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On LP-Sasakian manifold admitting a generalized symmetric metric connection

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Abstract

In this paper we study certain curvature properties of Lorentzian Para-Sasakian manifold (shortly, $LPSM$) with respect to the generalized symmetric metric connection. Here we discuss ξ-concircularly, ξ-conformally and ξ-projectively flat $LPSM$ with respect to the generalized symmetric metric connection and obtain various interesting results. Moreover, we study $LPSM$ with $\tilde{Z}(\xi,V).\tilde{S}=0$, where $\tilde{Z}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the generalized symmetric metric connection.

Keywords

LP-Sasakian manifold, quarter-symmetric connection, generalized symmetric connection, η-Einstein manifold, ξ-concircularly, projectively, conformally flat manifold.

AMS Subject Classification

53C15, 53C25.

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1. Introduction

The Lorentzian Para-Sasakian manifold (shortly, $LPSM$) has been introduced by Matsumoto [9]. Again, Mihai and Rosca [11] also introduced $LPSM$ and acquired various features. During the last three decades $LPSM$ has been studied by various authors and obtained several results. For this we refer the reader to see [1], [2], [3], [5], [10], [11], [12], [13], [17], [18], [19] and references therein. Among the study of $LPSM$, most of the research works of this manifold, admitting either semi-symmetric metric connection (see [6], [7], [14], [15], [21]) or quarter-symmetric metric connection (see [7], [8], [16], [20] and also references therein). It should be noted that the quarter-symmetric metric connection is more generalized form that of semi-symmetric metric connection.

In [1], Bahadir and Chaubey introduced a new type of linear connection called generalized symmetric metric connection which is a combination of semi-symmetric metric connection and quarter-symmetric metric connection.
A manifold M of dimension n is said to be $LPSM$ if it satisfies

$$\begin{aligned}
&\phi\xi=0, \qquad \eta(\phi U)=0, \qquad \phi^2U=U+\eta(U)\xi,\\
&g(\phi U,V)=g(U,\phi V),\ \ \eta(U)=g(U,\xi),\ \ \eta(\xi)=-1,\\
&g(\phi U,\phi V)=g(U,V)+\eta(U)\eta(V),\\
&\nabla_U\xi=\phi U \quad \text{and}\\
&(\nabla_U\phi)(U,V)=g(U,V)\xi+\eta(V)U+2\eta(U)V
\end{aligned}$$

for all vector fields U,V on M, where ∇, ϕ, ξ, η and g denote the levi-civita connection, a $(1,1)$ tensor field, a vector field, a 1-form and a Lorentzian metric respectively.
Relation between generalized symmetric metric connection $\tilde{\nabla}$ and ∇ is given by [1]

$$\begin{aligned}
\tilde{\nabla}_UV \;&=\; \nabla_UV+\alpha\{\eta(V)U-g(U,V)\xi\}\\
&\quad +\beta\{\eta(V)\phi U-g(\phi V,U)\xi\}.
\end{aligned}$$

In section 2 some preliminaries are discussed for further calculations in the subsequent sections. Section 3 is concerned with ξ-concircularly flat *LPSM*, which gives us an interesting result. Again, ξ-conformally flat *LPSM* is studied in section 4, where we find out a condition for which this manifold can be η Einstein. Section 5 deals with projectively flat *LPSM*. Section 6 is concerned with *LPSM* satisfying $\tilde{Z}(\xi,V).\tilde{S}=0$, where we establish a theorem as well as a corollary.

2. Preliminaries

Let R, S and r be the curvature tensor, Ricci tensor and scalar curvature of M respectively. $\chi(M)$ be the Lie algebra of vector fields of M.
The semi-symmetric linear connection have been introduced by Friedmann and Schouten [6].

Definition 2.1. *[14] ∇ is called semi-symmetric connection [6] if*

$$T(U,V)=\eta(V)U-\eta(U)V,$$

holds, where $T=$ the torsion tensor and $U,V\in\chi(M)$.

