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A note on meromorphic functions with positive coefficients associated wth Bessel function

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Abstract

The aim of the present paper is to introduce a new subclass of meromorphic functions with positive coefficients defined by a certain integral operator and obtain coefficient inequality, convex linear combinations, extreme point, radii of close-to-convexity, starlikeness, convexity, Hadamard product and integral transforms for the functions f in this class.

Keywords

Meromorphic, coefficient inequality, starlikeness, Hadamard product.

AMS Subject Classification

30C45.

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1. Introduction

Let $\sum$ be denote the class of functions $f(z)$ of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,\ n\in N=\{1,2,3,\cdots\} \tag{1.1}$$

which are analytic in the punctured unit disc
$U^*=\{z\in C:0<|z|<1\}=U\setminus\{0\}$.

Analytically a function $f\in\Sigma$ given by (1.1) is said to be meromorphically starlike of order α if it satisfies the following:

$$Re\left\{-\left(\frac{zf'(z)}{f(z)}\right)\right\}>\alpha,\ \ (z\in U)$$

for some $\alpha(0\le\alpha<1)$. We say that f is in the class $\sum^*(\alpha)$ of such functions.

Similarly a function $f\in\sum$ given by (1.1) is said to be meromorphically convex of order α if it satisfies the following:

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha,\ \ (z\in U)$$

for some $\alpha(0\le\alpha<1)$. We say that f is in the class $\Sigma_k(\alpha)$ of such functions.

For a function $f\in\Sigma$ given by (1.1) is said to be meromorphically close to convex of order β and type α if there exists a function $g\in\Sigma^*(\alpha)$ such that

$$Re\left\{-\left(\frac{zf'(z)}{g(z)}\right)\right\}>\beta,\ \ (0\le\alpha<1,\ 0\le\beta<1,\ z\in U).$$

We say that f is in the class $K(\beta,\alpha)$.

The class $\sum^*(\alpha)$ and various other subclasses of $\sum$ have been studied rather extensively by Clunie [5], Miller [10], Pommerenke [13], Royster [14], Akgul [1, 2], Sakar and Guney [11, 12] and Venkateswarlu et al. [15].

Recent years, many authors investigated the subclass of meromorphic functions with positive coefficient Juneja and Reddy [9] class Σ_p functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,a_n\ge 0 \tag{1.2}$$

which are regular and univalent in U. The functions in this class are said to be meromorphic functions with positive coefficients.

For functions $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

We recall here the generalized Bessel function of first kind of order γ (see [6]), denoted by

$$w(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(\gamma+n+\frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+\gamma} (z \in U)$$

(where Γ stands for the Gamma Euler function) which is the particular solution of the second order linear homogeneous differential equation (see, for details, [16])

$$z^2 w''(z) + bzw'(z) + [cz^2 - \gamma^2 + (1-b)\gamma]w(z) = 0$$

where $c, \gamma, b \in C$.

We introduce the function φ defined, in terms of the generalized Bessel function w by

$$\varphi(z) = 2^{\gamma}\Gamma\left(\gamma + \frac{b+1}{2}\right) z^{-\left(1+\frac{\gamma}{2}\right)} w(\sqrt{z}).$$

By using the well-known Pochhammer symbol $(x)_\mu$ defined, for $x \in C$ and in terms of the Euler gamma function, by
$(x)_\mu = \frac{\Gamma(x+\mu)}{\Gamma(x)} = 1, (\mu = 0)$
$= x(x+1)(x+2)\cdots(x+n-1), (\mu = n \in N = \{1,2,3\cdots\})$
We obtain the following series representation for the function $\varphi(z)$ $\varphi(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-c)^{n+1}}{4^{n+1}(n+1)!(\tau)_{n+1}} z^n$
$\left(\tau = \gamma + \frac{b+1}{2} \notin Z_0^- = \{0,-1,-2,\cdots\}\right)$.

Corresponding to the function φ define the Bessel operator S_τ^c by the following Hadamard product

$$\begin{aligned} S_\tau^c f(z) = (\varphi * f)(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^{n+1} a_n}{(n+1)!(\tau)_{n+1}} z^n \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \phi(n,\tau,c) a_n z^n, \end{aligned} \tag{1.3}$$

where $\phi(n,\tau,c) = \frac{\left(\frac{-c}{4}\right)^n}{(n)!(\tau)_n}$.

