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# Strong convergence of modified implicit hybrid S-iteration scheme for finite family of nonexpansive and asymptotically generalized $\Phi$-hemicontractive mappings

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**Abstract**

In this paper, we consider a modified implicit hybrid S-iteration scheme for finite family of nonexpansive and asymptotically generalized $\Phi$-hemicontractive mappings in the frame work of real Banach spaces. We remark that the iteration process of Kang et al. [17] can be obtained as a special case of our iteration process. Our result mainly improves and extends the result of Kang et al. [17] and several other results in the literation from the class of strongly psudocontractive mapping to the more general class asymptotically generalized $\Phi$-hemicontractive mappings. A different approach is used to obtain our result and the necessity of applying condition (C3) for the two mappings is weaken to only one mapping.

**Keywords**

Fixed point, Banach space, Implicit hybrid S-iteration process, nonexpansive mapping, asymptotically generalized $\Phi$-hemicontractive mapping.

**AMS Subject Classification**

39B82, 44B20, 46C05.

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## Contents



## 1. Introduction

Let $E$ be an arbitrary real Banach space with dual $E^*$. We denote by $J$ the *normalized duality* mapping from $E$ into $2^{E^*}$ defined by

$$J(x)=\{f^*\in E^*:\langle x,f^*\rangle=\|x\|^2=\|f^*\|^2\},\ \forall x\in E, \quad (1.1)$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. The single-valued-normalized duality mapping is denoted by $j$ and $F(T)$ denotes the set of fixed points of mapping $T$, i.e., $F(T)=\{x\in E: Tx=x\}$.

In the sequel, we give the following definitions which will be useful in this study.

**Definition 1.1.** *Let $K$ be a nonempty subset of real Banach space $E$. A mapping $T:K\to K$ is said to be:*

*(1) Non expansive if,*

$$\|Tx-Ty\|\leq\|x-y\|,\ \forall x,y\in K; \quad (1.2)$$

*(2) Strongly pseudocontractive (Kim et al. [20]) if for all $x,y\in K$, there exists a constant $k\in(0,1)$ and $j(x-y)\in J(x-y)$ satisfying*

$$\langle Tx-Ty,j(x-y)\rangle\leq k\|x-y\|^2; \quad (1.3)$$

*(3) $\phi$-strongly pseudocontractive (Kim et al. [20]) if for all $x,y\in K$, there exists a strictly increasing function $\phi:[0,\infty)\to[0,\infty)$ with $\phi(0)=0$ and $j(x-y)\in J(x-y)$ satisfying*

$$\langle Tx-Ty,j(x-y)\rangle\leq\|x-y\|^2-\phi(\|x-y\|)\|x-y\|; \quad (1.4)$$

*It has been proved (see [23]) that the class of $\phi$-strongly pseudocontractive mappings properly contains the class of strongly pseudocontractive mappings. By taking $\Phi(s) = s\phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $\phi(0) = 0$. However, the converse is not true.*

(4) *Generalized Φ-pseudocontractive (Albert et al. [1], Chidume and Chidume [4]) if for all $x, y \in K$, there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying*

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|); \quad (1.5)$$

*The class of generalized Φ-pseudocontractive mappings is also called uniformly pseudocontractive mappings (see [4]). Clearly, the class of generalized Φ-pseudo contractive mappings properly contains the class of $\phi$-pseudo contractive mappings.*

(5) *Generalized Φ-hemicontractive if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$, and there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$, such that for all $x \in K$, $p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that the following inequality holds:*

$$\langle Tx - p, j(x-p)\rangle \leq \|x-p\|^2 - \Phi(\|x-p\|); \quad (1.6)$$

*Clearly, the class of generalized Φ-hemicontractive mappings includes the class of generalized Φ-pseudo contractive mappings in which the fixed points set $F(T)$ is nonempty.*

(6) *Asymptotically generalized Φ-pseudocontractive (Kim et al. [20]) with sequence $\{h_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} h_n = 1$, if for each $x, y \in K$, there exist a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ satisfying*

$$\langle T^n x - T^n y, j(x-y)\rangle \leq h_n\|x-y\|^2 - \Phi(\|x-y\|). \quad (1.7)$$

*The class of asymptotically generalized Φ-pseudocontractive mappings is a generalization of the class of strongly pseudocontractive maps and the class of $\phi$-strongly peudocontractive maps. The class of asymptotically generalized Φ-pseudocontractive mappings was introduced by Kim et al. [20] in 2009.*