Again in 1975, Golab [7] introduced quarter-symmetric connection.

Definition 2.2. *[16] ∇ is said to be quarter-symmetric connection [7] if*

$$T(U,V)=\eta(V)\phi U-\eta(U)\phi V,$$

where T is the torsion tensor and $U,V\in\chi(M)$.

Definition 2.3. *A semi-symmetric and quarter-symmetric connection ∇ satisfying*

$$\nabla g=0,$$

is called a semi-symmetric metric connection (shortly, ssmc) and quarter-symmetric metric connection (shortly, qsmc) respectively.Otherwise it is called non-metric connection.

Definition 2.4. *[1] A metric connection ∇ is said to be a generalized symmetric metric connection (shortly, gsmc) if T satisfies*

$$T(U,V)=\alpha\{\eta(V)U-\eta(U)V\}+\beta\{\eta(V)\phi U-\eta(U)\phi V\},$$

where α and β are smooth functions.

We mention that if $\alpha=1$ and $\beta=0$, then the *gsmc* reduces to *ssmc*. Again if $\alpha=0$ and $\beta=1$, then *gsmc* turns into *qsmc*. For $M^n(\phi,\xi,\eta,g)$ the following results can be proved easily [2].

$$\phi\xi=0,\qquad \eta(\phi U)=0,\qquad \phi^2U=U+\eta(U)\xi,$$
$$\phi\xi=0,\quad \eta(\phi U)=0,\ ,\ rank\phi=n-1.$$

If we write

$$\tilde{\phi}=g(\phi U,V)$$

then we have

$$(\nabla_U\eta)(V)=\tilde{\phi}(U,V),\quad \tilde{\phi}(U,\xi)=0.$$
$$g(R(U,V)W,\xi)=g(V,W)\eta(U)-g(U,W)\eta(V),$$
$$R(U,V)\xi=\eta(V)U-\eta(U)V,$$
$$S(U,\xi)=(n-1)\eta(U),$$
$$S(\phi U,\phi V)=S(U,V)+(n-1)\eta(U)\eta(V)$$

for all vector fields U, V, $W\in\chi(M)$.

Definition 2.5. *An LPSM is said to be an η-Einstein if the Ricci sensor S satisfies*

$$S(U,V)=ag(U,V)+b\eta(U)\eta(V)$$

for all $U,V\in\chi(M)$, where a and b are scalar functions on M.

In *LPSM* admitting *gsmc* the following results holds [1].

$$\tilde{\nabla}_U\xi=(1-\beta)\phi U-\alpha U-\alpha\eta(U)\xi,$$

$$\begin{aligned}\tilde{R}(U,V)\xi &= (1-\beta+\beta^2)\{\eta(V)U-\eta(U)V\}\\ &\quad+\alpha(1-\beta)\{\eta(U)\phi V-\eta(V)\phi U\}\\ &= k_1\{\eta(V)U-\eta(U)V\}\\ &\quad+k_2\{\eta(U)\phi V-\eta(V)\phi U\},\end{aligned}\tag{2.1}$$

$$\tilde{R}(\xi,V)\xi=k_1\{\eta(V)\xi+V\}-k_2\phi V,\tag{2.2}$$

where $k_1=(1-\beta+\beta^2)$, $k_2=\alpha(1-\beta)$.

$$\begin{aligned}\tilde{S}(U,V) &= S(U,V)+\{-\alpha\beta+(n-2)(\alpha\beta-\alpha)\\ &\quad+(\beta^2-2\beta)\text{trace }\tilde{\phi}\}\tilde{\phi}(U,V)\\ &\quad+\{-2\alpha^2+\beta-\beta^2+n\alpha^2\\ &\quad+(\alpha\beta-\alpha)\text{trace }\tilde{\phi}\}g(U,V)\\ &\quad+\{-2\alpha^2+n(\alpha^2+\beta-\beta^2)\}\eta(U)\eta(V)\end{aligned}\tag{2.3}$$