It easy to verify from the definition (1.3) that

$$z[S_{\tau+1}^c f(z)] = \tau S_\tau^c f(z) - (\tau+1) S_{\tau+1}^c f(z). \tag{1.4}$$

Now we introduce the following subclass of $\sum_p$ associated with Bessel function .,

Definition 1.1. *A function $f \in \sum$ is said to be in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$ if and only if satisfies the inequality*

$$Re\left\{\frac{-z\left((\lambda\alpha z^2 S_\tau^c f(z))''+(\lambda-\alpha)z(S_\tau^c f(z))'+(1-\lambda+\alpha)S_\tau^c f(z)\right)'}{(\lambda\alpha z^2 S_\tau^c f(z))''+(\lambda-\alpha)z(S_\tau^c f(z))'+(1-\lambda+\alpha)S_\tau^c f(z)}\right\}$$
$$> \beta\left|\frac{z\left(\lambda\alpha z^2 (S_\tau^c f(z))''+(\lambda-\alpha)z(S_\tau^c f(z))'+(1-\lambda+\alpha)S_\tau^c f(z)\right)'}{(\lambda\alpha z^2 S_\tau^c f(z))''+(\lambda-\alpha)z(S_\tau^c f(z))'+(1-\lambda+\alpha)S_\tau^c f(z)} + 1\right| + \xi \tag{1.5}$$

where $0 \le \tau < 1, 0 \le\ c \le 1, 0 \le \xi < 1$,
$0 \le \alpha < \lambda < \frac{1}{2}, \beta \ge 0$.

Let $\Phi(z) = \lambda\alpha z^2 (S_\tau^c f(z))'' +$
$(\lambda-\alpha)z(S_\tau^c f(z))' + (1-\lambda+\alpha)S_\tau^c f(z)$. (1.6)

If we write equation (1.6) in the inequality (1.5), then by a simple calculation the inequality (1.5) can be written as

$$Re\left\{\frac{-z\Phi'(z)}{\Phi(z)}\right\} > \beta\left|\frac{z\Phi'(z)}{\Phi(z)} + 1\right| + \xi. \tag{1.7}$$

It is easily shown that there is following equality between these subclasses

$$\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta) = \Sigma(\alpha,\lambda,\tau,\ c,\xi,\beta) \cap \Sigma_p.$$

In order to prove our results wee need the following lemmas [3].

Lemma 1.2. *Let σ be a real number and $\omega = u+iv$ is a complex number. Then*

$$Re(\omega) \ge \sigma \Leftrightarrow |\omega - (1+\sigma)| \le |\omega + (1-\sigma)|.$$

Lemma 1.3. *Let $\omega = u+iv$ be a complex number and σ, γ are real numbers. Then*
$Re(-\omega) \ge \sigma|-\omega-1| + \gamma$
$\Leftrightarrow Re\left(-\omega(1+\sigma e^{i\ c}) - \sigma\gamma e^{i\ c}\right) \ge \gamma, (-\pi \le\ c \le \pi)$.

The purpose of this paper is to introduce a new subclass of meromorphic functions with positive coefficients and obtain the necessary and sufficient conditions for the functions defined by the relation (1.3) in this class. We also obtain coefficient inequality, convex linear combinations, extreme point, radii of close-to-convexity, starlikeness, convexity, Hadamard product and integral transforms for this class.

2. Coefficient estimates

We obtain in this section a necessary and sufficient condition for a function f to be in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. We employ the technique adopted by Aqlan et al. [3] and Athsan and Kulkarni [4] to find the coefficient estimates for the functions f defined by the equation (1.3) in the class

$\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. A subclass for meromorphic function $f\in\Sigma_p$ given by the equality (1.3) with positive coefficient was defined and investigated in [6]. In this study we modified and extended their subclass to the subclass of the functions $f\in\Sigma_p$ defined by the certain Bessel function.

Theorem 2.1. *A meromorphic function f defined by the equation (1.3) in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$ if and only if*

$$\sum_{n=1}^{\infty}[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)a_n \le (1-\xi)(2\alpha\lambda-2\lambda+2\alpha+1) \tag{2.1}$$

for some $0\le\xi<1, 0\le\tau<1, 0\le\ c\le 1, 0\le\alpha\le\lambda<\frac{1}{2}$ *and* $\beta\ge 0$.

