Malaya Journal of Matematik, Vol. 8, No. 4, 1614-1621, 2020
https://doi.org/10.26637/MJM0804/0049

Construction of larger singular and nonsingular graphs using a path

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Abstract
A singular graph G has an adjacency matrix $A(G)$ with nullity $\eta(G) > 0$. Vertices of singular graphs are classified as core and noncore vertices. There are two types of noncore vertices: noncore vertices of zero null spread and of null spread -1. Deletion of these vertices from a singular graph either changes the nullity or leave it unaltered. In this paper larger singular and nonsingular graphs were constructed by joining singular graphs by a path. As singular graphs have different types of vertices, the graphs constructed in this way differ in nullity depending on the vertex we are joining during construction. An attempt was made to construct singular graph of maximum nullity. Various spectral properties of the resulting graphs were studied.

Keywords
Singular graph, Path, Nullity, Core vertices, Coalescence.

AMS Subject Classification
05C38, 05C50, 15A18.

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Article History: Received **20** July **2020**; Accepted **26** September **2020**

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1. Introduction

Let $G = (V,E)$ be a finite, undirected simple graph of order n, with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose $(i, j)^{\text{th}}$ entry is equal to one if there is an edge between the vertices v_i and v_j, and is equal to zero otherwise. The characteristic polynomial of G, denoted by $\phi(G,x)$ is a polynomial of degree n in x. The roots of $\phi(G,x) = 0$ are called the eigenvalues of G. The collection of the eigenvalues together with their multiplicities is called the spectrum of G, and is denoted by spec(G). If zero is an eigenvalue of G, then G is a singular graph. The multiplicity of zero is called the nullity $\eta(G)$ of G. The nonzero vector X satisfying the equation $AX = 0$ is called kernel eigenvector of G. A singular graph on at least two vertices, with a kernel eigenvector having nonzero entries is called a core graph. Core graphs have nullity one or more. Let G be a singular graph of nullity one and X is a kernel eigenvector where $X = [x_1, x_2, \dots, x_m, 0, \dots, 0]^T$, with $x_m \neq 0, i = 1, 2, \dots, m$. Then the subgraph F of G induced by the non zero entries $x_1, x_2, \dots, x_m$ is called the core graph of G. The set of remaining vertices are called core-forbidden vertices or noncore vertices of G. (See Figure 1)

Definition 1.1 ([15]). *Let F be a core graph on at least two vertices, with nullity $\eta \geq 1$ and a kernel eigenvector X_F having no zero entries. If a graph N, of nullity one, having X_F as the non- zero part of a kernel eigenvector, is obtained, by adding $\eta - 1$ independent vertices, whose neighbours are vertices of F, then N is said to be a minimal configuration (MC).*

The set of $\eta - 1$ independent vertices added to F to produce N is said to be the periphery $\mathscr{P}(N)$ of N.

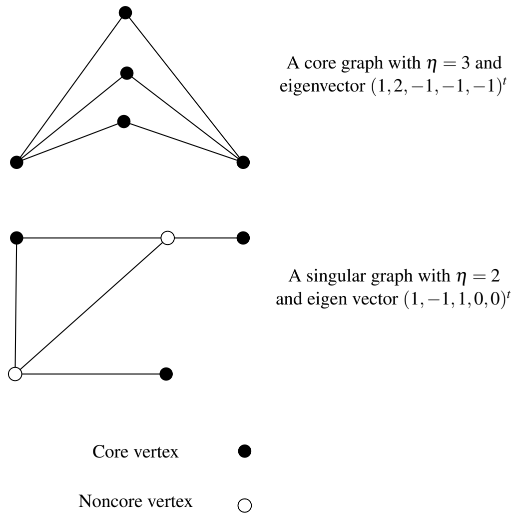


Figure 1. Two singular graphs

Now we have a very important result about eigenvalues of graphs, known as interlacing theorem .

Theorem 1.2 ([9]). *If G is an n-vertex graph with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and H is a vertex deleted sub graph of G with eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$, then $\lambda_i \leq \mu_i \leq \lambda_{i+1}$, $i = 1,2,\ldots,n-1$.*

Interlacing theorem states that the multiplicity of an eigenvalue and hence the multiplicity of nullity can change at most one upon deleting or adding a vertex of the graph.

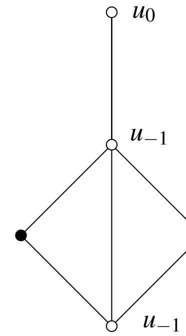
Definition 1.3 ([3]). *Let $G-u$ be the induced sub graph of the graph G obtained on deleting vertex u. The null spread of the vertex u is : $n_u(G) = \eta(G) - \eta(G-u)$.*

Clearly the null spread $n_u(G)$ of the vertex u of a graph G satisfies $-1 \leq n_u(G) \leq 1$. If u is a core vertex of G, then $n_u(G) = 1$. If G is a MC and u is a vertex in the periphery of G, then $n_u(G) = -1$. There are vertices with $n_u(G) = 0$ also. Thus noncore vertices (vertices other than core vertices) of a singular graph G can be classified as noncore vertices of null spread -1 and noncore vertices of zero null spread. (See Figure 2).

Definition 1.4 ([9]). *Let G_1 and G_2 be two graphs with disjoint vertex sets. If a vertex $u \in G_1$ is identified with a vertex $v \in G_2$, then the graph $G_1 \circ G_2$ obtained of order $|G_1| + |G_2| - 1$, is said to be the coalescence of G_1 and G_2 with respect to u and v.*

The following theorem gives an expression for the characteristic polynomial $\varphi(G,x)$ of the graph $G = G_1 \circ G_2$.