Or, $\tilde{S}(U,V)=S(U,V)+A\tilde{\phi}(U,V)+Bg(U,V)+C\eta(U)\eta(V)$,

where, $A=-\alpha\beta+(n-2)(\alpha\beta-\alpha)+(\beta^2-2\beta)\text{trace }\tilde{\phi}$,
$B=-2\alpha^2+\beta-\beta^2+n\alpha^2+(\alpha\beta-\alpha)\text{trace }\tilde{\phi}$
and $C=-2\alpha^2+n(\alpha^2+\beta-\beta^2)$.

$$\tilde{S}(U,\xi)=S(U,\xi)+Bg(U,\xi)-C\eta(U)\tag{2.4}$$

$$\text{Or,}\quad \tilde{S}(U,\xi)=\{n-1+B-C\}\eta(U),\tag{2.5}$$

$$\tilde{r}=r+A\ \text{trace}\tilde{\phi}+Bn-C,$$

$$\begin{aligned}\tilde{R}(\xi,U)V &= \{-\alpha\tilde{\phi}(U,V)+(1-\beta)g(U,V)\\ &\quad-\beta^2\eta(U)\eta(V)\}\xi\\ &\quad-k_1\eta(V)U+k_2\eta(V)\phi U.\end{aligned}$$

3. ξ-concircularly flat LPSM

An *LPSM* is said to be ξ-concircularly flat [4] w.r.t a *gsmc* $\tilde{\nabla}$ if $\tilde{Z}(U,V)\xi = 0$, where $\tilde{Z}$ is the concircular curvature tensor. Then

$$\tilde{R}(U,V)\xi - \frac{\tilde{r}}{n(n-1)}[\eta(V)U - \eta(U)V] = 0. \tag{3.1}$$

From (2.1) and (3.1) we find

$$k_1[\eta(V)U - \eta(U)V] + k_2[\eta(U)\phi V - \eta(V)\phi U] - \frac{\tilde{r}}{n(n-1)}[\eta(V)U - \eta(U)V] = 0.$$

Putting $V = \xi$ in above relation we get

$$k_1[-U - \eta(U)\xi] + k_2\phi U - \frac{\tilde{r}}{n(n-1)}[-U - \eta(U)\xi] = 0.$$

Taking inner product with V we have

$$k_1[-g(U,V) - \eta(U)\eta(V)] + k_2\tilde{\phi}(U,V).$$

Taking contraction over U and V we get

$$k_1[-n+1] + k_2\text{trace}\tilde{\phi} + \frac{\tilde{r}}{n} = 0$$

$$\text{Or,} \quad \tilde{r} = n[(n-1)k_1 - k_2\text{trace}\tilde{\phi}]$$

$$\text{Or,} \quad \tilde{r} = n[(n-1)(1-\beta+\beta^2) - \alpha(1-\beta)\text{trace}\tilde{\phi}].$$

The above results give us the following theorem:

Theorem 3.1. *The scalar curvature $\tilde{r}$ of a ξ-concircularly flat LPSM with respect to gsmc is given by $\tilde{r} = n[(n-1)(1-\beta+\beta^2) - \alpha(1-\beta)trace\tilde{\phi}]$.*

If we consider the manifold with respect to *qsmc* instead of *gsmc* i.e., $\alpha = 0$ and $\beta = 1$, then the following corollary holds:

Corollary 3.2. *The scalar curvature of ξ-concircularly flat LPSM with respect to qsmc is given by $\tilde{r} = n(n-1)$.*

Again, if we consider the manifold with respect to *ssmc* instead of *gsmc* i.e., $\alpha = 1$ and $\beta = 0$, then we have

Corollary 3.3. *The scalar curvature of ξ-concircularly flat LPSM with respect to ssmc is given by $\tilde{r} = n[(n-1) - trace\tilde{\phi}]$.*