Proof. Let $f\in\Sigma_p$ and suppose satisfies the condition (2.1). Then by applying Lemma 1.3, we have to show that

$$Re\left\{\frac{-z(\Phi(z))'}{\Phi(z)}(1+\beta e^{i\varphi})-\beta e^{i\varphi}\right\}>\xi,$$

$(-\Pi\le\varphi\le\Pi, 0\le\xi<1,\beta\ge 0)$ or equivalently

$$Re\left\{\frac{-z(\Phi(z))'(1+\beta e^{i\varphi})-\beta e^{i\varphi}\Phi(z)}{\Phi(z)}\right\}>\xi. \tag{2.2}$$

Let $\Psi(z)=-z\Phi'(z)[1+\beta e^{i\varphi}]-\beta e^{i\varphi}\Phi(z)$.

Thus, the equation (2.2) is equivalent to

$$Re\left\{\frac{\Psi(z)}{\Phi(z)}\right\}>\xi. \tag{2.3}$$

In view of Lemma 1.2, it is sufficient to prove that

$$|\Psi(z)+(1-\xi)\Phi(z)|-|\Psi(z)-(1+\xi)\Phi(z)|>0. \tag{2.4}$$

Therefore

$$\begin{aligned}&|\Psi(z)+(1-\xi)\Phi(z)|=\\&\Big|(2-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)\left(\tfrac{1}{z}\right)\\&+\sum_{n=1}^{\infty}[(1-n-\xi)+\beta(-n-1)e^{\varphi}]\\&[(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\phi(n,\tau,c)a_nz^n\Big|\\&\ge(2-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)\left|\tfrac{1}{z}\right|\\&-\sum_{n=1}^{\infty}[(n+\xi-1)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)\\&+1]\phi(n,\tau,c)a_n|z^n|.\end{aligned} \tag{2.5}$$

Thus, similarly we obtain

$$\begin{aligned}&|\Psi(z)-(1+\xi)\Phi(z)|\\&\le(2-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)\left|\tfrac{1}{z}\right|\\&+\sum_{n=1}^{\infty}[(n+\xi+1)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)\\&+1]\phi(n,\tau,c)a_n|z^n|.\end{aligned} \tag{2.6}$$

Thus from (2.5) and (2.6), we get

$$\begin{aligned}&|\Psi(z)+(1-\xi)\Phi(z)|-|\Psi(z)-(1+\xi)\Phi(z)|\\&\ge 2(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)\\&\quad-\sum_{n=1}^{\infty}[2(n+\xi)+2\beta(n+1)]\\&\qquad[(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\phi(n,\tau,c)a_n\\&\quad\ge 0.\end{aligned}$$

If we use the inequality (2.1) in last inequality then we obtain the desired result.

Conversely assume that that the function f defined by the equation (1.1) is in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. That is, the inequality (1.5) holds for the function f. By choosing the value of z on the positive real axis, where $0\le|z|=r<1$ the inequality (1.5) reduced to

$$\Re\left\{\frac{(1+\xi)(2\lambda\alpha-2\lambda+2\alpha+1)-\sum_{n=1}^{\infty}[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\phi(n,\tau,c)a_nr^{n+1}}{(2\lambda\alpha-2\lambda+2\alpha+1)+\sum_{n=1}^{\infty}[(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\phi(n,\tau,c)a_nr^{n+1}}\right\}$$

≥ 0, where $Re(-e^{i\varphi})\ge-|e^{i\varphi}|=-1$.

Letting $r\to 1^-$ through positive values, we obtain

$$\sum_{n=1}^{\infty}[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\phi(n,\tau,c)a_n \le (1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)$$

and this is desired result. □

Corollary 2.2. *Let the function f defined by the equation (1.3) be in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. Then*

$$a_n\le\frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\phi(n,\tau,c)}$$

for $n\ge 1$. The result is sharp for each n for the functions of the form

$$f_n(z)=\frac{1}{z}+\frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\phi(n,\tau,c)}z^n, \tag{2.7}$$

where $n\ge 1$.

3. Convex Linear Combination

In this section, we shall prove the closure theorem of the functions given by the form (1.3) in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$.