Theorem 1.5 ([9]). *The characteristic polynomial of the coalescence $G_1 \circ G_2$ of two rooted graphs (G_1,u) and (G_2,w) obtained by identifying the vertices u and w so that the vertex*



u_0 – vertex of zero null spread

u_{-1} – vertex of null spread -1

Figure 2. Graph with both types of noncore vertices.

$v = u = w$ become a cut vertex of $G_1 \circ G_2$ is given by

$$\varphi(G_1 \circ G_2) = \varphi(G_1)\varphi(G_2 - w) + \varphi(G_1 - u)\varphi(G_2) - x\varphi(G_1 - u)\varphi(G_2 - w). \tag{1.1}$$

2. Construction of Larger Singular Graphs by a Path

In this section, we construct larger singular graphs by joining two graphs by a path

We have the following theorems:

Theorem 2.1 ([6]). *The coalescence of two singular graphs of nullity η_1 and η_2 coalesced at a core vertex yield a singular graph of nullity $\eta_1 + \eta_2 - 1$.*

Theorem 2.2 ([7]). *Let G_1 be a nonsingular graph and G_2 be a singular graph of nullity n_2. If G_1 and G_2 are coalesced at a vertex $u \in G_1$ and a core vertex $v \in G_2$, then the nullity of $G_1 \circ G_2$ is $\eta_2 - 1$.*

Theorem 2.3 ([7]). *Let G_1 and G_2 be two singular graphs of order η_1 and η_2 respectively. If $G_1 \circ G_2$ is the coalescence of G_1 and G_2 at a noncore vertex of null spread -1, then $\eta(G_1 \circ G_0) = \eta_1 + \eta_2 + 1$.*

Theorem 2.4 ([7]). *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively. The nullity of the coalescence of G_1 and G_2 at a noncore vertex of zero null spread is $\eta_1 + \eta_2$.*

Theorem 2.5 ([7]). *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively. The coalescence of G_1 and G_2 at a core vertex of G_1 and at a noncore vertex (null spread* 0 *of -1) of G_2 or vice versa yield a singular graph of nullity $\eta_1 + \eta_2 - 1$.*

Theorem 2.6 ([7]). *Let G_1 be a non singular graph and G_2 be a singular graph of nullity η_2. Then the nullity of the coalescence of G_1 and G_2 with respect to any vertex of G_1 and a noncore vertex of zero null spread of G_2 is η_2.*

Theorem 2.7 ([7]). *Let G_1 be a nonsingular graph and G_2 be a singular graph of nullity η_2. Then the nullity of the coalescence of G_1 and G_2 with respect to any vertex $u \in G_1$ and a noncore vertex $w \in G_2$ of null spread -1 is*

1. *$\eta_2 + 1$, if $G_1 - u$ is singular*
2. *η_2, if $G_2 - u$ nonsingular.*

Theorem 2.8 ([8]). *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively. If a core vertex u of G_1 and a noncore vertex w (of null spread -1 or 0) is coalesced then in the coalesced graph, the coalesced vertex is a noncore vertex.*

Theorem 2.9 ([8]). *A singular graph with noncore vertices always satisfies the following conditions.*

1 *If one ore more neighbours of a noncore vertex v is the only neighbours of another vertex v', then v' will be a noncore vertex.*

2 *the vertices having core or noncore vertex neighbours whose neighbours are noncore vertices will be noncore vertices.*

Theorem 2.10 ([8]). *Let G_1 and G_2 be two singular graphs of nullity η_1 and η_2 respectively and $G_1 \circ G_2$ be the coalescence of G_1 and G_2 with respect to $u \in G_1$ and $w \in G_2$. Then, noncore vertices of G_1 and G_2 will remain as noncore vertices in $G_1 \circ G_2$.*

Theorem 2.11 ([7]). *Let G_1 and G_2 be two singular graphs and $G_1 \circ G_2$ be the coalescence of G_1 and G_2 with respect to $u \in G_1$ and $w \in G_2$. Then*

(i) $G_1 \circ G_2$ is singular if either $G_1 - u$ or $G_1 - w$ is nonsingular.

(ii) $G_1 \circ G_2$ is singular if $G_1 - u$ and $G_1 - w$ are nonsingular.

Definition 2.12. *Let (K,u) and (H,w) be two rooted graphs. The graph $KH + P_n$ is constructed such that u and w are joined to any two vertices of the path P_n as shown in Figure 3.*

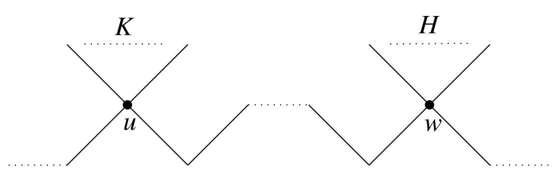


Figure 3. The graph of $KH + P_n$.

2.1 Construction using the path P_{2t}

Theorem 2.13. *Let (K,u) and (H,w) be the components of the graph obtained by deleting the path P_{2t} from $KH + P_{2t}$. If K, H and $K - u$ are nonsingular, then $KH + P_{2t}$ is nonsingular.*

Proof. We know that P_{2t} is a nonsingular graph. The graph $KH + P_{2t}$ can be constructed by coalescence as follows: First coalesce K and P_{2t} with respect to u and any vertex of P_{2t}. Then coalesce H and $K \circ P_{2t}$ with respect to w and any vertex of P_{2t}. Thus $(K \circ P_{2t}) \circ H = KH + P_{2t}$. Since $K - u$ is nonsingular, we have $K \circ P_{2t}$ is nonsingular. As $K - u$ is nonsingular and $K \circ P_{2t} - v$ for $v \in P_{2t}$ is nonsingular, it follows that $(K \circ P_{2t}) \circ H$ is nonsingular.