(7) *asymptotically generalized Φ–hemicontractive with sequence $\{h_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} h_n = 1$ if there exist a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$, such that for each $x \in K$, $p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that the following inequality holds:*

$$\langle T^n - p, j(x-p)\rangle \leq h_n\|x_n - p\|^2 - \Phi(\|x-p\|). \quad (1.8)$$

*Clearly, every asymptotically generalized Φ–pseudo contractive mapping with a nonempty fixed point set is an asymptotically generalized Φ–hemicontractive mapping. It follows that the class of asymptotically generalized Φ–hemicontractive mapping is most general of all the class of mappings mentioned above.*

On the other hand, the class of asymptotically generalized Φ-hemicontractive has been studied by several Authors (see for example, [3–5, 13, 14, 19, 22, 28, 32]).

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases} \quad \forall n \geq 1, \quad (1.9)$$

where $\{\alpha_n\}$ is a sequence in [0,1].

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \quad \forall n \geq 1, \quad (1.10)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. This iteration process reduces to Mann iteration when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Argawal et al. [2] introduced the following iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \quad \forall n \geq 1, \quad (1.11)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in [0,1]. They showed that their iteration process is independent of Mann and Ishikawa and converges faster than both for contractions.

In 2007, Sahu et al. [24], [25] introduced the following $S$-iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \quad \forall n \geq 1, \quad (1.12)$$

where $\{\beta_n\}$ is the sequence in [0,1].

In 1991, Schu [29] considered the modified Mann iteration process which is a generalization of the Mann iteration process as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \end{cases} \quad \forall n \geq 1, \quad (1.13)$$

where $\{\alpha_n\}$ is a sequence in [0,1].

In 1994, Tan and Xu [30] studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \quad \forall n \geq 1, \quad (1.14)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

Again, in 2007 Argawal et al. [2] introduced the modified

Argawal iteration process as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \quad \forall n \geq 1, \tag{1.15}$$

The above processes deal with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [7] gave and studied a two mappings process. Also see, for example, [16] and [27]. The problem of approximating common fixed points of finitely many mappings plays an important role in applied mathematics, especially in the theory of evolution equations and the minimization problems; see [8], [9], [10], [26], for example.

The following Ishikawa-type iteration process for two mappings has aslo been studied by many authors including [7], [16], [27], [28].

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n S^n x_n \end{cases} \quad \forall n \geq 1, \tag{1.16}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

In 2009, Khan et al. [18] modified the Argawal iteration process (1.15) to the case of two mappings as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \quad \forall n \geq 1, \tag{1.17}$$

$\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1].

In 2013, Kang et al. [15] considered the following iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = Sy_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \quad \forall n \geq 1, \tag{1.18}$$

where $\{\beta_n\}$ is the sequence in [0,1]. They proved the following results.

**Theorem 1.2** (see [15]). *Let $K$ be a nonempty closed convex subset of a real Banach space $E$, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a Lipschitz strongly pseudocontractive mapping such that $p \in F(S) \bigcap F(T) = \{x \in K : Sx = Tx = x\}$ and*

$$\|x - Sy\| \leq \|Sx - Sy\|, \quad \|x - Ty\| \leq \|Tx - Ty\| \tag{1.19}$$

*for all $x, y \in K$. Let $\{\beta_n\}$ be sequence in [0,1] satisfying*

*(i)* $\sum_{n=1}^{\infty} \beta_n = \infty$;

*(ii)* $\lim_{n\to\infty} \beta_n = 0$.

*For arbitrary $x_1 \in K$, the iteration process defined by (1.18) converges strongly to a fixed point $p$ of $S$ and $T$.*

In 2016, Gopinath et al. [11] considered the following modified S-iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = Sy_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \quad \forall n \geq 1, \tag{1.20}$$

where $\{\beta\}$ is the sequence in [0,1]. They proved the following result.

