



On minimal Hausdorff frames

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Abstract

The concept of minimal Hausdorff topological spaces was studied and characterized by M.P. Berri. A compact Hausdorff space is minimal Hausdorff and such spaces are reversible in the sense that every continuous self bijection is a homeomorphism. In this paper we study minimal Hausdorffness in the context of pointfree topology. We introduce the notion of minimal Hausdorff frames and characterize them in terms of convergence of filters in frames. We also study the association between minimal Hausdorff frames and minimal Hausdorff topological spaces. An application is to prove that a minimal Hausdorff frame is a reversible frame in the sense that every order preserving self bijection is a frame isomorphism.

Keywords

Frame, minimal Hausdorff frame, subframe, sublocale, filter, clustered filter.

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1. Introduction

Garrett Birkhoff, in 1936, pointed out the notion of the comparison of two different topologies on the same basic set. He had done this by ordering these topologies as a lattice under set inclusion. A topological space (X, T) with property R is said to be minimal(maximal) R if T is a minimal(maximal) element in the set $R(X)$ of all topologies on the set X having property R with the partial ordering of set inclusions. The set of all topologies sharing a given property may not have a least or greatest element. But it may have minimal or maximal elements. There has been a fairly good amount of research on minimal properties as compared to maximal properties in the lattice of topologies. For a background on the results concerning minimal property, we suggest the reader to refer

to *A survey of minimal topological structures* by M.P. Berri, J.R. Porter and R.M. Stephenson, Jr.[3].

The concept of minimal topologies was first introduced by A.S. Parhomeko [15] in 1939. He proved that compact Hausdorff spaces are minimal Hausdorff. Later E. Hewitt [9] proved that compact Hausdorff spaces are maximal compact as well as minimal Hausdorff. A. Ramanathan [20], [21] proved the existence of noncompact minimal Hausdorff spaces and characterized all minimal Hausdorff spaces. The concept of reversible topological spaces was introduced by M. Rajagopalan and A. Wilansky [14]. In this paper it is proved that a space that is maximal or minimal with respect to some topological property is reversible and vice-versa.

An intense research has been done on maximal and minimal topologies so far. The pointfree counterpart of topological spaces i.e. frames has not received due attention in this regard. Hence it is relevant to study such properties in the context of pointfree topology. The Hausdorff property for frames is not yet successfully defined as to become an extension or equivalent of the classical Hausdorff axiom for topological spaces. Many forms of it were defined by Dowker and Strauss[5], A. Pultr[19], J. Rosický and B. Šmarda[22] and by Isbell[11]. The aim of this paper is to study the property of minimal Hausdorffness in frames and to answer some related questions. We also characterize such frames in terms of con-

vergence of filters in frames. The concept of reversible frames was introduced and characterized in [12]. A related question is the reversibility of such frames. We also prove that such frames are reversible in the sense that every order preserving self bijection on such frames is a frame isomorphism.

2. Preliminaries

The term frame was coined by C.H. Dowker and studied by D. Strauss[6]. A *frame* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$ holds for all $a \in L, S \subseteq L$. A map between frames that preserves arbitrary joins and finite meets is called a *frame homomorphism*. Associated with a frame homomorphism $h : M \rightarrow L$ is its right adjoint $h_* : L \rightarrow M$ given by $h_*(b) = \bigvee \{x \in M : h(x) \leq b\}$. We denote the top element and the bottom element of a frame by 1 and 0 respectively. The category of frames and frame homomorphisms is denoted by **Frm**. The dual category **Frm**^{op} is referred to as the category of locales denoted by **Loc**. The morphisms in **Loc**, called *localic maps*, are given by the right adjoints of frame homomorphisms between two objects. A frame is said to be *spatial*, if it is isomorphic to the topology ΩX of a topological space $(X, \Omega X)$.