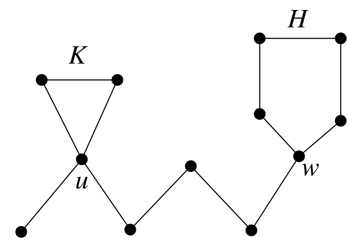


Figure 4. The nonsingular graph $KH + P_6$.

□

Theorem 2.14. *Let (K,u) and (H,w) be the components of the graph obtained by deleting the path P_{2t} from $KH + P_{2t}$. If K, H, $H - w$ are nonsingular and $K - u$ is singular, then $KH + P_{2t}$ is singular with nullity one or nonsingular depends on the vertex of P_{2t} we are joining to K and H.*

Proof. First coalesce K and P_{2t} with respect to u and any vertex v_j of P_{2t}. Since $P_{2t} - v_j$ is singular with nullity one and $K - u$ is singular, we have $K \circ P_{2t}$ is singular with nullity one. So $K \circ P_{2t}$ is a singular or minimal configuration of some graph. Obviously $K \circ P_{2t}$ contains core and noncore (of null spread -1) vertices. When we coalesce $K \circ P_{2t}$ and H with respect to w and any core vertex v_j of $K \circ P_{2t}$ ($v_j \in P_{2t}$), $KH + P_{2t}$ is nonsingular. If v_j is a noncore vertex of null spread -1, then $KH + P_{2t}$ is singular of nullity one. □

Theorem 2.15. *Let (K,u) and (H,w) be the components of the graph obtained by deleting the path P_{2t} from $KH + P_{2t}$. If K, H are nonsingular and $H - w$, $K - u$ are singular, then $KH + P_{2t}$ is singular with nullity two or nonsingular depends on the vertex of P_{2t} we are joining to K and H.*

Proof. $K \circ P_{2t}$ is the coalescence of K and P_{2t} at any vertex u of K and v_j of P_{2t}. Clearly $K \circ P_{2t}$ is singular with nullity one as $P_{2t} - v_j$ is singular with nullity one and $K - u$ is singular. So $K \circ P_{2t}$ is a singular or minimal configuration of some graph. Clearly $K \circ P_{2t}$ contains core and noncore (of null spread -1) vertices. When we coalesce $K \circ P_{2t}$ and H with respect to w and any core vertex v_j of $K \circ P_{2t}$ ($v_j \in P_{2t}$), $KH + P_{2t}$ is nonsingular. If v_j is a noncore vertex of null spread -1, then $KH + P_{2t}$ is singular of nullity two. □

Example 2.16. *In Figure 5, K, H are nonsingular and $K - u$, $H - w$ are singular. The graph $K \circ P_6$, the coalescence of K and P_6 with respect to the vertex u of K and v_3 of P_6 is a singular graph of nullity one. The graph $KH + P_6$ is obtained* *by coalescing $K \circ P_6$ with H at a core vertex v_4 of $K \circ P_6$ and w of H. The graph $KH + P_6$ is a nonsingular graph.*

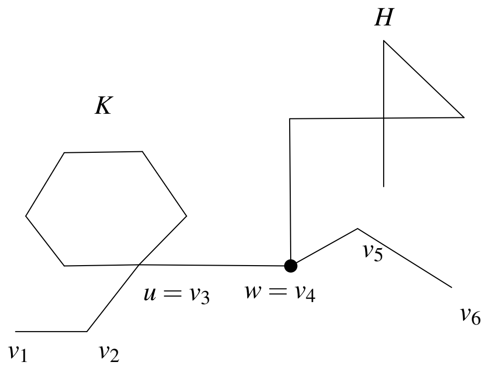


Figure 5. Graph of $KH + P_6$ with core vertex v_4.

Example 2.17. *In Figure 6 and 7, the graphs $K, H, H - w$ are nonsingular and $K - u$ is singular. In both figures the graph $K \circ P_6$, the coalescence of K and P_6 with respect to the vertex u of K and v_3 of P_6 is a singular graph of nullity one. In Figure 6, the graph $KH + P_6$ is obtained by coalescing $K \circ P_6$ with H at a core vertex v_4 of $K \circ P_6$ and w of H. The graph $KH + P_6$ is a nonsingular graph. In Figure 7, the graph $KH + P_6$ is obtained by coalescing $K \circ P_6$ with H at a nonecore vertex v_5 of $K \circ P_6$ and w of H. The graph $KH + P_6$ is a singular graph of nullity one*

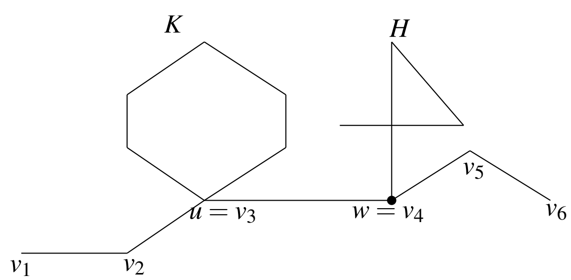


Figure 6. Graph of $KH + P_6$ with core vertex v_4

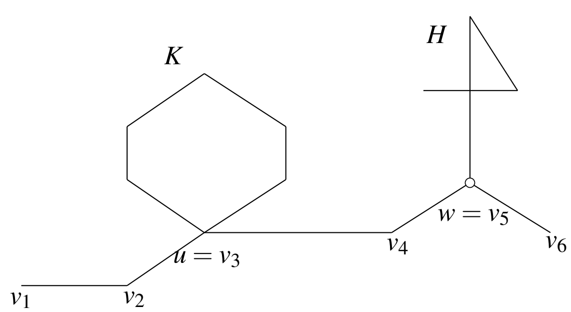


Figure 7. Graph of $KH + P_6$ with noncore vertex v_5.