4. ξ-conformally flat LPSM

The *LPSM* is said to be ξ-conformally flat w.r.t a *gsmc* $\tilde{\nabla}$ if $\tilde{C}(U,V)\xi = 0$, where $\tilde{C}$ is the conformal curvature tensor. Then

$$\tilde{R}(U,V)\xi - \frac{1}{(n-2)}[\tilde{S}(V,\xi)U - \tilde{S}(U,\xi)V + \eta(V)\tilde{Q}U - \eta(U)\tilde{Q}V] + \frac{\tilde{r}}{(n-1)(n-2)}[\eta(V)U - \eta(U)V] = 0.$$

Using (2.1), (2.3) and (2.4) in above we get

$$\begin{aligned} &k_1[\eta(V)U - \eta(U)V] + k_2[\eta(U)\phi V - \eta(V)\phi U] \\ &-\frac{1}{n-2}[U\{S(U,\xi) + B\eta(V) - C\eta(V)\} \\ &-V\{S(U,\xi) + B\eta(U) - C\eta(U)\} \\ &+\eta(V)\{QU + A\phi U + BU + C\eta(U)\xi\} \\ &-\eta(U)\{QV + A\phi V + BV + C\eta(V)\xi\}] \\ &+\frac{\tilde{r}}{n(n-1)}[\eta(V)U - \eta(U)V] = 0 \end{aligned}$$

$$\begin{aligned} \text{Or,}\ &k_1[\eta(V)U - \eta(U)V] + k_2[\eta(U)\phi V - \eta(V)\phi U] \\ &-\frac{1}{n-2}[\{(n-1) + B - C\}\eta(V)U \\ &-\{(n-1) + B - C\}\eta(U)V \\ &+\eta(V)\{QU + A\phi U + BU + C\eta(U)\xi\} \\ &-\eta(U)\{QV + A\phi V + BV + C\eta(V)\xi\}] \\ &+\frac{\tilde{r}}{n(n-1)}[\eta(V)U - \eta(U)V] = 0. \end{aligned}$$

Putting $V = \xi$ in above we have

$$\begin{aligned} &k_1[-U - \eta(U)\xi] + k_2[\phi U] \\ - \ &\frac{1}{n-2}[U\{-(n-1) - B + C\} - \xi\{(n-1) + B - C\}\eta(U) \\ - \ &\{QU + A\phi U + BU + C\eta(U)\}\xi - \eta(U)\{\phi\xi + B\xi - C\xi\}] \\ + \ &\frac{\tilde{r}}{n(n-1)}[-U - \eta(U)\xi] = 0 \end{aligned}$$

$$\begin{aligned} \text{Or,}\quad QU &= (n-2)[U + \eta(U)\xi][k_1 + \frac{\tilde{r}}{n(n-1)} - B^1] \\ &+BU - [k_2(n-2) + A]\phi U + [B - C]\eta(U)\xi, \end{aligned}$$

where $B^1 = n - 1 + B - C$ or,

$$QU = [U + \eta(U)\xi]k^1 + BU - [k_2(n-2) + A]\phi U + [B - C]\eta(U)\xi,$$

where $k^2 = (n-2)[k_1 + \frac{\tilde{r}}{n(n-1)} - B^1]$.
Taking inner product with V we get

$$\begin{aligned} S(U,V) = k^2 g(U,V) + k^2\eta(U)\eta(V) - [k_2(n-2) + A]\tilde{\phi}(U,V) \\ +Bg(U,V) + [B - C]\eta(U)\eta(V) \end{aligned}$$

$$\begin{aligned} \text{Or,}\quad S(U,V) = [k^2 + B]g(U,V) - [k_2(n-2) + A]\tilde{\phi}(U,V) \\ +[B - C + k^2]\eta(U)\eta(V) \end{aligned}$$