A subset of a frame which is closed under arbitrary joins and finite meets in that frame is called a *subframe*. A *sublocale* M of a locale L can be represented in terms of an onto frame homomorphism $h : L \rightarrow M$ in the sense that the image of M under the right adjoint $h_* : M \rightarrow L$ will represent that sublocale. For a locale L , denote $\uparrow a = \{x \in L : x \geq a\}$ and $\downarrow b = \{x \in L : x \leq b\}$. Then the sublocale given by the frame homomorphism $j : L \rightarrow \uparrow a$ defined by $x \rightarrow a \vee x$ for any $a \in L$ is called a *closed sublocale* of L . A *cover* in a frame L is a subset S of L with $\bigvee S = 1_L$. A sublocale $h : L \rightarrow M$ is said to be *extension closed* if for every cover C of M there is a cover D of L such that $h(D) = C$.

A frame L is *almost compact* if whenever $\bigvee \{x_i : i \in I\} = 1$ then there exists a finite subset $K \subseteq I$ of the index set I such that $(\bigvee \{x_i : i \in K\})^* = 0$ where “*” denotes the pseudo-complementation operator in L . A frame L is said to be *compact* if each cover A of L has a finite subcover.

Let $\{L_i : i \in I\}$ be any class of locales for an index set I . Then $\bigoplus_{i \in I} L_i$ denotes the *locale product* or the *frame co-product*. For finite systems of two we write $L \oplus M$. Also denote $\bigoplus_{i \in I} a_i = \downarrow (a_i)_{i \in I} \cup 0$, where 0 is the bottom of the co-product. For finite systems of two we write $a \oplus b$. For a frame L set $d_L = \bigvee \{x \oplus y : x \wedge y = 0\} \in L \oplus L$.

The concept of convergence of filters in frames was introduced by S.S. Hong [10]. A *filter* in a frame A is a nonempty subset F with the property that $0 \notin F, a \geq b \in F$ implies $a \in F$, and $a \wedge b \in F$ whenever a and b are in F . A maximal filter is called an *ultrafilter*. A filter F in a frame L is said to be *clustered* if for any cover S of L , $sec F = \{x \in L : \text{for all } a \in F, a \wedge x \neq 0\}$ meets S . A filter F in a frame L is said to be *convergent* if for any cover S of L , F meets S .

A subset B of L is a *base* for L if for any $x \in L$ there exists a subset C of B with $\bigvee C = x$. A frame L is called a *separated*

frame if, for every filter F in L , $\bigvee(L - F)$ is either 1 or a dual atom.

For a detailed reading of the above concepts we refer the reader to [18].

3. Minimal Hausdorff Frames

A topological space (X, τ) is minimal Hausdorff if it is Hausdorff and there is no strictly weaker Hausdorff topology than τ on X . The property of being minimal Hausdorff is a topological property. A characterization for a minimal Hausdorff space is given in [4].

Theorem 3.1. [4] *A necessary and sufficient condition that a Hausdorff space (X, τ) be minimal Hausdorff is that τ satisfies the following property:*

1. Every open filter-base has an adherent point;
2. If an open filter-base has a unique adherent point, then it converges to this point.

Some results regarding minimal Hausdorff spaces are given in M. P. Berri [2]. The pointfree counterpart of topological spaces i.e. frames has not yet considered for carrying out a study on such maximal or minimal frame isomorphic properties such as Hausdorff property. The definition for Hausdorff frame as given by Isbell[11] is as follows. A frame L is called a *Hausdorff frame* if the diagonal $\Delta : L \rightarrow L \oplus L$ defined by $\Delta(a) = \{(x, y) : x \wedge y \leq a\}$ is a closed localic map. In this section, we introduce the concept of minimal Hausdorff frames and look forward to obtain a characterization for such frames. We know that $\{0, 1\}$ is a subframe of any frame and it is Hausdorff. We adopt the following convention to define what is called a proper subframe.