Example 2.18. *In Figure 8, K, H are nonsingular and $K - u$, $H - w$ are singular. The graph $K \circ P_6$, the coalescence of K and P_6 with respect to the vertex u of K and v_3 of P_6 is a singular graph of nullity one. The graph $KH + P_6$ is obtained by coalescing $K \circ P_6$ with H at a nonecore vertex v_5 of $K \circ P_6$ and w of H. The graph $KH + P_6$ is a nonsingular graph of nullity two.*

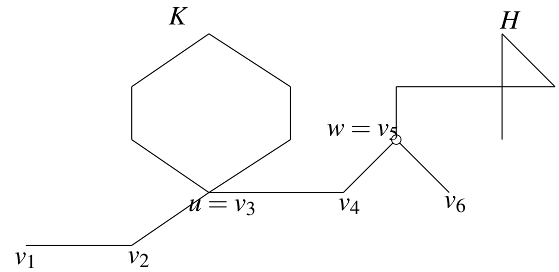


Figure 8. Graph of $KH + P_6$ with noncore vertex v_5.

Theorem 2.19. *Let (K, u) and (H, w) be the components of the graph obtained by deleting the path P_{2t} from $KH + P_{2t}$. If K is singular with nullity 1, u is a core vertex, H and $H - w$ are nonsingular, then $KH + P_{2t}$ is nonsingular.*

Proof. Since u is a core vertex, $K \circ P_{2t}$, the coalescence of K and P_{2t} with respect to u and any vertex v_j of P_{2t} is a nonsingular graph. As $H - w$ is nonsingular, $(K \circ P_{2t}) \circ H = KH + P_{2t}$ is nonsingular. □

Theorem 2.20. *Let (K, u) and (H, w) be the components of the graph obtained by deleting the path P_{2t} from $KH + P_{2t+1}$. If K is singular with nullity 1, u is a noncore vertex, H and $H - w$ are nonsingular, then $KH + P_{2t+1}$ is a singular graph of nullity one or two.*

Proof. $K \circ P_{2t+1}$, the coalescence of K and P_{2t+1} with respect to the noncore vertex u and any vertex v_j of P_{2t} is a singular graph of nullity two. To construct the graph $KH + P_{2t}$, we coalesce the graph $K \circ P_{2t}$ and H with respect to any vertex v_j of P_{2t} and w of H. Being a singular graph $K \circ P_{2t}$ has core and noncore vertices (of null spread 0 and -1). If v_j is a core vertex, $KH + P_{2t+1}$ is a singular graph of nullity one. If v_j is a noncore vertex (of null spread 0 and -1), then $KH + P_{2t}$ is a singular graph of nullity two (by Theorem 2.6 and 2.7). □

Theorem 2.21. *Let (K, u) and (H, w) be the components of the graph obtained by deleting the path P_{2t} from $KH + P_{2t}$. If K and H are singular with nullity one, then*

(i) $KH + P_{2t}$ is nonsingular when u and w are core vertices.

(ii) $KH + P_{2t}$ is singular of nullity greater than or equal to two when u and w are noncore vertices of null spread -1.

(iii) $KH + P_{2t}$ is singular of nullity greater than or equal to one when u is a core vertex and w is a noncore vertex or vice versa.

Proof. (i) If u is a core vertex, then $K-u$ is nonsingular. So $K\circ P_{2t}$ is nonsingular. Since w is a core vertex, we have $H-w$ is nonsingular and so $H\circ(K\circ P_{2t})$ is nonsingular.

(ii) If u is a noncore vertex of null spread -1, then $K-u$ is singular of nullity two. So $K\circ P_{2t}$ is singular of nullity two. Now, coalesce $K\circ P_{2t}$ and H with respect to any vertex v_j of P_{2t} and w of H. Since $K\circ P_{2t}$ is a graph of nullity two, it contains core and noncore (of null spread zero and null spread -1) vertices. If v_j is a core vertex, $(K\circ P_{2t})\circ H$ has nullity two. If v_j is a noncore vertex of null spread -1, $K\circ(P_{2t}\circ H)$ has nullity four. If v_j is a noncore vertex of null spread 0, then $K\circ(P_{2t}\circ H)$ has nullity three.

(iii) Without loss of generality suppose that u is a core vertex and w is a noncore vertex (of null spread -1). Clearly $K\circ P_{2t}$ is nonsingular as $K-u$ is nonsingular. Next Coalesce $K\circ P_{2t}$ and H with respect to any vertex v_j of P_{2t} and w of H. If $(K\circ P_{2t})-v_j$ is nonsingular, we have $K\circ(P_{2t}\circ H)$ is singular of nullity one. If $(K\circ P_{2t})-v_j$ is singular, then $(K\circ P_{2t})\circ H$ is singular with nullity two.