Or,

$$\begin{aligned} S(U,V) &= [k^2 + B]g(U,V) \\ &-\beta[-\alpha + (\beta - 2)\text{trace}\tilde{\phi}]\tilde{\phi}(U,V) \\ &+[B - C + k^2]\eta(U)\eta(V). \end{aligned} \tag{4.1}$$

If $(n-2)k_2 + A = 0$ i.e., if

$$\begin{aligned} &(n-2)\alpha(1-\beta) + \{-\alpha\beta + (n-2)(\alpha\beta - \alpha) \\ &+(\beta^2 - 2\beta)\text{trace }\tilde{\phi}\} = 0, \end{aligned} \tag{4.2}$$

then we have from (4.1)

$$S(U,V) = [k^2+B]g(U,V) + [B-C+k^2]\eta(U)\eta(V)$$

i.e., the manifold is η Einstein.
From (4.2) we have

$$\beta\{-\alpha+(\beta-2)\text{trace } \tilde{\phi}\} = 0. \tag{4.3}$$

Therefore in view of above the following theorem exists.

Theorem 4.1. *A ξ-conformally flat LPSM with respect to gsmc is η Einstein if and only if (4.3) holds.*

If we consider the *ssmc* instead of *gsmc* i.e., $\alpha = 1$ and $\beta = 0$, then the following corollary holds:

Corollary 4.2. *A ξ-conformally flat LPSM with respect to ssmc is η Einstein.*

Again, if we consider the *qsmc* instead of *gsmc* i.e., $\alpha = 0$ and $\beta = 1$, then there is a corollary given as:

Corollary 4.3. *A ξ-conformally flat LPSM with respect to gsmc is η Einstein iff trace$\tilde{\phi} = 0$.*

5. ξ-projectively flat LPSM

We consider an *LPSM* with *gsmc* which is ξ-projectively flat then

$$\tilde{R}(U,V)\xi = \frac{1}{n-1}[\tilde{S}(V,\xi)U - \tilde{S}(U,\xi)V].$$

Putting $U = \xi$ in above we have

$$\tilde{R}(\xi,V)\xi = \frac{1}{n-1}[\tilde{S}(V,\xi)\xi - \tilde{S}(\xi,\xi)V].$$

Using (2.2) and (2.5) in above

$$\begin{aligned} k_1\{\eta(V)\xi+V\} - k_2\phi V &= \frac{1}{n-1}[\{(n-1)\eta(V) \\ &\quad +B\eta(V) - C\eta(V)\}\xi \\ &\quad -\{-(n-1)-B+C\}V] \end{aligned}$$

Or, $(n-1)k_2\phi V = [(n-1)k_1 - B^1][\eta(V)\xi+V]$,
where $B^1 = (n-1)+B-C$, or,

$$\alpha(1-\beta)[(n-1)\phi U - \text{trace}\tilde{\phi}\ U - \text{trace}\tilde{\phi}\ \eta(U)\xi] = 0.$$

If $\alpha(1-\beta) \neq 0$, then

$$(n-1)\phi U - \text{trace}\tilde{\phi}\ U - \text{trace}\tilde{\phi}\ \eta(U)\xi = 0.$$

Hence we come to the following conclusion.

Theorem 5.1. *If an LPSM is ξ -projectively flat with respect to gsmc, then $(n-1)\phi U = trace\tilde{\phi}U - trace\tilde{\phi}\eta(U)\xi$, provided $\alpha(1-\beta) \neq 0$.*

If we consider *ssmc* instead of *gsmc* i.e., $\alpha = 1$ and $\beta = 0$, then the following corollaries holds:

Corollary 5.2. *If an LPSM is ξ-projectively flat with respect to ssmc, then $(n-1)\phi U = trace\tilde{\phi}U - trace\tilde{\phi}\eta(U)\xi$.*

Corollary 5.3. *In an LPSM with respect to ssmc, if trace$\tilde{\phi} =$ 0 then, it can not be ξ -projectively flat.*

