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Integral inequalities involving $(\mathbf{k},\mathbf{s})-$ fractional moments of a continuous random variables

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Abstract
In this work, we establish some new integral inequalities of $(k,s)-$fractional moment of continuous random variables by using the $(k,s)-$Riemann-Liouville integral operator.

Keywords
$(k,s)-$Riemann-Liouville integral, integral inequalities, random variable, $(k,s)-$fractional moment.

AMS Subject Classification
26D15, 26A33, 60E15.

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Contents

1. Introduction

The integral inequalities play a fundamental role in the theory of probability and statistical problems. For details, we refer to [2–5, 14–16] and the references therein. The study of the integral inequalities using fractional calculus is also of great importance, we refer the reader to [1, 6–11] for more information and applications. Recently, by using the different fractional integral operators such as Riemann-Liouville fractional integral, $k-$Riemann-Liouville fractional integrals, $(k,s)-$Riemann-Liouville fractional integral, many researchers presented several fractional applications for continuous random variable whose the probability density function $(p.d.f)$. For some applications on the subject, one may refer to [1, 6, 7, 9–12, 19], and the references cited therein. Z. Dahmani [6], established some inequalities for the fractional dispersion and variance functions of continuous random variables by using the Riemann-Liouville fractional integrals. In [1] A. Akkurt *et* al. proposed some generalizations of the results in [6] by applying the generalized Riemann-Liouville fractional integrals. In [12] M. Houas established some integral inequalities of expectation and variance of continuous random variables by applying the $k-$Riemann-Liouville fractional integral operators. Also in [8, 9] Z. Dahmani *et* al. presented new $W-$weighted concepts for continuous random variables with applications involving Riemann-Liouville fractional integral operators. M. Tomar *et* al. [19] presented some new integral inequalities for the $(k,s)-$fractional expectation and variance functions for the $(k,s)-$Riemann-Liouville fractional integral . Z. Dahmani [7] presented new applications of Riemann-Liouville fractional operators for continuous random variables. Then, he established several integral inequalities for the fractional dispersion and the fractional variance functions, also some corollaries on the paper [6], were corrected. Recently, M. Houas *et* al. [10] derived certain integral inequalities for the $(r,\alpha)-$moments involving Riemann-Liouville fractional integral operators. Very recently, M. Houas [13] established new estimations on continuous random variables for $(k,s)-$fractional operators. Motivated by the above works, in this paper, we present new fractional results integral for the $(k,s)-$fractional moments.

2. Preliminaries

We recall the notations and definitions of the $(k,s)-$fractional integration theory [17, 18].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha>0$, for a continuous function f on $[a,b]$ is defined by*

$$J_a^{\alpha}\left[f(t)\right]=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau;\ a<t\leq b, \quad (2.1)$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

Definition 2.2. *The $k-$Riemann-Liouville fractional integral of order $\alpha>0$, for a continuous function f on $[a,b]$ is defined by*

$$_kJ_a^{\alpha}\left[f(t)\right]=\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}f(\tau)\,d\tau;\ a<t\le b, \tag{2.2}$$

where $\Gamma_k(\alpha)=\int_0^{\infty}e^{-\frac{u^k}{k}}u^{\alpha-1}du.$

Definition 2.3. *The $(k,s)-$Riemann-Liouville fractional integral of order $\alpha>0$, for a continuous function f on $[a,b]$ is defined as*

$$^k_sJ_a^{\alpha}\left[f(t)\right]=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s f(\tau)\,d\tau;\ a<t\le b, \tag{2.3}$$

where $k>0,\ s\in\mathbb{R}\backslash\{-1\}.$

For $t=b$, we have

$$^k_sJ_a^{\alpha}\left[f(b)\right]=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^b\left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s f(\tau)\,d\tau. \tag{2.4}$$

where $k>0,\ s\in\mathbb{R}\backslash\{-1\}.$

Theorem 2.4. *Let f be continuous on $[a,b]$, $k>0$, and $s\in\mathbb{R}\backslash\{-1\}$. Then,*

$$^k_sJ_a^{\alpha}\left[^k_sJ_a^{\beta}\left[f(t)\right]\right]={}^k_sJ_a^{\alpha+\beta}\left[f(t)\right]={}^k_sJ_a^{\beta}\left[^k_sJ_a^{\alpha}\left[f(t)\right]\right], \tag{2.5}$$

for all $\alpha>0,\ \beta>0, a<t\le b.$

We recall also the following definitions [6, 19].