Theorem 3.1. *The class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$ is closed under convex linear combination.*

Proof. Let the functions $f(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}a_nz^n$ and $g(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}b_nz^n$ be in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$.
Then by Theorem 2.1, we have

$$\sum_{n=1}^{\infty}[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\times\phi(n,\tau,c)a_n \leq (1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1) \tag{3.1}$$

and

$$\sum_{n=1}^{\infty}[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\times\phi(n,\tau,c)a_n \leq (1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1). \tag{3.2}$$

For $0\leq\tau\leq1$, define the function h as $h(z)=\tau f(z)+(1-\tau)g(z)$.
Then, we get $h(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}[\tau a_n+(1-\tau)b_n]z^n$.
Now, we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}[(n+\xi)+\beta(n+1)]\\&\times[(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)[\tau a_n+(1-\tau)b_n]\\&=\tau[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)a_n\\&+(1-\tau)[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\\&\times\phi(n,\tau,c)b_n\\&\leq\ \tau(1-\alpha)+(1-\tau)(1-\alpha)\\&=\ (1-\alpha).\end{aligned}$$

So, $h(z)\in\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. ☐

4. Convex Linear Combination

Theorem 4.1. *Let* $f_0(z)=\frac{1}{z}$ *and*
$f_n(z)=\frac{1}{z}+\frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}z^n$, $n\in N$. *Then* $f\in\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$ *if and only if it can be represented in the form*
$f(z)=\sum\limits_{n=0}^{\infty}\omega_nf_n(z)$, *where* $\omega_n\geq0$ *and* $\sum\limits_{n=0}^{\infty}\omega_n=1$.

Proof. Assume that

$$f(z)=\sum_{n=0}^{\infty}\omega_nf_n(z),\left(\omega_n\geq0,n=0,1,2\cdots,\ \sum_{n=0}^{\infty}\omega_n=1\right).$$

Then, we have

$$\begin{aligned}f(z)&=\sum_{n=0}^{\infty}\omega_nf_n(z)\\&=\omega_0f_0(z)+\sum_{n=1}^{\infty}\omega_nf_n(z)\\&=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}z^n.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{n=1}^{\infty}\omega_n\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)\sum\limits_{n=1}^{\infty}}\\&\times\frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}\\&=\sum_{n=1}^{\infty}\omega_n=1-\omega_0\leq1.\end{aligned}$$

Hence, by Theorem 2.1, $f\in\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$.

Conversely, suppose that $f\in\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. Since by Corollary 2.2,
$a_n\leq\frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)},\ \ n\geq1$
if we set
$\omega_n=\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}a_n,\ n\geq1$
and $\omega_0=1-\sum\limits_{n=1}^{\infty}\omega_n$, then we obtain

$$f(z)=\omega_0f_0(z)+\sum_{n=1}^{\infty}\omega_nf_n(z).$$

This completes the proof of the theorem. ☐

5. Radii of Starlikeness and Convexity

In this section, we find the radii of meromorphically close-to-convexity, starlikeness and convexity for functions f in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$.

Theorem 5.1. *Let* $f\in\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. *Then* f *is meromorphically close-to-convex of order* $\delta(0\leq\delta<1)$ *in the disk* $|z|<r_1$, *where*
$r_1=\inf\limits_{n\in N}\left[\frac{(1-\delta)[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{n(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}\right]^{\frac{1}{n+1}}$
and the result is sharp.

Proof. Let $f\in\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. It is sufficient to prove that

$$|z^2f'(z)+1|<1-\delta. \tag{5.1}$$

By Theorem 2.1, we have
$\sum\limits_{n=1}^{\infty}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}a_n\leq1.$
So the inequality
$|z^2f'(z)+1|=\left|\sum\limits_{n=1}^{\infty}na_nz^{n+1}\right|\leq\sum\limits_{n=1}^{\infty}na_n|z|^{n+1}<1-\delta$
holds true if

$$\frac{n|z|^{n+1}}{1-\delta}\leq\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}.$$

Then, (5.1) holds true if

$$|z|^{n+1}\leq\frac{[(1-\delta)(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{n(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)},$$

which yields the close-to-convexity of the function and completes the proof.
Also, the result is sharp for the functions of the form (2.7). ☐

Theorem 5.2. *Let* $f\in\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. *Then* f *is meromorphically starlike of order* $\delta(0\leq\delta<1)$
in the disk $|z|<r_2$, *where* $r_2=$
$\inf\limits_{n\in N}\left[\left(\frac{1-\delta}{n+2-\delta}\right)\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}\right]^{\frac{1}{n+1}}$
and the result is sharp.