A *proper subframe* here means a frame which is a strict subframe of the frame under consideration other than the trivial frame $\{0, 1\}$.

Definition 3.1. *A frame L is said to be minimal Hausdorff if L is Hausdorff and no proper subframe of L is Hausdorff.*

The four element frame $B_4 = \{0, a, b, 1\}$ where $a \parallel b, a \vee b = 1, a \wedge b = 0$ is a Hausdorff frame as it is regular and is minimal Hausdorff by definition. Then any Boolean frame other than B_4 is not minimal Hausdorff since any such frame contains B_4 as a subframe. Thus B_4 is the only finite frame that is minimal Hausdorff.

When we come to infinite spatial frames, there are Hausdorff frames containing no Boolean frames as a proper subframe. For example, the set of all real numbers with usual topology, denoted by $(\mathbb{R}, \Omega\mathbb{R})$, is regular and hence the frame $\Omega\mathbb{R}$ is regular. Since every regular frame is Hausdorff, $\Omega\mathbb{R}$ is Hausdorff. But \mathbb{R} is a connected space and there are no *open and closed sets* other than \mathbb{R} and \emptyset . Hence $\Omega\mathbb{R}$ contains no Boolean frame as a proper subframe. Also $(\mathbb{R}, \Omega\mathbb{R})$ is coarser than discrete topological space. We will prove later that $\Omega\mathbb{R}$ is not



minimal Hausdorff. This reveals the presence infinite spatial Hausdorff frames which contains Hausdorff frames other than Boolean ones. The task of verifying minimal Hausdorffness is very complex in infinite case as verifying the presence of Hausdorff frames in such frames is practically very difficult. Hence we need some characterizations for minimal Hausdorffness in frames.

Let $\mathcal{D}(F) = \{A \subseteq F : \emptyset \neq A = \downarrow A\}$, where $\downarrow A = \{x \in L : x \leq u, u \in A\}$ and F is any bounded meet semilattice on a frame L . Then $\mathcal{D}(F)$ is a frame under set inclusion. The following remarks on $\mathcal{D}(F)$ are used in the proof of the characterization theorem for minimal Hausdorff frames.

Remark 3.1. Let $A, B \in \mathcal{D}(F)$. Then $\downarrow A = A$ and $\downarrow B = B$. Therefore $\downarrow (A \cap B) = A \cap B$. Hence $\downarrow ((A \cap B) \oplus (A \cap B)) = (A \cap B) \oplus (A \cap B)$. Now $d_{\mathcal{D}(F)} = \bigvee \{U \oplus V : U \cap V = \{0\}\}$. Then

$$\begin{aligned} \downarrow d_{\mathcal{D}(F)} &= \bigvee \{\downarrow (U \oplus V) : U \cap V = \{0\}\} \\ &= \bigvee \{\downarrow U \oplus \downarrow V : U \cap V = \{0\}\} = \bigvee \{U \oplus V : U \cap V = \{0\}\} \\ &= d_{\mathcal{D}(F)}. \end{aligned}$$

Therefore,

$$\downarrow [(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)} = [(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)}.$$

Remark 3.2. Let $A, B \in \mathcal{D}(F)$, Then $x \in A \cap B$ implies $x \in \{a \wedge b : a \in A, b \in B\}$. Conversely if $a \in A, b \in B$ and since $a \wedge b \leq a, b$, we have $a \wedge b \in \downarrow_L A = A, a \wedge b \in \downarrow_L B = B$. Thus $a \wedge b \in A \cap B$ and hence $A \cap B = \{a \wedge b : a \in A, b \in B\}$.

For the following theorem, see [18].

Theorem 3.2. A frame L is Hausdorff if and only if for any $a, b \in L, a \oplus b \leq ((a \wedge b) \oplus (a \wedge b)) \vee d_L$.

Now we prove the following theorem which helps to prove our main result in this section.