Definition 2.5. *The $(k,s)-$fractional expectation of order $\alpha>0$ for a random variable X with a positive $p.d.f.$ f defined on $[a,b]$ is defined as*

$$^k_sE_{X,\alpha}=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^b\left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^{s+1}f(\tau)\,d\tau, \tag{2.6}$$

where $k>0,\ s\in\mathbb{R}\backslash\{-1\}.$

Definition 2.6. *The $(k,s)-$fractional variance of order $\alpha>0$ for a random variable X having a positive $p.d.f.$ f on $[a,b]$ is defined as*

$$^k_s\sigma^2_{X,\alpha}=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^b\left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^{s}\left(\tau-E(X)\right)^2 f(\tau)\,d\tau, \tag{2.7}$$

where $k>0,\ s\in\mathbb{R}\backslash\{-1\}.$

For the $(k,s)-$fractional moment of orders of X, we recall next definition [11].

Definition 2.7. *The $(k,s)-$fractional moment of orders (r,α), $r>0$, $\alpha>0$ for a continuous random variable X having a $p.d.f.$ f defined on $[a,b]$ is defined as*

$$^k_sM_{r,\alpha}=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^b\left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^{s+r}f(\tau)\,d\tau, \tag{2.8}$$

where $k>0,\ s\in\mathbb{R}\backslash\{-1\}.$

Remark 2.8. (i^*) *If we take $\alpha=k=1$ and $s=0$ in Definition 2.7, we obtain the classical moment of order $r>0$ given by $M_r=\int_a^b\tau^r f(\tau)\,d\tau.$*

(i^{**}) *If we take $k=1$ and $s=0$ in Definition 2.7, we obtain the fractional moment of order (r,α), $r>0,\ \alpha>0$ given by $M_{r,\alpha}=J_a^{\alpha}\left[f(b)\right].$*

Also, we recall the following generalized property of the $p.d.f.$ of X [13].

Theorem 2.9. *Let X be a continuous random variable having a $p.d.f.$ $f:[a,b]\to\mathbb{R}^+$. Then we have*

$$^k_sJ_a^{\alpha+k}[f(b)]=\frac{\Gamma_k(\alpha-kn+k)}{\Gamma_k(\alpha+k)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\times{}^k_sM_{i(s+1),\alpha-kn+k}, \tag{2.9}$$

where $\alpha\ge 0, k>0$ and $s\in\mathbb{R}\backslash\{-1\}, n=[\frac{\alpha}{k}].$

3. Main Results

In this section, we present new results for the $(k,s)-$fractional moments of continuous random variables. The first main result is the following theorem.

Theorem 3.1. *Let X be a continuous random variable with a $p.d.f.$ $f:[a,b]\to\mathbb{R}^+$ and $\alpha\ge k, k>0, s\in\mathbb{R}\backslash\{-1\}; n=\left[\frac{\alpha}{k}-1\right].$*

(i) If $f\in L_{\infty}[a,b]$, then

$$\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\,{}^k_sM_{i(s+1),\alpha-kn}\times{}^s_kJ_a^{\alpha}[b^{r-1}(b-E(X))f(b)]-{}^s_kJ_a^{\alpha}[[(b-E(X))f(b)]\,{}^s_kM_{r-1,\alpha}\le\|f\|^2_{\infty}\left(\frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)(s+1)^{\frac{\alpha}{k}}}\,{}^s_kJ_a^{\alpha}[b^r]-{}^s_kJ_a^{\alpha}[b]\,{}^s_kJ_a^{\alpha}[b^{r-1}]\right), \tag{3.1}$$

(ii) *We have also*

$$\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\,{}^k_sM_{i(s+1),\alpha-kn}\times{}^s_kJ_a^{\alpha}[b^{r-1}(b-E(X))f(b)]-{}^s_kJ_a^{\alpha}[[b-E(X)]f(b)]\,{}^s_kM_{r-1,\alpha}\le\frac{1}{2}(b-a)(b^{r-1}-a^{r-1})\left(\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\,{}^k_sM_{i(s+1),\alpha-kn}\right)^2. \tag{3.2}$$