Proof. Let $f \in \Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. It is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)}+1\right| < 1-\delta \tag{5.2}$$

where $0 \le \delta < 1,\ |z| < r_2$. For the function $f \in \Sigma_p$ given by the equation (1.3), we get

$$\left|\frac{zf'(z)}{f(z)}+1\right| \le \frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}a_n|z|^{n+1}} \le 1-\delta$$

holds true if

$$\frac{\sum_{n=1}^{\infty}(n+2-\delta)}{(1-\delta)}a_n|z|^{n+1}.$$

By Theorem 2.1, we have
$\sum_{n=1}^{\infty}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}a_n \le 1.$
Then inequality (5.2) holds true if

$\frac{(n+2-\delta)}{(1-\delta)}|z|^{n+1} \le \frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}$

which is equivalent to

$|z| \le \left[\frac{(1-\delta)}{(n+2-\delta)}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}\right]^{\frac{1}{n+1}}$

which yields the starlikeness of the function and completes the proof.
Also, the result is sharp for the functions of the form (2.7). □

Theorem 5.3. *Let $f \in \Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. Then f is meromorphically convex of order $\delta(0 \le \delta < 1)$ in the disk $|z| < r_3$, where*

$$r_3 = \inf_{n\in N}\left[\frac{(1-\delta)}{n(n+2-\delta)}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}\right]^{\frac{1}{n+1}}.$$

The result is sharp for the extremal function f given by

$$f_n(z) = \frac{1}{z}+\frac{n(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}z^n,\ n \ge 1.$$

Proof. By using the technique employed in the proof of Theorem 5.1 and 5.2, we can show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| < 1-\delta$$

for $|z| < r_3$ and prove that the assertion of the theorem is true. The result is sharp for the functions given by the equation (2.7). □

6. Hadamard Product

Theorem 6.1. *For functions $f,g \in \Sigma_p$ defined by (1.1), let $f,g \in \Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. Then the Hadamard product*

$$f*g \in \Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta),$$

where

$$\rho \le \frac{\left[[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\right]^2\phi(n,\tau,c)-n(1-\alpha)^2}{\left[[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\right]^2\phi(n,\tau,c)(1-\alpha)^2[1-\delta(n+1)]}.$$

Proof. From Theorem 2.1, we have

$$\sum_{n=1}^{\infty}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}a_n \le 1 \tag{6.1}$$

$$\sum_{n=1}^{\infty}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}b_n \le 1. \tag{6.2}$$

From (6.1) and (6.2) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{n=1}^{\infty}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)} \times\sqrt{a_nb_n} \le 1. \tag{6.3}$$

We need to find the largest ρ such that
$\sum_{n=1}^{\infty}\frac{[(n+\rho)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\rho)(2\lambda\alpha-2\lambda+2\alpha+1)}a_nb_n \le 1.$
Thus it is enough to show that
$\sum_{n=1}^{\infty}\frac{[(n+\rho)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\rho)(2\lambda\alpha-2\lambda+2\alpha+1)}a_nb_n \le \sum_{n=1}^{\infty}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}\sqrt{a_nb_n}$
that is,

$$\sqrt{a_nb_n} \le \frac{(1-\rho)[(n+\xi)+\beta(n+1)]}{(1-\xi)[(n+\xi)+\beta(n+1)]}. \tag{6.4}$$

On the other hand, from (6.3), we have

$$\sqrt{a_nb_n} \le \frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}. \tag{6.5}$$

Therefore in view of (6.4) and (6.5), it is enough to show that

$$\frac{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)} \le \frac{(1-\rho)[(n+\xi)+\beta(n+1)]}{(1-\xi)[(n+\xi)+\beta(n+1)]}$$

$$\rho \le 1-\frac{2(1-\xi)^2(2\lambda\alpha-2\lambda+2\alpha+1)\phi(n,\tau,c)}{[(n+\xi)+\beta(n+1)]^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)+2(1-\xi)^2(2\lambda\alpha-2\lambda+2\alpha+1)}.$$

Let $\Phi(n) = \frac{2(1-\xi)^2(2\lambda\alpha-2\lambda+2\alpha+1)\phi(n,\tau,c)}{[(n+\xi)+\beta(n+1)]^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)+2(1-\xi)^2(2\lambda\alpha-2\lambda+2\alpha+1)}.$

Clearly $\Phi(n)$ is an increasing function of $n(n \ge 1)$.
Letting $n=1$, we have prove the assertion. □

Theorem 6.2. *For functions $f,g \in \Sigma_p$ defined by (1.1), let $f,g \in \Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. Then the function $k(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)z^n$ is in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$, where*
$\rho \leq 1 - \frac{4(1-\xi)^2[(2\lambda\alpha-2\lambda+2\alpha+1)]\phi(n,\tau,c)}{[(n+\xi)+\beta(n+1)]^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)-2(1-\xi)^2[n+\beta(n+1)]} \times \frac{1}{[2\lambda\alpha-2\lambda+2\alpha+1]}$.

