructed isospectral manifolds by this method strongly isospectral manifold.

6.3 Third method : (Riemannian submersions method)

The recent development is Riemannian submersions method, the surjective differentiable map $\pi : M \to N$ between differentiable manifolds is submersion, this submersion is said to be Riemannian if $ker(\pi^*)^1 \to TN$ is isometry. TN is tangent bundle, we called $ker(\pi^*)^1$ vertical space and its orthogonal complement is horizontal space. Geodesics are curves which minimize length and distance between points on manifold, totally geodesic is any geodesic in M which starts tangent to fiber stays in the fiber.

Proposition 6.3. *Let $\pi : M \to N$ be a Riemannian submersion with totally geodesic then the Laplace on M and N Δ_M, Δ_N satisfy $\pi^*(\triangle_M(f)) = \pi^*(\triangle_N(f))$, where f is function on N in particular $spec(\triangle_M) = spec(\triangle(N))$.*
The main idea is giving by theorem
Let T be a torus of dimension greater than one, suppose that M_1, M_2 are compact Riemannian Manifolds with induced Riemannian metrics ,are totally geodesic flat tori. M_1/S and M_2/S are isospectral for every subtorus S and T of codimension ≤ 1 then M_1, M_2 are isospectral.

We will introduce example of isospectral manifolds.
$j : S^3 \times S^3 \to So(6)$, $So(6)$ the space of skew-symmetric 6×6 matrices.
$(x,y) \in R^3 \times R^3$, $z \in R^3$ define $j(z)(x,y) = (z \times x, z \times y)$ for $j^{'}(z)(x,y) = (z \times x, -z \times y)$
Where $u \times v$ denotes the vector cross product of $u, v \in R^3$, so j is isospectral to $j^{'}$.

7. Conclusion

We covered most of analytic and geometric aspects of spectrum of Laplace on Riemannian Manifold by the solution of direct problems which is typified by Cheeger, Cheng, Szegö inequalities.

This paper mainly illustrates the inverse problems and the way to discover the geometry of Riemannian manifold from spectral data this study still a very active field of research till now.

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