**Theorem 1.3** (see [11]). *Let $K$ be a nonempty closed convex subset of a real Banach space $E$, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a uniform L-Lipschitzian, asymptotically demicontractive mapping with sequence $\{h_n\} \in [0,1)$, $\lim_{n\to\infty} h_n = 1$ such that*

$$\|x - Sy\| \leq \|Sx - Sy\|, \ x, y \in K \tag{1.21}$$

$$\|x - Ty\| \leq \|Tx - Ty\|, \ x, y \in K. \tag{1.22}$$

*Assume that $F(S) \bigcap F(T) = \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $p \in F(S) \bigcap F(T)$ and $\{\beta_n\}$ be sequences in [0,1] satisfying*

*(i)* $\sum_{n=1}^{\infty} \beta_n = \infty$;

*(ii)* $\lim_{n\to\infty} \beta_n = 0$.

*For arbitrary $x_1 \in K$, the iteration process defined by (1.20) converges strongly to a fixed point $p$ of $S$ and $T$.*

In 2014, Khan [17] proved the following result:

**Theorem 1.4** (see [17]). *Let $K$ be a nonempty closed convex subset of a real Banach space $E$, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a Lipschitz strongly pseudocontractive mapping such that $p \in F(S) \bigcap F(T) = \{x \in K : Sx = Tx = x\}$ and*

$$\|x - Sy\| \leq \|Sx - Sy\|, \quad \|x - Ty\| \leq \|Tx - Ty\| \tag{1.23}$$

*for all $x, y \in K$. Let $\{\beta_n\}$ be a sequence in [0,1] satisfying*

*(i)* $\sum_{n=1}^{\infty} \beta_n = \infty$;

*(ii)* $\lim_{n\to\infty} \beta_n = 0$.

*For arbitrary $x_0 \in K$, the iteration process defined by*

$$\begin{cases} x_n = Sy_n, \\ y_n = (1-\beta_n)x_{n-1} + \beta_n T x_n \end{cases} \quad \forall n \geq 1, \tag{1.24}$$

*converges strongly to a fixed point $p$ of $S$ and $T$.*

Recently, Gopinath et al. [12] proved the following results:

**Theorem 1.5** (see [12]). *Let $K$ be a nonempty closed convex subset of a real Banach space $E$, let $S : K \to K$ be a* *nonexpansive mapping, and let $T : K \to K$ be a uniform L-Lipschitzian, asymptotically demicontractive mappin with sequence $\{a_n\} \in [0,1)$, $\lim_{n\to\infty} a_n = 1$ such that*

$$\|x - Sy\| \leq \|Sx - Sy\|, \ x, y \in K \tag{1.25}$$

$$\|x - Ty\| \leq \|Tx - Ty\|, \ x, y \in K. \tag{1.26}$$

*Assume that $F(S) \bigcap F(T) = \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $p \in F(S) \bigcap F(T)$ and $\{\beta_n\}$ be sequences in [0,1] satisfying*

*(i)* $\sum_{n=1}^{\infty} \beta_n = \infty$;

*(ii)* $\lim_{n\to\infty} \beta_n = 0$.

*For arbitrary $x_0 \in K$, the iteration process defined by*

$$\begin{cases} x_n = Sy_n, \\ y_n = (1-\beta_n)x_{n-1} + \beta_n T^n x_n \end{cases} \quad \forall n \geq 1, \tag{1.27}$$

*converges strongly to a fixed point $p$ of $S$ and $T$.*

In [15], Kang et al. introduced the following condition.

**Remark 1.6.** *Let $S, T : K \to K$ be two mappings. The mappings $S$ and $T$ are said to satisfy condition (C3) if*

$$\|x - Sy\| \leq \|Sx - Sy\|, \quad \|x - Ty\| \leq \|Tx - Ty\| \tag{1.28}$$

*for all $x, y \in K$.*

Inspired and motivated by the above results, we modify (1.20) for finite families of nonexpansive and asymptotically generalized Φ-hemicontractive mappings in Banach spaces. The result in this paper can be view as generalization and extension of the corresponding results of Kang et al. [15], Gopinath et al. [11] and several others in the literature.

**Definition 1.7.** *Let $\{S_i\}_{i=1}^N : K \to K$ be finite family of nonexpansive mappings and $\{T_i\}_{i=1}^N : K \to K$ be finite family of asymptotically generalized Φ–hemicontractive mappings. Define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} x_0 \in K, \\ x_n = S_{i(n)} y_n, \\ y_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n \end{cases} \quad \forall n \geq 1, \tag{1.29}$$

*where $\{\alpha_n\}$ is a sequence in [0,1] and $n = (k-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.*

**Remark 1.8.** *If we take $N = 1$, then (1.29) reduces to (1.20). Again, if we take $N = 1$ and $T^n = T$ for all $n \geq 1$, then (1.29) reduces to (1.18).*

The purpose of this paper is to study the strong convergence of the new modified implicit hybrid S-iteration process (1.29) for the finite families of nonexpansive and asymptotically generalized Φ-hemicontractive mappings in Banach space.