Proof. In [3, 4], it has been proved that

$$\begin{aligned}&\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b(b^{s+1}-\tau^{s+1})^{\frac{\alpha}{k}-1}\\&(b^{s+1}-\rho^{s+1})^{\frac{\alpha}{k}-1}\tau^s\rho^s p(\tau)p(\tau)\\&\times(g(\tau)-g(\rho))(h(\tau)-h(\rho)d\rho d\tau\\&=2{}_k^sI_a^\alpha[p(b)]\ {}_k^sI_a^\alpha[pgh(b)]-2\ {}_k^sI_a^\alpha[pg(b)]\ {}_k^sI_a^\alpha[ph(b)].\end{aligned} \tag{3.3}$$

If we choose in (3.3) $p(b)=f(b), g(b)=b-E(X)$, $h(b)=b^{r-1}$, then we get

$$\begin{aligned}&\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b(b^{s+1}-\tau^{s+1})^{\frac{\alpha}{k}-1}(b^{s+1}-\rho^{s+1})^{\frac{\alpha}{k}-1}\\&\times\tau^s\rho^s(\rho-\tau)(\tau^{r-1}-\rho^{r-1})f(\tau)f(\rho)d\rho d\tau\\&=2{}_k^sI_a^\alpha[f(b)]\ {}_k^sI_a^\alpha[b^{r-1}(b-E(X))f(b)]\\&-2\ {}_k^sI_a^\alpha[[b-E(X)]f(b)]\ {}_k^sM_{r-1,\alpha}.\end{aligned} \tag{3.4}$$

We have also

$$\begin{aligned}&\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b(b^{s+1}-\tau^{s+1})^{\frac{\alpha}{k}-1}(b^{s+1}-\rho^{s+1})^{\frac{\alpha}{k}-1}\\&\times\tau^s\rho^s(\rho-\tau)(\tau^{r-1}-\rho^{r-1})f(\tau)f(\rho)d\rho d\tau\\&\leq\| f\|_\infty^2\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t(t^{s+1}-\tau^{s+1})^{\frac{\alpha}{k}-1}(t^{s+1}-\rho^{s+1})^{\frac{\alpha}{k}-1}\\&\times\tau^s\rho^s(\rho-\tau)(\tau^{r-1}-\rho^{r-1}\\&=\| f\|_\infty^2\left(2\,{}_k^sI_a^\alpha[1]\,{}_k^sI_a^\alpha[b^r]-2{}_k^sI_a^\alpha[b]\,{}_k^sI_a^\alpha[b^{r-1}]\right).\end{aligned} \tag{3.5}$$

By (3.4), (3.5) and (2.9), we obtain (3.1).

To prove (ii), we remark that

$$\begin{aligned}&\frac{(s+1)^{(2-\frac{2\alpha}{k})}}{k^2\Gamma_k(\alpha)\Gamma_k(\alpha)}\int_a^t\int_a^t(b^{s+1}-\tau^{s+1})^{\frac{\alpha}{k}-1}\\&(b^{s+1}-\rho^{s+1})^{\frac{\beta}{k}-1}\tau^s\rho^s f(\tau)f(\rho)(\tau-\rho)^2d\rho d\tau\\&\leq\sup_{\tau,\rho\in[a,b]}|\tau-\rho|^2\,{}_k^sI_a^\alpha[f(b)]^2\leq(b-a)^2\,{}_k^sI_a^\alpha[f(b)]^2\\&\times\tau^s\rho^s(\rho-\tau)(\tau^{r-1}-\rho^{r-1}\\&=\| f\|_\infty^2\left(2\,{}_k^sI_a^\alpha[1]\,{}_k^sI_a^\alpha[b^r]-2{}_k^sI_a^\alpha[b]\,{}_k^sI_a^\alpha[b^{r-1}]\right).\end{aligned} \tag{3.6}$$

Then, thanks to Theorem 2.8 and using (3.4), we get (3.2). □

Using two fractional parameters, we consider the following generalization of the above results:

Theorem 3.2. *Let X be a continuous random variable having a probability density function f defined on $[a,b]$. Then we have*

(i^*) For all $k>0, s\in\mathbb{R}\backslash\{-1\}, \alpha\geq k$ and $\beta\geq k$

$$\begin{aligned}&\frac{\Gamma_k(\beta-km)}{\Gamma_k(\alpha)(s+1)^m}\sum_{i=0}^n(-1)^iC_m^ib^{(s+1)(m-i)}\ {}_s^kM_{i(s+1),\beta-km}\\&\times\left({}_k^sM_{2,\alpha}-2E(X)\,{}_k^sE_{X,\alpha}+\frac{\Gamma_k(\alpha-kn)E^2(X)}{\Gamma_k(\alpha)(s+1)^n}\right.\\&\left.\times\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\ {}_s^kM_{i(s+1),\alpha-kn}\right)\\&+\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\ {}_s^kM_{i(s+1),\alpha-kn}\\&\times\left({}_k^sM_{2,\beta}-2E(X)\,{}_k^sE_{X,\beta}+\frac{\Gamma_k(\beta-km)E^2(X)}{\Gamma_k(\beta)(s+1)^m}\right.\\&\left.\sum_{i=0}^m(-1)^iC_n^ib^{(s+1)(m-i)}\ {}_s^kM_{i(s+1),\beta-km}\right)\\&\leq2\left({}_s^kE_{X-E(X),\alpha}\right)\left({}_s^kE_{X-E(X),\beta}\right)+\|f\|_\infty^2\left[\frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{\Gamma_k(\beta+k)(s+1)^{\frac{\beta}{k}}}\right.\\&\left.\times{}_k^sI_a^\alpha[b^2]+\frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)(s+1)^{\frac{\alpha}{k}}}\,{}_k^sI_a^\beta[b^2]-2{}_k^sI_a^\alpha[b]\,{}_k^sI_a^\beta[b^2]\right],\end{aligned} \tag{3.7}$$

where $f\in L_\infty[a,b], n=\left[\frac{\alpha}{k}-1\right]$ and $m=\left[\frac{\beta}{k}-1\right]$.

(ii^*) For any $\alpha\geq k,\beta\geq k\ k>0, s\in\mathbb{R}\backslash\{-1\}$ the inequality

$$\begin{aligned}&\frac{\Gamma_k(\beta-kn)}{\Gamma_k(\alpha)(s+1)^m}\sum_{i=0}^n(-1)^iC_m^ib^{(s+1)(m-i)}\ {}_s^kM_{i(s+1),\beta-km}\\&\times\left({}_k^sM_{2,\alpha}-2E(X)\,{}_k^sE_{X,\alpha}+\right.\\&\left.\frac{\Gamma_k(\alpha-kn)E^2(X)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\ {}_s^kM_{i(s+1),\alpha-kn}\right)\\&+\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\ {}_s^kM_{i(s+1),\alpha-kn}\\&\times\left({}_k^sM_{2,\beta}-2E(X)\,{}_k^sE_{X,\beta}+\right.\\&\left.\frac{\Gamma_k(\beta-kn)E^2(X)}{\Gamma_k(\beta)(s+1)^m}\sum_{i=0}^m(-1)^iC_n^ib^{(s+1)(m-i)}\ {}_s^kM_{i(s+1),\beta-kn}\right)\\&\leq2\left({}_s^kE_{X-E(X),\alpha}\right)\left({}_s^kE_{X-E(X),\beta}\right)+(b-a)^2\\&\times\left(\frac{\Gamma_k(\beta-kn)}{\Gamma_k(\alpha)(s+1)^m}\sum_{i=0}^n(-1)^iC_m^ib^{(s+1)(m-i)}\ {}_s^kM_{i(s+1),\beta-km}\right)\\&\times\left(\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\ {}_s^kM_{i(s+1),\alpha-kn}\right),\end{aligned} \tag{3.8}$$

hold with $n=\left[\frac{\alpha}{k}-1\right], m=\left[\frac{\beta}{k}-1\right]$.