□

Theorem 2.21 can be generalized as follows:

Theorem 2.22. *Let (K,u) and (H,w) be the components of the graph obtained by deleting the path P_{2t} from $KH+P_{2t}$. If K and H are singular with nullity η_1 and η_2, then*

i $KH+P_{2t}$ is singular of nullity $\eta_1+\eta_2-2$ when u and w are core vertices.

ii $KH+P_{2t}$ is singular of nullity greater than or equal to $\eta_1+\eta_2$ when u and w are noncore vertices of null spread -1.

iii $KH+P_{2t}$ is singular of nullity greater than or equal to $\eta_1+\eta_2-1$ when u is a core vertex and w is a noncore vertex or vice versa.

Example 2.23. *In Figure 9, K is a singular graph of nullity 3 and H is a singular graph of nullity 2. K and H are joined by P_4 at the noncore vertices of null spread -1 of K and H. The graph $KH+P_4$ is singular of nullity 7.*

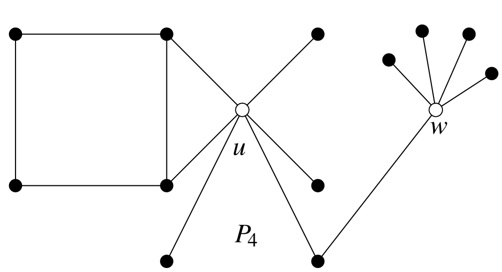


Figure 9. Larger singular graph constructed by an even Path

2.2 Construction using the path P_{2t+1}

In this section we construct the graph $KH+P_{2t+1}$ using the path P_{2t+1} of odd number of vertices. We have the following obvious result about paths having odd number of vertices.

Theorem 2.24. *In a path $P_{2t+1}=v_1,v_2,v_3,\ldots,v_{2t+1}$, the vertices $v_2,v_4,\ldots,v_{2t}$ are noncore vertices of null spread -1 and $v_1,v_3,\ldots,v_{2t+1}$ are core vertices.*

The path P_{2t+1} contains core vertices and noncore vertices of null spread -1. Also, there is no edges between noncore verices. Being a graph of nullity one, it is clearly a minimal configuration.

Lemma 2.25. *Let K be a nonsingular graph such that $K-u$ is singular, where $u\in K$. If $K\circ P_{2t+1}$ is the coalescence of K and P_{2t+1} with respect to $u\in K$ and a core vertex $v\in P_{2t+1}$, then $K\circ P_{2t+1}$ is nonsingular. Moreover, $K\circ P_{2t+1}-v'$ for any $v'\in P_{2t+1}$ is singular.*

Proof. By Theorem 2.7, we have $K\circ P_{2t+1}$ is nonsingular. Now it remains to prove that $K\circ P_{2t+1}-v'$ for any $v'\in P_{2t+1}$ is singular. Suppose that v' is a non-pendant core vertex of P_{2t+1}. Then $P_{2t+1}-v'$ will split into two points of even number of vertices, each of which is nonsingular. As $K-u$ is singular, the coalescence of K with any of these paths of even number of vertices yield a singular graph of nullity one. If v' is a pendant core vertex, then $P_{2t+1}-v'$ is a path of even number of vertices and coalescence of this path with K again yield a singular graph of nullity one. Next suppose that v' is a noncore vertex of P_{2t+1}. In this case $P_{2t+1}-v'$ will split into two parts of odd number of vertices. The coalescence of K with any if these paths at its core vertices yield a nonsingular graph. Since the other odd component is singular, $K\circ P_{2t+1}-v'$ is singular. □

Theorem 2.26. *Let $KH+P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at core vertices of P_{2t+1}. If K,H and $H-w$ are nonsingular and $K-u$ is singular, then $KH+P_{2t+1}$ is nonsingular.*

Proof. We know that $KH+P_{2t+1}=(K\circ P_{2t+1})\circ H$. Since K is a nonsingular and the coalescence is done at a core vertex of P_{2t+1}, by Lemma 2.2, we have $K\circ P_{2t+1}$ is nonsingular. Now $(K\circ P_{2t+1})\circ H$ is nonsingular as H and $H-w$ are nonsingular. □

Theorem 2.27. *Let $KH+P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at core vertices of P_{2t+1}. If K,H are nonsingular and $K-u$, $H-w$ are singular, then $KH+P_{2t+1}$ is singular.*

Proof. We know that $KH+P_{2+1t}=(K\circ P_{2t+1})\circ H$. Since K is nonsingular and the coalescence of K and P_{2t+1} is done at the core vertices of P_{2t+1}, by Lemma 2.25, we have $K\circ P_{2t+1}$ is nonsingular. As $K\circ P_{2t+1}-v'$ for any $v'\in P_{2t+1}$ and $H-w$ are singular, it follows that $KH+P_{2t+1}$ is singular. □

Remark 2.28. *The graph $KH+P_{2t+1}$ is singular if H in Theorem 2.27 is also singular. However, if it is singular and $H-w$ is nonsingular, then $KH+P_{2t+1}$ is nonsingular.*

Lemma 2.29. *Let K be a nonsingular graph such that $K-u$ is singular, where $u \in K$. If $K \circ P_{2t+1}$ is the coalescence of K and P_{2t+1} with respect to $u \in K$ and a noncore vertex $v \in P_{2t+1}$, then $K \circ P_{2t+1}$ is singular of nullity two. Moreover, $K \circ P_{2t+1} - v'$ for any $v' \in P_{2t+1}$ is singular.*