## 2. Preliminaries

In order to prove our main results, we also need the following lemmas.

**Lemma 2.1** ([3]). *Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall j(x+y) \in J(x+y). \tag{2.1}$$

**Lemma 2.2** ([31]). *Let $\{\rho_n\}$ and $\{\theta_n\}$ be nonnegative sequences satisfying*

$$\rho_{n+1} \leq (1-\theta_n)\rho_n + \omega_n \tag{2.2}$$

*where $\theta_n \in [0,1]$, $\sum_{n\geq 1} \theta_n = \infty$ and $\omega_n = o(\theta_n)$. Then $\lim_{n\to\infty} \rho_n = 0$.*

## 3. Main Results

**Theorem 3.1.** *Let $K$ be a nonempty closed convex subset of a real Banach space $E$. Let $\{S_i\}_{i=1}^N : K \to K$ be finite family of nonexpansive mappings and let $\{T_i\}_{i=1}^N : K \to K$ be finite family of asymptotically generalized Φ–hemicontractive mappings with $\{T_i(K)\}_{i=1}^N$ bounded and the sequence $\{h_{in}\} \subset [1,\infty)$, where $\lim_{n\to\infty} h_{in} = 1$ for each $1 \leq i \leq N$. Furthermore, let $\{T_i\}_{i=1}^N$ be uniformly continuous. Assume that*

$$p \in \boldsymbol{F} = \bigcap_{i=1}^N F(S_i) \bigcap \bigcap_{i=1}^N F(T_i) = \{x \in K : S_i x = T_i x = x\} \neq \emptyset,$$

*for each $1 \leq i \leq N$ such that for all $x, y \in K$*

$$\|x - S_i y\| \leq \|S_i x - S_i y\|, \text{ for each } 1 \leq i \leq N. \tag{3.1}$$

*Let $h_n = \max\{h_{in} : 1 \leq i \leq N\}$ and $\{\alpha_n\}$ be a sequence in [0,1] satisfying the following conditions:*

*(i)* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

*(ii)* $\lim_{n\to\infty} \alpha_n = 0$.

*For arbitrary $x_0 \in K$, let $\{x_n\}$ be the sequence iteratively defined by (1.29). Then the sequence $\{x_n\}$ converges strongly at common fixed $p$ of $S_i$ and $T_i$ for each $1 \leq i \leq N$.*

*Proof.* Let $p \in \mathbf{F}$ and since $T_i(K)$ bounded, we set

$$M_1 = \|x_0 - p\| + \sup_{n\geq 1} \|T_{i(n)}^{k(n)} x_n - p\|, \ 1 \leq i \leq N.$$

It is clear that $\|x_0 - p\| \leq M_1$. Let $\|x_{n-1} - p\| \leq M_1$. Next we will prove that $\|x_n - p\| \leq M_1$. From (1.29), we have

$$\begin{aligned} \|x_n - p\| &= \|S_{n(i)} y_n - p\| \\ &= \|S_{i(n)} y_n - S_{i(n)} p\| \\ &\leq \|y_n - p\| \\ &= \|(1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n - p\| \\ &= \|(1-\alpha_n)(x_{n-1} - p) + \alpha_n (T_{i(n)}^{k(n)} x_n - p)\| \\ &\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|T_{i(n)}^{k(n)} x_n - p\| \\ &\leq (1-\alpha_n)M_1 + \alpha_n M_1 = M_1. \end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is bounded.
Let

$$M_2 = \sup_{n\geq 1}\|x_n - p\| + M_1. \tag{3.2}$$

From (1.29) and condition (ii), we obtain

$$\begin{aligned}\|x_{n-1} - y_n\| &= \|x_{n-1} - (1-\alpha_n)x_{n-1} - \alpha_n T_{i(n)}^{k(n)} x_n\| \\ &= \alpha_n \|x_{n-1} - T_{i(n)}^{k(n)} x_n\| \\ &\leq \alpha_n(\|x_{n-1} - p\| + \|T_{i(n)}^{k(n)} x_n - p\|) \\ &\leq \alpha_n(M_2 + M_1) \to 0 \text{ as } n\to\infty, \quad (3.3)\end{aligned}$$