Proof. We have (see [3, 4])

$$\begin{aligned}&\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\\&\left(b^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s(\tau-\rho)^2f(\tau)f(\rho)d\tau d\rho\\&=\ {}_k^sI_a^\alpha[f(b)]\,{}_k^sI_a^\beta[f(b)](b-E(X))^2]\\&+{}_k^sI_a^\beta[f(b){}_k^sI_a^\alpha[f(b)(b-E(X))^2]\\&-2{}_k^sI_a^\alpha[f(b)](b-E(X))]\,{}_k^sI_a^\beta[f(b)(b-E(X))].\end{aligned} \tag{3.9}$$

By the hypothesis $f \in L_\infty([a,b])$, we obtain

$$\begin{aligned}
&\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b \left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\\
&\left(b^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s(\tau-\rho)^2 f(\tau)f(\rho)\,d\tau d\rho\\
&\le \|f\|_\infty^2\,\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b \left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\\
&\left(b^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s(\tau-\rho)^2\,d\tau d\rho\\
&\le \|f\|_\infty^2\left({}_k^sI_a^\alpha[1]\,{}_k^sI_a^\beta\left[f(b^2)\right] + {}_k^sI_a^\beta[1]\,{}_k^sI_a^\alpha[b^2]\right.\\
&\left.-2\,{}_k^sI_a^\alpha[b]\,{}_k^sI_a^\beta[b]\right).
\end{aligned} \tag{3.10}$$

Thanks to (3.9), (3.10) and (2.9), we obtain (3.7).

For the second part of Theorem 3.1, we remark that,

$$\begin{aligned}
&\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b \left(b^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\\
&\left(b^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s(\tau-\rho)^2 f(\tau)f(\rho)\,d\tau d\rho\\
&\le (b-a)\;{}_s^kI_a^\alpha\left[f(b)\right]\;{}_s^kI_a^\beta\left[f(b)\right].
\end{aligned} \tag{3.11}$$

So, applying Theorem 2.8 and thanks to (3.9), (3.11), we get (3.8). □

Remark 3.3. *Taking $\alpha=\beta$ in the above theorem, we obtain Theorem 3.1.*

We give also the following estimation for the $(k,s)-$fractional moment:

Theorem 3.4. *Let X be a continuous random variable having a p.d.f. $f:[a,b]\to\mathbb{R}^+$. Then, for all $k>0, s\in\mathbb{R}/\{-1\}$ and $\alpha\ge k$, we have*

$$\begin{aligned}
&\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\;{}_s^kM_{i(s+1),\alpha-kn}\\
&\times\,{}_s^kM_{2r,\alpha}-\,{}_s^kM_{r,\alpha}^2\\
&\le\frac{1}{4}\left(\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\right.\\
&\left.\times\,{}_s^kM_{i(s+1),\alpha-kn}^2\right)^2\left(b^r-a^r\right)^2,
\end{aligned} \tag{3.12}$$

where $n=\left[\frac{\alpha}{k}-1\right]$.

Proof. Using a $(k,s)-$fractional Gruss result [17], we can write

$$\begin{aligned}
&\left|{}_s^kJ_a^\alpha[p(b)]\;{}_s^kJ_a^\alpha[phg(b)]-\,{}_s^kJ_a^\alpha[ph(b)]\;{}_s^kJ_a^\alpha[pg(b)]\right|\\
&\le\frac{1}{4}\left({}_s^kJ_a^\alpha[p(b)]\right)^2(L-l)^2.
\end{aligned} \tag{3.13}$$

In (3.13), we replace h by g, we will have

$$\begin{aligned}
&\left|\,{}_s^kJ_a^\alpha[p(t)]\;{}_s^kJ_a^\alpha\left[pg^2(t)\right]-\left({}_s^kJ_a^\alpha[pg(t)]\right)^2\right|\\
&\le\frac{1}{4}\left({}_s^kJ_a^\alpha[p(b)]\right)^2(L-l)^2.
\end{aligned} \tag{3.14}$$