Proof. By Theorem 2.7, we have $K \circ P_{2t+1}$ is singular of nullity two. Now, we prove that $K \circ P_{2t+1} - v'$ for $v' \in P_{2t+1}$ is singular. First suppose that v' is a non-pendant core vertex of P_{2t+1}. Then $P_{2t+1} - v'$ will split into two paths of even number of vertices each of which is nonsingular. Since $K-u$ is singular, the coalescence of K with any of these paths yield a singular graph of nullity one. So $K \circ P_{2t+1} - v'$ is singular if v' is a non-pendant core vertex of P_{2t+1}. If v' is a pendant core vertex, then $P_{2t+1} - v'$ is a path of even number of vertices and coalescence of this path with K again yield a singular graph of nullity one. Finally suppose that v' is a noncore vertex of P_{2t+1}. In this case, $P_{2t+1} - v'$ will split into two paths of odd number of vertices. The coalescence of K with any of these at its noncore vertices yield a nonsingular graph of nullity two. Since the other odd component is singular of nullity one, it follows that $K \circ P_{2t+1} - v'$ is singular of nullity three. ☐

Theorem 2.30. *Let $KH+P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at noncore vertices of P_{2t+1}. If K,H and $H-w$ are nonsingular and $K-u$ is singular, then $KH+P_{2t+1}$ is singular.*

Proof. We know that $KH+P_{2t+1} = (K \circ P_{2t+1}) \circ H$. By Lemma 2.29, we know $K \circ P_{2t+1}$ is singular of nullity two. Since the coalescence is done at noncore vertices of P_{2t+1} and $K \circ P_{2t+1}$ is singular of nullity two, we have $KH+P_{2t+1}$ is singular. ☐

Lemma 2.31. *Let K be a singular graph such that $K-u$ is nonsingular, where $u \in K$. If $K \circ P_{2t+1}$ is the coalescence of K and P_{2t+1} with respect to $u \in K$ and a core vertex $v \in P_{2t+1}$, then $K \circ P_{2t+1}$ is singular. Moreover, $K \circ P_{2t+1} - v'$ for any $v' \in P_{2t+1}$ is nonsingular if v' is a core vertex and singular if v' is a noncore vertex.*

Proof. Since K and P_{2t+1} are singular, and $K-u$, $P_{2t+1}-v$ are nonsingular, it follows by Theorem 2.11 that $K \circ P_{2t+1}$ is a singular graph of nullity one. First suppose that v' is a pendant core vertex of P_{2t+1}. Clearly $P_{2t+1} - v'$ is a path of even number of vertices and the coalescence of this path with K yield a nonsingular graph. If v' is a non-pendant core vertex of P_{2t+1}, $P_{2t+1} - v'$ will split into two paths of even number of vertices. The coalescence of K with any of these even paths gives a nonsingular graph. Finally if v' is a noncore vertex, then $P_{2t+1} - v'$ with split into two paths of odd number of vertices. The coalescence of any one of these path at its core vertices with K yield a singular graph of nullity one. Since the other component is also singular of nullity one, we have $K \circ P_{2t+1} - v'$ is a singular graph of nullity two, if v' is a noncore vertex. ☐

Theorem 2.32. *Let $KH+P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at core vertices of P_{2t+1}. If K is singular and $K-u$, H, $H-w$ are nonsingular, then $KH+P_{2t+1}$ is nonsingular.*

Proof. We know that $KH+P_{2t+1} = (K \circ P_{2t+1}) \circ H$. As K is singular and $K-u$ is nonsingular and the coalescence is done at the core vertex of P_{2t+1}, by Lemma 2.31, we can see that $K \circ P_{2t+1}$ is singular. We are coalescing $K \circ P_{2t+1}$ with H at the core vertices of P_{2t+1} to construct $KH+P_{2t+1}$. Since $K \circ P_{2t+1} - v'$ is nonsingular for core vertex $v' \in P_{2t+1}$, it follow that $KH+P_{2t+1}$ is nonsingular. ☐

Theorem 2.33. *Let $KH+P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at core vertices of P_{2t+1}. If K, H are singular and $K-u$, $H-w$ are nonsingular, then $KH+P_{2t+1}$ is singular.*

Proof. We know that $KH+P_{2t+1} = (K \circ P_{2t+1}) \circ H$. Here also, we have by Lemma 2.31, $K \circ P_{2t+1}$ is singular. Since H is singular and $K \circ P_{2t+1}$ is singular of nullity one, it is clear that $(K \circ P_{2t+1}) \circ H$ is a singular graph of nullity one. ☐

Lemma 2.34. *Let K be a singular graph such that $K-u$ is nonsingular, where $u \in K$. If $K \circ P_{2t+1}$ is the coalescence of K and P_{2t+1} with respect to $u \in K$ and a noncore vertex $v \in P_{2t+1}$, then $K \circ P_{2t+1}$ is a singular graph of nullity one. Moreover, $K \circ P_{2t+1} - v'$ for any $v' \in P_{2t+1}$ is nonsingular if v' is a core vertex and singular of nullity two if v' is a noncore vertex.*

Proof. Proof of the lemma is similar to the proof of Lemma 2.29. ☐

Theorem 2.35. *Let $KH+P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at noncore vertices of P_{2t+1}. If K is singular and $K-u$, H, $H-w$ are nonsingular, then $KH+P_{2t+1}$ is singular of nullity one.*

Proof. We know that $KH+P_{2t+1} = (K \circ P_{2t+1}) \circ H$. By Lemma 2.34, $K \circ P_{2t+1}$ is a singular graph of nullity one. As $K \circ P_{2t+1} - v'$ is singular of nullity two if v' is a nocore vertex of P_{2t+1}, it follows that $KH+P_{2t+1}$ is a singular graph of nullity one. ☐