6. LPSM with $\tilde{Z}(\xi.V).\tilde{S} = 0$

An *LPSM* with $\tilde{Z}(\xi,V).\tilde{S} = 0$ imply

$$\tilde{S}(\tilde{Z}(\xi,V)W,U) + \tilde{S}((W,\tilde{Z}(\xi,V)U) = 0.$$

Putting $W = \xi$ we get

$$\tilde{S}(\tilde{Z}(\xi,V)\xi,U) + \tilde{S}((\xi,\tilde{Z}(\xi,V)U) = 0$$

$$\begin{aligned} \text{Or, } &\tilde{S}(\tilde{R}(\xi,V)\xi - m[\eta(V)\xi+V],U) \\ &+\tilde{S}(\xi,\tilde{R}(\xi,V)U - m[g(V,U)\xi-\eta(U)V]) = 0, \end{aligned}$$

where $m = \frac{\tilde{r}}{n(n-1)}$. Now

$$\begin{aligned} &\tilde{S}(\tilde{R}(\xi,V)\xi - m[\eta(V)\xi+V],U) \\ = &\ \tilde{S}(k_1[\eta(V)\xi+V] - k_2\phi V - m[\eta(V)\xi+V],U) \\ = &\ k_1\eta(V)\tilde{S}(\xi,U) + k_1\tilde{S}(V,U) - k_2\tilde{S}(\phi V,U) \\ - &\ m\eta(V)\tilde{S}(\xi,U) - m\tilde{S}(V,U) \\ = &\ k_1B^1\eta(V)\eta(U) + k_1[S(V,U) + A\tilde{\phi}(V,U) \\ + &\ Bg(V,U) + C\eta(V)\eta(U)] \\ - &\ k_2[S(\phi V,U) + Ag(\phi^2V,U) + B\tilde{\phi}(V,U)] - m[B^1\eta(V)\eta(U)] \\ - &\ m[S(V,U) + A\tilde{\phi}(V,U) + Bg(V,U) + C\eta(V)\eta(U)] \\ = &\ k_1B^1\eta(V)\eta(U) + k_1[S(V,U) + A\tilde{\phi}(V,U) \\ + &\ Bg(V,U) + C\eta(V)\eta(U)] \\ - &\ k_2[S(\phi V,U) + A\eta(V)\eta(U) + Ag(V,U) + B\tilde{\phi}(V,U)] \\ - &\ m[B^1\eta(V)\eta(U)] - m[S(V,U) + A\tilde{\phi}(V,U) + \\ + &\ Bg(V,U) + C\eta(V)\eta(U)] \\ = &\ [k_1-m]S(V,U) - k_2[S(\phi V,U)] + [Ak_1 - Bk_2 - Am]\tilde{\phi}(V,U) \\ + &\ [Bk_1 - Ak_2 - Bm]g(V,U) + [k_1B^1 + k_1C - Ak_2 \\ - &\ \{B^1+C\}m]\eta(V)\eta(U) \\ = &\ D_1S(V,U) - k_2S(\phi V,U) + D_2\tilde{\phi}(V,U) \\ + &\ D_3g(V,U) + D_4\eta(V)\eta(U), \end{aligned} \tag{6.1}$$

where $D_1 = [k_1-m]$, $D_2 = Ak_1 - Bk_2 - Am$, $D_3 = Bk_1 - Ak_2 - Bm$, $D_4 = k_1B^1 + Ck_1 - Ak_2 - \{B^1+C\}m.$
Now

$$\tilde{S}(W,\tilde{Z}(\xi,V)U) = \tilde{S}(W,\tilde{R}(\xi,V)U - m[g(V,U)\xi - g(\xi,V)U]). \tag{6.2}$$