Taking $p(t)=f(t)$ and $g(t)=t^r, t\in[a;b]$; we obtain $l=a^r, L=b^r$. Hence, the inequality (3.14) allows us to obtain

$$\begin{aligned}
0&\le\,{}_s^kJ_a^\alpha[f(b)]\;{}_s^kJ_a^\alpha\left[t^{2r}f(b)\right]-\left({}_s^kJ_a^\alpha[t^rf(b)]\right)^2\\
&\le\frac{1}{4}\left({}_s^kJ_a^\alpha[f(b)]\right)^2(b^r-a^r)^2,
\end{aligned} \tag{3.15}$$

this implies that

$${}_s^kJ_a^\alpha[f(b)]\;{}_s^kM_{2r,\alpha}-\,{}_s^kM_{r,\alpha}^2\le\frac{1}{4}\left({}_s^kJ_a^\alpha[f(b)]\right)^2(b^r-a^r)^2. \tag{3.16}$$

By Theorem 2.8, we get (3.12). □

Remark 3.5. *Applying Theorem 3.3, for $\alpha=k=1$ and $s=0$, we obtain Theorem 3.2 in [13].*

We present also the following estimation:

Theorem 3.6. *Let X be a continuous random variable having a probability density function f defined on $[a,b]$. Then, for all $k>0, s\in\mathbb{R}/\{-1\}$, $\alpha\ge k,\beta\ge k$, we have*

$$\begin{aligned}
&\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\\
&{}_s^kM_{i(s+1),\alpha-kn}M_{2r,\beta}\\
&+\frac{\Gamma_k(\beta-km)}{\Gamma_k(\beta)(s+1)^m}\sum_{i=0}^m(-1)^iC_m^ib^{(s+1)(m-i)}\\
&{}_s^kM_{i(s+1),\beta-km}M_{2r,\alpha}\\
&\le(a^r+b^r)\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\\
&\times\,{}_s^kM_{i(s+1),\alpha-kn}-2a^rb^r\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^n(-1)^iC_n^ib^{(s+1)(n-i)}\\
&\times\,{}_s^kM_{i(s+1),\alpha-kn}\times M_{r,\beta}\\
&+\frac{\Gamma_k(\beta-km)}{\Gamma_k(\beta)(s+1)^m}\sum_{i=0}^m(-1)^iC_m^ib^{(s+1)(m-i)}\\
&\times\,{}_s^kM_{i(s+1),\beta-km}M_{r,\alpha}\\
&\times\frac{\Gamma_k(\beta-km)}{\Gamma_k(\beta)(s+1)^m}\sum_{i=0}^m(-1)^iC_m^ib^{(s+1)(m-i)}\;{}_s^kM_{i(s+1),\beta-km},
\end{aligned} \tag{3.17}$$

where $n=\left[\frac{\alpha}{k}-1\right]$ and $m=\left[\frac{\beta}{k}-1\right]$.

Proof. We take $p(b)=f(b), g(b)=b^r$ and by Theorem 2.4 of [17], we can write

$$\begin{aligned}
&\left[{}_s^kJ_a^\alpha[f(b)]\;{}_s^kJ_a^\beta\left[b^{2r}f(b)\right]+\,{}_s^kJ_a^\beta[f(b)]\;{}_s^kJ_a^\alpha\left[b^{2r}f(b)\right]\right.\\
&\left.-2\,{}_s^kJ_a^\alpha[b^rf(b)]\;{}_s^kJ_a^\beta[b^rf(b)]\right]^2\\
&\le\left[\left(L\,{}_s^kJ_a^\alpha[f(b)]-\,{}_s^kJ_a^\alpha[b^rf(b)]\right)\right.\\
&\left({}_s^kJ_a^\beta[b^rf(b)]-l\,{}_s^kJ_a^\beta[f(b)]\right)\\
&+\left({}_s^kJ_a^\alpha[b^rf(b)]-l\,{}_s^kJ_a^\alpha[f(b)]\right)\\
&\left.\left(L\,{}_s^kJ_a^\beta[f(b)]-\,{}_s^kJ_a^\beta[t^rf(b)]\right)\right],
\end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned}
&{}_s^kJ_a^\alpha[f(b)]M_{2r,\beta}+\,{}_s^kJ_a^\beta[f(b)]M_{2r,\alpha}\\
&\le(L+l)\left[{}_s^kJ_a^\alpha[f(b)]M_{r,\beta}+\,{}_s^kJ_a^\beta[f(b)]M_{r,\alpha}\right]\\
&-2lL\,{}_s^kJ_a^\alpha[f(b)]\;{}_s^kJ_a^\beta[f(b)].
\end{aligned} \tag{3.19}$$

Substituting the values of l and L in (3.19) and by Theorem 2.9, we obtain (3.17). □

We also prove the following result.