Theorem 3.3. Let L be a Hausdorff frame and let F be a bounded meet semilattice in L . Then the frame $(\mathcal{D}(F), \subseteq)$ is Hausdorff.

Proof. $\{(a \wedge b) \oplus (a \wedge b) : a \in A, b \in B\}$ is a join basis for $[(A \cap B) \oplus (A \cap B)]$, since $\{a \oplus b : a \in A, b \in A\}$ forms a join basis for $A \oplus B$.

If $U \cap V = \{0\}$ where $U, V \in \mathcal{D}(F)$,

by Remark 3.2 $U \cap V = \{u \wedge v : u \in U, v \in V\}$ and hence

$$\{0\} = U \cap V \Leftrightarrow u \wedge v = 0, \forall u \in U, v \in V$$

Thus $U \oplus V$ has join basis $\{u \oplus v : u \wedge v = 0, u \in U, v \in V\}$. Since L is Hausdorff, by Theorem 3.2, for any $a, b \in L$,

$$a \oplus b \leq [(a \wedge b) \oplus (a \wedge b)] \vee d_L \quad (3.1)$$

Now $[(a \wedge b) \oplus (a \wedge b)] \vee_{L \oplus L} d_L$

$$\begin{aligned} &= [(a \wedge b) \oplus (a \wedge b)] \vee_{L \oplus L} \left[\bigvee_{L \oplus L} \{x \oplus y : x \wedge y = 0\} \right] \\ &= \bigvee_{L \oplus L} \{[(a \wedge b) \oplus (a \wedge b)] \vee (x \oplus y) : x \wedge y = 0\} \end{aligned}$$

Then $[(A \cap B) \oplus (A \cap B)] \vee_{\mathcal{D}(F) \oplus \mathcal{D}(F)} d_{\mathcal{D}(F)}$

$$\begin{aligned} &= [(A \cap B) \oplus (A \cap B)] \vee_{\mathcal{D}(F) \oplus \mathcal{D}(F)} \\ &\quad \left[\bigvee_{\mathcal{D}(F) \oplus \mathcal{D}(F)} \{U \oplus V : U \cap V = \{0\}\} \right] \\ &= \bigvee_{\mathcal{D}(F) \oplus \mathcal{D}(F)} \{[(A \cap B) \oplus (A \cap B)] \vee_{\mathcal{D}(F) \oplus \mathcal{D}(F)} \\ &\quad (U \oplus V) : U \cap V = \{0\}\} \\ &= \bigvee_{L \oplus L} \left\{ \bigvee_{L \oplus L} \{[(a \wedge b) \oplus (a \wedge b)] \vee (x \oplus y) : x \wedge y = 0\} \right\} \\ &= \bigvee_{L \oplus L} \{[(a \wedge b) \oplus (a \wedge b)] \vee d_L : a \in A, b \in B\} \quad (3.2) \end{aligned}$$

Hence $[(a \wedge b) \oplus (a \wedge b)] \vee d_L$ forms a join basis for $[(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)}$. Since $[(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)}$ is a downset by Remark 3.1, we have

$$[(a \wedge b) \oplus (a \wedge b)] \vee d_L \in [(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)}$$

and hence $a \oplus b \in [(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)}$ by (3.1). Thus $A \oplus B \subseteq [(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)}$. Hence $\mathcal{D}(F)$ is Hausdorff, by Theorem 3.2. \square

We state the following theorem from [10] which we use to prove our next result.

Theorem 3.4. A filter F is convergent if and only if for any $C \subseteq B$ with $\bigvee C = 1$ where B a base for L, F meets C .

The following result is proved in [18].

Theorem 3.5. Let F be a semilattice and L be a frame. Let $f : F \rightarrow L$ be a semilattice homomorphism (L viewed, for a moment, as the semilattice $(L, \wedge, 1)$). Then there exists precisely one frame homomorphism $h : \mathcal{D}(F) \rightarrow L$ such that $h \circ \lambda_F = f$ where $\lambda_F : F \rightarrow \mathcal{D}(F)$ defined by $\lambda_F(x) = \downarrow x$.