Therefore

$$\begin{aligned} &\tilde{S}(\xi,\tilde{Z}(\xi,V)U) \\ = &\ \tilde{S}(\xi,\tilde{R}(\xi,V)U - m[g(V,U)\xi - g(\xi,V)U]) \\ = &\ \frac{\tilde{r}}{n}g(V,U) + m\eta(U)\tilde{S}(\xi,V) + \tilde{S}(\xi,\tilde{R}(\xi,V)U) \\ = &\ \frac{\tilde{r}}{n}g(V,U) + B^1m\eta(U)\eta(V) \\ + &\ \tilde{S}(\xi,\{-\alpha\tilde{\phi}(U,V) + (1-\beta)g(V,U) - \beta^2\eta(U)\eta(V)\}\xi \\ - &\ k_1\eta(U)V + k_2\eta(U)\phi V) \\ = &\ \frac{\tilde{r}}{n}g(V,U) + B^1m\eta(U)\eta(V) \\ - &\ (n-1)\{-\alpha\tilde{\phi}(U,V) + (1-\beta)g(U,V) - \beta^2\eta(U)\eta(V)\} \\ - &\ k_1\eta(U)\tilde{S}(\xi,V). \end{aligned} \tag{6.3}$$

Using (6.2) and (6.3) in (6.1) we get

$$D_1 S(U,V) - k_2 S(\phi V,U) + \{D_2 + (n-1)\alpha\}\tilde{\phi}(V,U)$$
$$+ g(U,V)\{D_3 + \frac{\tilde{r}}{n} - (n-1)(1-\beta)\}$$
$$+ \{D_4 + B^1 m + \beta^2 - K_1 B^1\}\eta(U)\eta(V) = 0,$$

or,

$$D_1 S(U,V) - k_2 S(\phi U,V)$$
$$= D_5 \tilde{\phi}(U,V) + D_6 g(U,V) + D_7 \eta(U)\eta(V),$$

where $D_5 = -\{D_2 + (n-1)\alpha\}$,
$D_6 = -\{D_3 + \frac{\tilde{r}}{n} - (n-1)(1-\beta)\}$,
$D_7 = -\{D_4 + B^1 m + (n-1)\beta^2 - K_1 B^1\}$. Again

$$D_1 S(U,V) - k_2 S(\phi V,U) \tag{6.4}$$
$$= D_5 \tilde{\phi}(V,U) + D_6 g(U,V) + D_7 \eta(U)\eta(V).$$

Putting $V = \phi V$, we get

$$-k_2 S(U,V) + D_1 S(\phi V,U) \tag{6.5}$$
$$= D_6 \tilde{\phi}(V,U) + D_5 g(U,V) + \{k_2(n-1) + D_5\}\eta(U)\eta(V).$$

Solving (6.4) and (6.5) we get

$$(D_1^2 - k_2^2)S(U,V) = (D_1 D_5 + k_2 D_6)\tilde{\phi}(V,U)$$
$$+ g(U,V)(D_1 D_6 + k_2 D_5)$$
$$+ [D_1 D_7 + k_2\{k_2(n-1) + D_5\}]\eta(U)\eta(V).$$

If $(D_1 D_5 + k_2 D_6) = 0$ then M is of η Einstein. Hence the following theorem can be stated.

Theorem 6.1. *If an LPSM admitting gsmc satisfies*

$$(i)\tilde{Z}(\xi,V).\tilde{S} = 0,$$
$$(ii) D_1 D_5 + k_2 D_6 = 0,$$

is η Einstein.

If we consider *qsmc* instead of *gsmc* i.e., $\alpha = 0$ and $\beta = 1$, then $k_2 = 0$ and hence $(D_1 D_5 + k_2 D_6) = 0$ reduces to$\{1 - \frac{\tilde{r}}{n(n-1)}\}^2 \text{trace}\tilde{\phi} = 0$ and hence we can state the corollary as:

Corollary 6.2. *If an LPSM with respect to qsmc satisfies*

$$(i)\tilde{Z}(\xi,V).\tilde{S} = 0,$$
$$(ii)\tilde{r} \neq n(n-1),$$

is η Einstein if and only if trace$\tilde{\phi} = 0$.

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