Theorem 3.7. *Let X be a continuous random variable having a p.d.f. $f:[a,b]\to\mathbb{R}^+$. Then, for all $k>0, s\in\mathbb{R}/\{-1\}$ and $\alpha\geq k$, we have*

$$\Bigg| \frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\alpha}{k}-2}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)} M_{r,\alpha} - \frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^{n}(-1)^i C_n^i b^{(s+1)(n-i)}\, {}_s^kM_{i(s+1),\alpha-kn}\times {}_s^kJ_a^{\alpha}\left[b^r\right]\Bigg| \leq (L-l)\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\alpha}{k}-2}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)} \times\left(\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\alpha}{k}-2}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}\,{}_s^kJ_a^{\alpha}\left[b^{2r}\right]-\left({}_s^kJ_a^{\alpha}\left[b^r\right]\right)^2\right)^{\frac{1}{2}}, \tag{3.20}$$

where $n=\left[\frac{\alpha}{k}-1\right]$.

Proof. Using Theorem 3.8 of [17], we can write

$$\left|{}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\alpha}[fg(b)]-{}_s^kJ_a^{\alpha}[f(b)]\,{}_s^kJ_a^{\alpha}[g(b)]\right| \leq (L-l)\frac{{}_s^kJ_a^{\alpha}[1]}{2}. \times\left({}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\alpha}\left[g^2(b)\right]-\left({}_s^kJ_a^{\alpha}[g(b)]\right)^2\right)^{\frac{1}{2}}. \tag{3.21}$$

If we put $g(b)=b^r$, in (3.21), we obtain

$$\left|{}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\alpha}[b^rf(b)]-{}_s^kJ_a^{\alpha}[f(b)]\,{}_s^kJ_a^{\alpha}[b^r]\right| \leq (L-l)\frac{{}_s^kJ_a^{\alpha}[1]}{2} \left({}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\alpha}\left[b^{2r}\right]-\left({}_s^kJ_a^{\alpha}[b^r]\right)^2\right)^{\frac{1}{2}}. \tag{3.22}$$

Then, by (2.9), we get (3.20). □

Using two fractional parameters, we establish the following generalization.

Theorem 3.8. *Let f be the p.d.f. of X on $[a,b]$. Then, for all $k>0, s\in\mathbb{R}/\{-1\}$, $\alpha\geq k, \beta\geq k$, we have*

$$\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\alpha}{k}-2}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}M_{r,\alpha}+\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\beta}{k}-2}}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+1)}M_{r,\beta} -\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^{n}(-1)^iC_n^ib^{(s+1)(n-i)} \times{}_s^kM_{i(s+1),\alpha-kn}\,{}_s^kJ_a^{\beta}\left[b^r\right] -\frac{\Gamma_k(\beta-km)}{\Gamma_k(\beta)(s+1)^m}\sum_{i=0}^{m}(-1)^iC_m^ib^{(s+1)(m-i)}\,{}_s^kM_{i(s+1),\beta-km}$$
$$\leq\Bigg[\Bigg(L\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\alpha}{k}-2}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)} -\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^{n}(-1)^iC_n^ib^{(s+1)(n-i)}\,{}_s^kM_{i(s+1),\alpha-kn}\Bigg) \tag{3.23}$$
$$\times\Bigg(\frac{\Gamma_k(\beta-km)}{\Gamma_k(\beta)(s+1)^m}\sum_{i=0}^{m}(-1)^iC_m^ib^{(s+1)(m-i)}\,{}_s^kM_{i(s+1),\beta-km} -l\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\beta}{k}-2}}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+1)}\Bigg) +\Bigg(\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^{n}(-1)^iC_n^ib^{(s+1)(n-i)}\,{}_s^kM_{i(s+1),\alpha-kn} -l\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\alpha}{k}-2}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}\Bigg)\times\Bigg(L\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\beta}{k}-2}}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+1)} -\frac{\Gamma_k(\beta-km)}{\Gamma_k(\beta)(s+1)^m}\sum_{i=0}^{m}(-1)^iC_m^ib^{(s+1)(m-i)}\,{}_s^kM_{i(s+1),\beta-km}\Bigg)\Bigg]$$
$$-\frac{\Gamma_k(\alpha-kn)}{\Gamma_k(\alpha)(s+1)^n}\sum_{i=0}^{n}(-1)^iC_n^ib^{(s+1)(n-i)}\,{}_s^kM_{i(s+1),\alpha-kn}\Bigg) \times\Bigg[\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\alpha}{k}-2}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}\,{}_s^kJ_a^{\beta}\left[b^{2r}\right] +\frac{\left(b^{(s+1)}-a^{(s+1)}\right)^{\frac{\beta}{k}-2}}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+1)}\,{}_s^kJ_a^{\alpha}\left[b^{2r}\right]-2\,{}_s^kJ_a^{\alpha}\left[b^r\right]\,{}_s^kJ_a^{\beta}\left[b^r\right]\Bigg]^{\frac{1}{2}},$$