From now a filter means any filter other than $\{1\}$ unless stated otherwise.

Lemma 3.1. Let L be any Hausdorff frame which contains a clustered filter that is not convergent. Then there exists a proper subframe of L which is Hausdorff.

Proof. Let F' be the clustered filter in L that is not convergent. Then $F = F' \cup \{0\}$ is a bounded meet semilattice in L . By Theorem 3.5, for every meet semilattice homomorphism $f : F \rightarrow L$, there is exactly one frame homomorphism $h : \mathcal{D}(F) \rightarrow L$ such that $h \circ \lambda_F(a) = f(a)$, namely the mapping given by $h(A) = \bigvee \{f(a) : a \in A\}$. Take $f = i : F \rightarrow L$, the inclusion map, then $h(A) = \bigvee A$.

Claim: $h : \mathcal{D}(F) \rightarrow L$ is not onto.

Suppose $h : \mathcal{D}(F) \rightarrow L$ is onto. Then for any $x \in L$ there exists $A \in \mathcal{D}(F)$ such that $\bigvee A = x$. Thus F is a base for the frame L . As any cover C of F meets F' , by Theorem 3.4, the filter F' is convergent which is a contradiction. Also if $h(\mathcal{D}(F)) = \{0, 1\}$, then $F = h \circ \lambda_F(F) \subseteq h(\mathcal{D}(F)) = \{0, 1\}$. Then $F = \{1\}$, a contradiction. Hence $h(\mathcal{D}(F))$ is a proper subframe of L that is Hausdorff by Theorem 3.3. \square



An element $x \in L$ is *dense*, if $a \wedge x \neq 0$, for all $0 \neq x \in L$. That x is dense if and only if $x^* = 0$. Denote by $D(L) = \{l \in L : l^* = 0\}$, the set of all dense elements of L , is a filter in L . Recall from the [10] that a filter F in L clusters if and only if $\bigvee \{x^* : x \in F\} < 1$. Since $\bigvee \{x^* : x \in D(L)\} < 1$, $D(L)$ clusters.

We state the following theorem from [17] for proving our main theorem on characterization of minimal Hausdorff frames.

- Theorem 3.6.** 1. A compact frame is almost compact.
2. A frame L is not almost compact if and only if an ideal Q in L exists such that $Q \subseteq S_L = \{l \in L : l^* = 0_L\}$ and $\bigvee Q = 1_L$.

We prove the following lemma using the above theorem.

Lemma 3.2. A frame in which every clustered filter converges is almost compact.

Proof. Let L be such a frame and suppose, for contradiction, that L is not almost compact. Then by *Theorem 3.6*, $L - D(L)$ is a cover of L . Since $D(L)$ misses this cover, $D(L)$ does not converge. But this is a contradiction because $D(L)$ clusters. \square

Now we prove the main theorem on characterization of minimal Hausdorff frames in terms of convergence of filters.

Theorem 3.7. A frame L is minimal Hausdorff if and only if every clustered filter in L converges.

Proof. We assume that L is not boolean because the only boolean frame that is minimal Hausdorff is B_4 for which the result is true. Suppose that L is minimal Hausdorff. If there exists a clustered filter that does not converge, then by *Lemma 3.1*, there exists a proper subframe of L that is Hausdorff, contradicting the minimality of L .