Proof. Since $f,g \in \Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$, we have

$$\sum_{n=1}^{\infty}\left[\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}a_n\right]^2 \leq 1 \tag{6.6}$$

and

$$\sum_{n=1}^{\infty}\left[\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}b_n\right]^2 \leq 1. \tag{6.7}$$

Combining the inequalities (6.6) and (6.7), we get
$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}\right]^2 \cdot (a_n^2+b_n^2) \leq 1.$
But, we need to find the largest ρ such that

$$(a_n^2+b_n^2) \leq 1. \tag{6.8}$$

The inequality (6.8) would hold if

$$\left[\frac{[(n+\rho)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\rho)(2\lambda\alpha-2\lambda+2\alpha+1)}a_n\right] \leq \frac{1}{2}\left[\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}\right]^2.$$

Then we have
$\rho \leq 1 - \frac{4(1-\xi)^2[(2\lambda\alpha-2\lambda+2\alpha+1)]\phi(n,\tau,c)}{[(n+\xi)+\beta(n+1)]^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)-2(1-\xi)^2[n+\beta(n+1)][2\lambda\alpha-2\lambda+2\alpha+1]}$.
Let $\Phi(n) = \frac{4(1-\xi)^2[(2\lambda\alpha-2\lambda+2\alpha+1)]\phi(n,\tau,c)}{[(n+\xi)+\beta(n+1)]^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)-2(1-\xi)^2[n+\beta(n+1)][2\lambda\alpha-2\lambda+2\alpha+1]}$.

A simple computation shows that $\Phi(n+1)-\Phi(n) > 0$ for all n. This means that $\Phi(n)$ is increasing and $\Phi(n) \geq \Phi(1)$. Letting $n=1$, we prove the assertion. □

7. Integral Operators

In this section, we consider integral transforms of functions in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$ of the type considered by Goel and Sohi [8] .

Theorem 7.1. *Let the function $f \in \Sigma_p$ defined by the equation (1.3) is in the class of $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. Then the integral operator*

$$F(z) = d\int_0^z u^d f(uz)du,\ 0<u\leq 1,\ 0<d<\infty \tag{7.1}$$

is in $\Sigma_p(\alpha,\lambda,\tau,\ c,\rho,\beta)$, where $\rho \leq 1 - \frac{2d(1-\xi)[(2\lambda\alpha-2\lambda+2\alpha+1)]}{d(1-\xi)[(2\lambda\alpha-2\lambda+2\alpha+1)]+(d+2)(2k+\xi+1)}$ and the result is sharp.

Proof. Let the function $f \in \Sigma_p$ given by (1.3) is in the class $\Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$. Then by a simple computation we have

$$F(z) = d\int_0^1 u^d f(uz)du = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{d}{d+n+1}a_n z^n. \tag{7.2}$$

We have to show that

$$\sum_{n=1}^{\infty}\frac{d[(n+\rho)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(d+n+1)(1-\rho)(2\lambda\alpha-2\lambda+2\alpha+1)} \times a_n \leq 1. \tag{7.3}$$

Since $f \in \Sigma_p(\alpha,\lambda,\tau,\ c,\xi,\beta)$, we have
$\sum_{n=1}^{\infty}\frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}a_n \leq 1.$
We note that the inequality (7.3) is satisfied if
$\frac{d[(n+\rho)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(d+n+1)(1-\rho)(2\lambda\alpha-2\lambda+2\alpha+1)} \leq \frac{[(n+\xi)+\beta(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)]\phi(n,\tau,c)}{(1-\xi)(2\lambda\alpha-2\lambda+2\alpha+1)}.$
Then we get $\rho \leq 1 - \frac{2d(1-\xi)[n+k(n+1)]}{[n+\xi-\xi\beta(n+1)](n+d+1)+(1-\xi)[1-\beta(n+1)]}$.
By a simple computation, we can show that the function $\phi(n) = 1 - \frac{(1-\xi)[1+\beta(n+1)]+dn}{(d+n+1)[n+\beta+k(n+1)]-d(1-\beta)[n+k(n+1)]}$ is an increasing function of $n (n \geq 1)$ and $\phi(n) \geq \phi(1)$. Using this, we obtain the desired result. □

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