where $m=\left[\frac{\beta}{k}-1\right]$ and $n=\left[\frac{\alpha}{k}-1\right]$.

Proof. Thanks to Theorem 3.10 of [17], we can state that

$$\Big|{}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\beta}[fg(b)]+{}_s^kJ_a^{\beta}[1]\,{}_s^kJ_a^{\alpha}[fg(b)] -{}_s^kJ_a^{\alpha}[f(b)]\,{}_s^kJ_a^{\beta}[g(b)]-{}_s^kJ_a^{\beta}[f(b)]\,{}_s^kJ_a^{\alpha}[g(b)]\Big| \leq\Big[\left(L\,{}_s^kJ_a^{\alpha}[1]-{}_s^kJ_a^{\alpha}[f(b)]\right)\left({}_s^kJ_a^{\beta}[f(b)]-l\,{}_s^kJ_a^{\beta}[1]\right) +\left({}_s^kJ_a^{\alpha}[f(b)]-l\,{}_s^kJ_a^{\alpha}[1]\right)\left(L\,{}_s^kJ_a^{\beta}[1]-{}_s^kJ_a^{\beta}[f(b)]\right)\Big] \times\Big({}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\beta}\left[g^2(b)\right]+{}_s^kJ_a^{\beta}[1]\,{}_s^kJ_a^{\alpha}\left[g^2(b)\right] -2\,{}_s^kJ_a^{\alpha}[g(b)]\,{}_s^kJ_a^{\beta}[g(b)]\Big)^{\frac{1}{2}}. \tag{3.24}$$

In (3.24), if we take $g(t)=b^r$, we obtain

$$\Big|{}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\beta}[b^rf(b)]+{}_s^kJ_a^{\beta}[1]\,{}_s^kJ_a^{\alpha}[b^rf(b)] -{}_s^kJ_a^{\alpha}[f(b)]\,{}_s^kJ_a^{\beta}[b^r]-{}_s^kJ_a^{\beta}[f(b)]\,{}_s^kJ_a^{\alpha}[b^r]\Big| \leq\Big[\left(L\,{}_s^kJ_a^{\alpha}[1]-{}_s^kJ_a^{\alpha}[f(b)]\right)\left({}_s^kJ_a^{\beta}[f(b)]-l\,{}_s^kJ_a^{\beta}[1]\right) +\left({}_s^kJ_a^{\alpha}[f(b)]-l\,{}_s^kJ_a^{\alpha}[1]\right)\left(L\,{}_s^kJ_a^{\beta}[1]-{}_s^kJ_a^{\beta}[f(b)]\right)\Big] \times\Big({}_s^kJ_a^{\alpha}[1]\,{}_s^kJ_a^{\beta}\left[b^{2r}\right]+{}_s^kJ_a^{\beta}[1]\,{}_s^kJ_a^{\alpha}\left[b^{2r}\right] -2\,{}_s^kJ_a^{\alpha}[b^r]\,{}_s^kJ_a^{\beta}[b^r]\Big)^{\frac{1}{2}}. \tag{3.25}$$

Finally, by Theorem 2.9, we obtain (33. □

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