Conversely assume that every clustered filter in L converges. Now by *Lemma 3.2*, L is almost compact. Let A is a proper subframe of L and let “ $*_A$ ” be the pseudo-complementation operator in A . Then A is almost compact and by *Proposition 2.1* [17] there exists no ideal $Q \subseteq S_A = \{l \in A : l^{*A} \neq 0\}$ in A with $\bigvee Q = 1$. Let M be the ultrafilter in A containing $F = \{l \in A : l^{*A} = 0\}$. Then by *corollary 3* [7], $A - M$ is a minimal prime ideal in A and $\bigvee (A - M) \neq 1$ because $A - M \subseteq S_A$. Now suppose that $\bigvee (A - M) = u$, where u is a dual atom in A . Then either $u \vee u^{*A} = u$ or $u \vee u^{*A} = 1$. If $u \vee u^{*A} = u$, then $u^{*A} \leq u$ and we get $u^{*A} = 0$. Thus $u \in F \subseteq M$. Since M is an ultrafilter $u^{*A} \in A - M$ and hence $u^{*A} \neq 0$, a contradiction. If $u \vee u^{*A} = 1$, then $u^{*A} = u^c$, the complement of u in L . Now $u, u^c \notin F' = \{l \in L : l^* = 0\}$ as $u \neq 1$ and $u \neq 0$. Now the cover $\{u, u^c\}$ of L does not intersect F' and hence F' is not convergent. If $x \wedge y = 0$ for some $y \in F', 0 \neq x \in L$, then $x \leq y^* = 0$ and hence $x = 0$, a contradiction. Thus $x \wedge y = 0$ for no $y \in F'$ if $0 \neq x \in L$. Therefore $\text{sec } F' = L - \{0\}$. Since every cover of L contains at least one element different from 0, every cover of L intersects $\text{sec } F'$ and hence F' is clustered.

This contradicts our assumption that every clustered filter in L converges. Thus $\bigvee (A - M) = u$, where u is neither 1 nor a dual atom. Thus A is not a separated frame and hence not Hausdorff since every Hausdorff frame is separated by *Proposition 4* [23]. Hence L is minimal Hausdorff. \square

Corollary 3.8. Every filter in a minimal Hausdorff frame is clustered.

Proof. The proof follows from *Theorem 3.7*, *Lemma 3.2* and *Theorem 3.9*. \square

We state the following theorem before we prove the next corollary.

Theorem 3.9. [17] For a frame L , the following are equivalent:

1. L is almost compact.
2. Every filter in L is clustered.
3. Every maximal filter in L is convergent.

Corollary 3.10. A minimal Hausdorff frame is almost compact.

Proof. By *Theorem 3.9*, if every filter in a frame is clustered, then it is almost compact. \square

The converse that an almost compact frame is minimal Hausdorff need not be true, as a finite Boolean frame other than B_4 is compact and hence almost compact but not minimal Hausdorff.

We state the following Lemma due to Banaschewski[1].

Lemma 3.3. A frame is compact if and only if each of its prime upsets converges.

Corollary 3.11. A minimal Hausdorff frame is compact

Proof. The result follows from *Corollary 3.8* and *Lemma 3.3*. \square

Corollary 3.12. Let L be a minimal Hausdorff frame then the clustered filters in L are exactly the convergent filters in L .

Proof. If L is minimal Hausdorff, then every clustered filter is convergent by *Theorem 3.7*. Also every convergent filter is clustered. \square

Theorem 3.13. An extension closed sublocale $h : L \rightarrow M$ of a minimal Hausdorff frame L is minimal Hausdorff.

Proof. Let F be any clustered filter in M . Then by *Proposition 1.6* [10], $h^{-1}(F)$ is clustered in L and is convergent as L is minimal Hausdorff. Thus F is a filter in M such that $h^{-1}(F)$ converges in L . Since every filter is an upset and $h : L \rightarrow M$ is extension closed, by *Proposition 3.3* [8], F converges in M . Hence the theorem. \square

Corollary 3.14. A closed sublocale of a minimal Hausdorff frame is minimal Hausdorff.

Proof. Since every closed sublocale is extension closed, the proof directly follows. \square



4. Association with topological spaces

We know that Hausdorffness in frames is not equivalent to Hausdorffness defined in topological spaces, but an imitation of the classical Hausdorffness axiom in topological spaces. Also, if ΩX is a Hausdorff frame, then $(X, \Omega X)$ need not be a Hausdorff space by the following example.