Theorem 2.36. *Let $KH+P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at noncore vertices of P_{2t+1}. If K, $H-w$ are singular and $K-u$, H are nonsingular, then $KH+P_{2t+1}$ is a singular graph of nullity two.*

Proof. We know that $KH+P_{2t+1} = (K \circ P_{2t+1}) \circ H$. By Lemma 2.34, $K \circ P_{2t+1}$ is a singular graph of nullity one. As $K \circ P_{2t+1} - v'$ is a singular graph of nullity two if v' is a noncore vertex of P_{2t+1} and $H-w$ is singular, we have $(K \circ P_{2t+1}) \circ H$ is a singular graph of nullity two. ☐

Lemma 2.37. *Let K be a nonsingular graph such that $K-u$ is nonsingular, where $u \in K$. If $K \circ P_{2t+1}$ is the coalescence of K and P_{2t+1} with respect to $u \in K$ and a noncore vertex $v \in P_{2t+1}$, then $K \circ P_{2t+1}$ is a singular graph of nullity one. Moreover, $K \circ P_{2t+1} - v'$ for any $v' \in P_{2t+1}$ is nonsingular if v' is a core vertex and singular if v' is a noncore vertex.*

Proof. Proof of lemma is similar to the proof of Lemma 2.29. □

Theorem 2.38. *Let K and H be two nonsingular graphs and $KH + P_{2t+1}$ is the graph obtained by joining (K,u) to P_{2t+1} at a noncore vertex of P_{2t+1} and (H,w) to P_{2t+1} at a core vertex of P_{2t+1}. Then $KH + P_{2t+1}$ is nonsingular or singular of nullity one according as $K-u$ is nonsingular or singular.*

Proof. We know that $KH + P_{2t+1} = (K \circ P_{2t+1}) \circ H$. By Lemma 2.37, if $K-u$ is nonsingular, then $K \circ P_{2t+1}$ is a singular graph of nullity one. Also, by Lemma 2.37, $K \circ P_{2t+1} - v'$ for $v' \in P_{2t+1}$ is nonsingular, for core vertex v'. Since H is nonsingular and (H,w) is joining to P_{2t+1} at a core vertex, we have $KH + P_{2t+1}$ is nonsingular. By Lemma 2.29, if $K-u$ is singular, then $K \circ P_{2t+1}$ is singular of nullity two. Also, by Lemma 2.29, $K \circ P_{2t+1} - v'$ for any $v' \in P_{2t+1}$ is singular. So, clearly we have, $KH + P_{2t+1}$ is singular. □

Lemma 2.39. *Let K be a singular graph of nullity η such that $K-u$ is singular of nullity $\eta - 1$ for $u \in K$ (u is a core vertex). If $K \circ P_{2t+1}$ is the coalescence of K and P_{2t+1} with respect to $u \in K$ and a core vertex of P_{2t+1}, then $K \circ P_{2t+1}$ is a singular graph of nullity η. Moreover, $K \circ P_{2t+1} - v'$ is singular of nullity $\eta - 1$, if v' is a core vertex and singular of nullity $\eta + 1$, if v' is a noncore vertex of P_{2t+1}.*

Proof. Proof is similar to Lemma 2.37. □

Theorem 2.40. *Let (K,u) and (H,w) be two singular graphs of nullity η_1 and η_2 respectively and u is a core vertex and w is a noncore vertex of null spread -1. If the graph $KH + P_{2t+1}$ is obtained by joining (K,u) to a core vertex of P_{2t+1} and (H,w) to a noncore vertex of P_{2t+1}, then $KH + P_{2t+1}$ is a singular graph of nullity $\eta_1 + \eta_2 + 1$.*

Proof. We know that $KH + P_{2t+1} = (K \circ P_{2t+1}) \circ H$. By Lemma 2.39, we have $K \circ P_{2t+1}$ is a singular graph of nullity η_1. Since $K \circ P_{2t+1} - v'$ for noncore $v' \in P_{2t+1}$ is singular of nullity $\eta_1 + 1$, by Lemma 2.39 and $H-w$ is singular of nullity $\eta_2 + 1$, it follows from Theorem 2.3 that $KH + P_{2t+1}$ is singular of nullity $\eta_1 + \eta_2 + 1$. □

Theorem 2.41. *Let (K,u) and (H,w) be two singular graphs of nullity η_1 and η_2 respectively and u is a core vertex and w is a noncore vertex of null spread -1. If the graph $KH + P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at the core vertices of P_{2t+1}, then $KH + P_{2t+1}$ is a singular graph of nullity $\eta_1 + \eta_2 - 1$.*

Proof. We know that $KH + P_{2t+1} = (K \circ P_{2t+1}) \circ H$. By Lemma 2.39, $KH + P_{2t+1}$ is a singular graph of nullity η_1. As $K \circ P_{2t+1} - v'$ for core $v' \in P_{2t+1}$ is singular of nullity $\eta_1 - 1$ and $H-w$ is singular of nullity $\eta_2 + 1$ by Theorem 2.5, we have $KH + P_{2t+1}$ is singular of nullity $\eta_1 + \eta_2 - 1$. □