which implies that $\{\|x_{n-1} - y_n\|\}$ is bounded.
Again, let

$$M_3 = \sup_{n\geq 1}\|x_{n-1} - y_n\| + M_2.$$

Since,

$$\begin{aligned}\|y_n - p\| &= \|y_n - x_{n-1} + x_{n-1} - p\| \\ &\leq \|y_n - x_{n-1}\| + \|x_{n-1} - p\| \\ &\leq M_3\end{aligned}$$

therefore, $\{\|y_n - p\|\}$ is bounded. Set

$$M_4 = \sup_{n\geq 1}\|y_n - p\| + \sup_{n\geq 1}\|T_{i(n)}^{k(n)} y_n - p\|.$$

Denote

$$M = M_1 + M_2 + M_3 + M_4, \text{ obviously, } M < \infty.$$

Now from (1.29) for all $n \geq 1$, we obtain

$$\begin{aligned}\|x_n - p\|^2 &= \|S_{i(n)} y_n - p\|^2 = \|S_{i(n)} y_n - S_{i(n)} p\|^2 \\ &\leq \|y_n - p\|^2, \end{aligned}\tag{3.4}$$

thus by Lemma 2.1 and (1.8), we get

$$\begin{aligned}&\|y_n - p\|^2 \\ =\ & \|(1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_{n-1} - p\|^2 \qquad (3.5)\\ =\ & \|(1-\alpha_n)(x_{n-1} - p) + \alpha_n(T_{i(n)}^{k(n)} x_n - p)\|^2 \\ \leq\ & (1-\alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\langle T_{i(n)}^{k(n)} x_n - p, j(y_n - p)\rangle \\ =\ & (1-\alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\langle T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n \\ & + T_{i(n)}^{k(n)} y_n - p, j(y_n - p)\rangle \\ =\ & (1-\alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\langle T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n, j(y_n - p)\rangle \\ & + 2\alpha_n\langle T_{i(n)}^{k(n)} y_n - p, j(y_n - p)\rangle \\ \leq\ & (1-\alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\|T_{i(n)}^{k(n)} x_n \\ & - T_{i(n)}^{k(n)} y_n\|\|y_n - p\| + 2\alpha_n\{h_n\|y_n - p\|^2 - \Phi(\|y_n - p\|)\} \\ =\ & (1-\alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\delta_{in} \\ & + 2\alpha_n h_n\|y_n - p\|^2 - 2\alpha_n\Phi(\|y_n - p\|), \qquad (3.6)\end{aligned}$$

where

$$\delta_{in} = M\|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n\|, \ (1 \leq i \leq N).$$

From (1.29), we have

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - x_{n-1} + x_{n-1} - y_n\| \\ &= \|S_{i(n)} y_n - x_{n-1}\| + \|x_{n-1} - y_n\| \\ &\leq \|S_{i(n)} x_{n-1} - S_{i(n)} y_n\| + \|x_{n-1} - y_n\| \\ &\leq 2\|x_{n-1} - y_n\| \\ &= 2\alpha_n\|x_{n-1} - T_{i(n)}^{k(n)} x_n\| \\ &\leq 2\alpha_n(\|x_{n-1} - p\| + \|T_{i(n)}^{k(n)} x_n - p\|) \\ &\leq 2\alpha_n(M_2 + M_1) \\ &\leq 2\alpha_n M,\end{aligned}$$

thus from (ii), we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.7}$$

From the uniform continuity of $T_i$ , $(1 \leq i \leq N)$ leads to

$$\lim_{n\to\infty}\|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n\| = 0,$$

thus, we have

$$\lim_{n\to\infty}\delta_{in} = 0.$$

Also,

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n - p\|^2 \\ &= \|(1-\alpha_n)(x_{n-1} - p) + \alpha_n(T_{i(n)}^{k(n)} x_n - p)\|^2 \\ &\leq (1-\alpha_n)\|x_{n-1} - p\|^2 + \alpha_n\|T_{i(n)}^{k(n)} x_n - p\|^2 \\ &\leq \|x_{n-1} - p\|^2 + M^2\alpha_n, \end{aligned}\tag{3.8}$$

where the first inequality holds by the convexity of $\|\cdot\|^2$. Now substituting (3.8) into (3.6), we obtain