Example 4.1. Let $X = \{a, b, c, d\}$ and $\Omega X = \{X, \phi, \{a, b\}, \{c, d\}\}$. Then ΩX is a Hausdorff frame, but $(X, \Omega X)$ is not a Hausdorff space.

The space $(X, \Omega X)$ need not be Hausdorff when ΩX is a Hausdorff frame. But it is true when X is a T_0 space. We have a similar result on minimal Hausdorffness by the following theorem.

Theorem 4.1. Let $(X, \Omega X)$ be a T_0 topological space. If ΩX is a minimal Hausdorff frame, then $(X, \Omega X)$ is a minimal Hausdorff space.

Proof. Let \mathcal{B} be an open filter-base in the topological space X having a unique cluster point p . Let \mathcal{F} be the filter in the frame ΩX generated by \mathcal{B} . Let \mathcal{A} be a cover of the frame ΩX and let G be an open neighbourhood of p in \mathcal{A} . Since p is a cluster point of \mathcal{B} , the neighbourhood G intersects every element of \mathcal{B} and consequently every member of \mathcal{F} . Thus \mathcal{F} is a clustered filter in ΩX . Since ΩX is minimal Hausdorff, by *Theorem 3.7*, \mathcal{F} is convergent in ΩX . Let \mathcal{U} be the filter with base \mathcal{B} in the topological space X . Then \mathcal{U} is convergent in X and assume that it converges to a point $x \neq p$. Consider the open filter-base \mathcal{C} which contains all sets that are the finite intersection of elements of $\{B \cup N_p : B \in \mathcal{B}, N_p \in \mathcal{N}_p\}$ where \mathcal{N}_p is the open neighbourhood system at p . Consequently this generates a filter that converges to both p and x which is not possible in a Hausdorff space. Hence the open filter-base \mathcal{B} must converge to p . Thus $(X, \Omega X)$ is minimal Hausdorff by *Theorem 3.1*. \square

Corollary 4.2. The set of all real numbers with usual topology is denoted by $(\mathbb{R}, \Omega\mathbb{R})$. Then the frame $\Omega\mathbb{R}$ is not minimal Hausdorff.

Proof. $(\mathbb{R}, \Omega\mathbb{R})$ is T_0 and is not minimal Hausdorff. Hence $\Omega\mathbb{R}$ cannot be a minimal Hausdorff frame. \square

Remark 4.1. The converse of the above theorem need not be true. Let $X = \{a, b, c\}$. Consider the topological space $(X, \mathcal{P}(X))$ where $\mathcal{P}(X)$ is the power set of X . It is a minimal Hausdorff space as it is compact and Hausdorff. But the frame $\mathcal{P}(X)$ is a Boolean frame containing B_4 and hence not a minimal Hausdorff frame.

Remark 4.2. In the category **Sp** of all topological spaces and continuous mappings, a compact Hausdorff space is minimal Hausdorff. But in the category **Frm** this need not happen. The frame in the example provided in *Remark 4.1* is a compact frame, but not minimal Hausdorff.

5. Application

A frame is said to be *reversible*[12], if every order preserving self bijection is a frame isomorphism. A characterization for reversible frames is given in [12]. It is also proved that a frame that is maximal or minimal with respect to some frame isomorphic property is reversible. This leads to the following result.

Theorem 5.1. A minimal Hausdorff frame is reversible

Proof. By *Theorem 3.7* [12] a minimal Hausdorff frame is reversible \square

A reversible frame need not be minimal Hausdorff. A boolean frame strictly containing B_4 is not minimal Hausdorff but reversible by *Theorem 3.14* of [12]. Hence the characterization for minimal Hausdorff frames can be used as one method to identify reversible frames.

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