Lemma 2.42. *Let K be a singular graph of nullity η such that $K-u$ is singular of nullity $\eta + 1$ (u is a noncore vertex of null spread -1). If $K \circ P_{2t+1}$ is the coalescence of K and P_{2t+1} with respect to $u \in K$ and a noncore vertex of P_{2t+1}, then $K \circ P_{2t+1}$ is a singular graph of nullity $\eta + 2$. Moreover, $K \circ P_{2t+1} - v'$ is singular of nullity $\eta + 1$, if v' is a core vertex and singular of nullity $\eta + 3$, if v' is a noncore vertex of P_{2t+1}.*

Proof. Proof of the lemma is similar to the proofs of other lemmas. □

Theorem 2.43. *Let $KH + P_{2t+1}$ is obtained by joining (K,u) and (H,w) by P_{2t+1} at noncore vertices of P_{2t+1}. If K and H are singular graphs of nullity η_1 and η_2 respectively and u,w are noncore vertices, then $KH + P_{2t+1}$ is singular of nullity $\eta_1 + \eta_2 + 3$.*

Proof. We know that $KH + P_{2t+1} = (K \circ P_{2t+1}) \circ H$. By Lemma 2.42, we have, $K \circ P_{2t+1}$ is singular of nullity $\eta_1 + 2$. Since by Lemma 2.42, $K \circ P_{2t+1} - v'$ for noncore $v' \in P_{2t+1}$ is singular of nullity $\eta_1 + 3$ and $H-w$ is singular of nullity $\eta_2 + 1$, it follows by theorem 2.3 that $KH + P_{2t+1}$ is singular of nullity $\eta_1 + \eta_2 + 3$. □

Remark 2.44. *In the above theorem if (K,u) and (H,w) are connected by P_{2t+1} at core vertices of P_{2t+1} then $KH + P_{2t+1}$ is a singular graph of nullity $\eta_1 + \eta_2 + 1$. If (K,u) and (H,w) are connected by P_{2t+1} at a core vertex to (K,u) and at a noncore vertex to (H,w), then $KH + P_{2t+1}$ is again a singular graph of nullity $\eta_1 + \eta_2 + 1$. Also, if u and w are core vertices in theorem 2.43, then $KH + P_{2t+1}$ is a singular graph of nullity $\eta_1 + \eta_2 + 1$.*

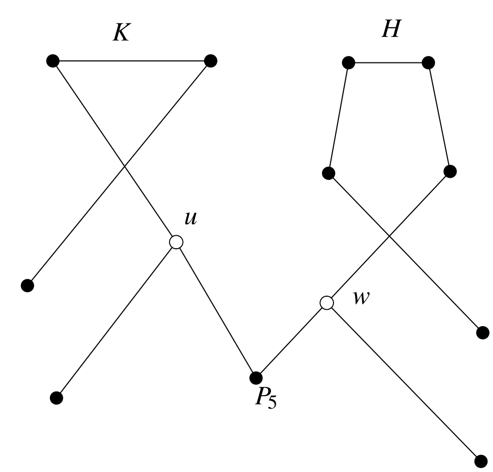


Figure 10. The graph $KH + P_5$ with $K-u$ and $H-w$ nonsingular.

Example 2.45. *In Figure 10, the graph K, H are singular and $K-u$, $H-w$ are nonsingular. The graph $KH+P_5$ is a singular graph of nullity one. Note that K and H are joined at the noncore vertices of P_5.*

Example 2.46. *In Figure 11, the graphs K, $H-w$ are singular of nullity one and $K-u$, H are nonsingular. The graph $KH+P_5$ is a singular graph of nullity two. Obviously K and H joined at noncore vertices of P_5.*

Example 2.47. *Theorem 2.43 can be used to construct singular graph of larget nullity. In Figure 12, we have a singular graph of nullity 23.*

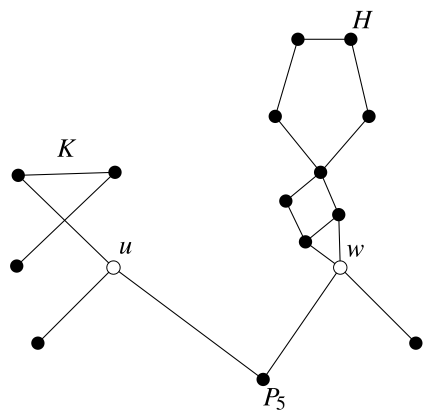


Figure 11. The graph $KH+P_5$ with $K-u$ and H nonsingular.

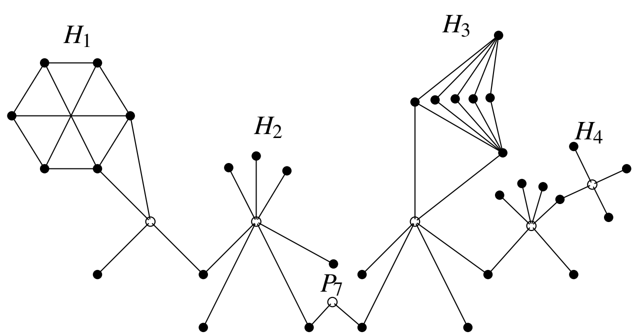


Figure 12. A singular graph of larger nullity.

3. Conclusion

In this paper, we have constructed larger singular and nonsingular graphs using a path. The presence of three types of vertices in a singular graph is the key in the construction. We have established that how a graph of largest nullity can be constructed using a path. Singular graphs have important applications in mathematics and science. Singular graphs are widely used in Chemistry. If the nullity of the molecular graph is greater than zero, then the corresponding chemical compound is highly reactive or unstable. Because of this reason, the Chemists have a great interest in singular graphs. So we are sure that, in some way, our findings will have application in Chemistry.

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ISSN(P):2319 − 3786
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