$$\begin{aligned}&\|y_n - p\|^2 \\ \leq\ & (1-\alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\delta_{in} \\ & + 2\alpha_n h_n(\|x_{n-1} - p\|^2 + M^2\alpha_n) - 2\alpha_n\Phi(\|y_n - p\|) \\ =\ & (1 - 2\alpha_n + \alpha_n^2)\|x_{n-1} - p\|^2 + 2\alpha_n h_n\|x_{n-1} - p\|^2 \\ & + 2h_n M^2\alpha_n^2 + 2\alpha_n\delta_{in} - 2\alpha_n\Phi(\|y_n - p\|) \\ =\ & (1 - 2\alpha_n)\|x_{n-1} - p\|^2 + (\alpha_n^2 + 2\alpha_n h_n)\|x_{n-1} - p\|^2 \\ & + 2h_n M^2\alpha_n^2 + 2\alpha_n\delta_{in} - 2\alpha_n\Phi(\|y_n - p\|) \\ \leq\ & (1 - 2\alpha_n)\|x_{n-1} - p\|^2 + (\alpha_n^2 + 2\alpha_n h_n)M^2 \\ & + 2h_n M^2\alpha_n^2 + 2\alpha_n\delta_{in} - 2\alpha_n\Phi(\|y_n - p\|) \\ \leq\ & (1 - 2\alpha_n)\|x_{n-1} - p\|^2 + \alpha_n[M^2(\alpha_n + 2h_n + 2\alpha_n h_n) \\ & + 2\delta_{in}]. \end{aligned}\tag{3.9}$$

Hence, from (3.4) and (3.9) we obtain

$$\begin{aligned}\|x_n - p\|^2 &\leq (1 - 2\alpha_n)\|x_{n-1} - p\|^2 \\ &+ \alpha_n[M^2(\alpha_n + 2h_n + 2\alpha_n h_n) + \delta_{in}].\end{aligned}$$

For all $n \geq 1$, put

$$\begin{aligned} \rho_n &= \|x_{n-1} - p\|, \\ \theta_n &= 2\alpha_n, \\ \omega_n &= \alpha_n[M^2(\alpha_n + 2h_n + 2\alpha_n h_n) + \delta_{in}], \end{aligned}$$

then according to Lemma 2.2, we obtain that

$$\lim_{n\to\infty} \|x_n - p\| = 0. \tag{3.10}$$

Completing the proof of Theorem 3.1. □

**Corollary 3.2.** *Let $K$ be a nonempty closed convex subset of a real Banach space $E$. Let $S : K \to K$ be a nonexpansive mapping and let $T : K \to K$ be an asymptotically generalized $\Phi$–hemicontractive mappings with $T(K)$ bounded and the sequence $\{h_n\} \subset [1,\infty)$, where $\lim_{n\to\infty} h_n = 1$. Furthermore, let $T$ be uniformly continuous. Assume that*

$$p \in \boldsymbol{F} = F(S) \bigcap F(T) = \{x \in K : Sx = Tx = x\} \neq \emptyset,$$

*such that for all $x, y \in K$,*

$$\|x - Sy\| \leq \|Sx - Sy\|. \tag{3.11}$$

*Let $\{\alpha_n\}$ be a sequence in [0,1] satisfying the following conditions:*

*(i)* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

*(ii)* $\lim_{n\to\infty} \alpha_n = 0$.

*For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by*

$$\begin{cases} x_0 \in K, \\ x_n = Sy_n, & \forall n \geq 1. \\ y_n = (1-\alpha_n)x_{n-1} + \alpha_n T x_n \end{cases} \tag{3.12}$$

*Then the sequence $\{x_n\}$ converges strongly at common fixed $p$ of $S$ and $T$.*

*Proof.* Taking $N = 1$ and $T^n = T$ in Theorem 3.1, the conclusion can be obtained immediately. □

**Remark 3.3.**

*(i) Corollary 3.2 captures the result of Kang et al. [17]. It follows that the result Kang et al. [17] is a special case of our result. Hence, our result extends and improves the results of Kang et al [17] and many others in the literature.*

*(ii) In our result the necessity of applying condition (C3) for the two classes of mappings (nonexpansive mappings and strongly pseudocontractive mappings) to prove strong convergence as considered in [17] and [12] is weaken by applying it to only one family of mapping which is the nonexpansive mappings.*

The above results are also valid for Lipschitz asymptotically generalized Φ-hemicontractive mappings.


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