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On cartesian product of commutative, self-distributive and transitive BE-algebra

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Abstract
In this paper we develop the idea of cartesian product of BE- algebras. Furthermore we introduced the cartesian product on commutative, self-distributive and transitive BE-algebras.

Keywords
BE-algebra, commutative BE-algebra, self-distributive BE-algebra, transitive BE-algebra.

AMS Subject Classification
06F35, 03G25, 08A30, 03B52.

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Contents

1. Introduction

After the introduction of the concepts of BCK and BCI algebras ([4,5]) by K. Iseki in 1966, some more systems of similar type have been introduced and discussed by a number of authors in the last two twenty years. K. H. Kim and Y.H. Yon studied dual BCK algebra and M.V. algebra in 2007 ([6]). It is known that BCK-algebras is a proper subclass of BCI-algebras. There are so many generalizations of BCK/BCI-algebras, such as BCH-algebras ([9]), dual BCK-algebras ([6]), d-algebras ([3]), etc. In ([2]), H. S. Kim and Y. H. Kim introduced the concept of BE-algebra as a generalization dual BCK-algebra. A. Walendziak ([1]) introduced the notion of commutative BE-algebras and discussed some of its properties.

2. preliminaries

Definition 2.1. *Let* $(\mathrm{A};*,1)$ *be a system of type* $(2,0)$ *consisting of a non-empty set* A, a *binary operation* "$*$" *and a fixed element* 1. *The system* $(\mathrm{A};*,1)$ *is called a* BE$-$ *algebra* $([2,7,8])$ *if the following conditions are satisfied:*

(i) $a*a=1$

(ii) $a*1=1$

(iii) $1*\mathrm{a}=\mathrm{a}$

(iv) $\mathrm{a}*(\mathrm{b}*\mathrm{c})=\mathrm{b}*(\mathrm{a}*\mathrm{c}), \forall \mathrm{a},\mathrm{b},\mathrm{c}\in \mathrm{A}.$

Note 2.2. *In any BE-algebra we can define a binary relation* "$\leq$" *as* $a \leq b$ *if and only if* $a*b=1, \forall a,b,\in A.$

Lemma 2.3. *In a BE-algebra the following identities are true [2]:*

1. $\mathrm{a}*(\mathrm{b}*\mathrm{a})=1$

2. $a*((a*b)*b)=1.$

Definition 2.4. *Let* $(\mathrm{A};*,1)$ *be a* BE *-algebra. An element a* $\in \mathrm{A}$ *is said to commute with* $\mathrm{b}\in\mathrm{A}$ *if* $(\mathrm{a}*\mathrm{b})*\mathrm{b}=(\mathrm{b}*\mathrm{a})*a.$ *If this condition is true for all a,* $\mathrm{b}\in\mathrm{A}$, *then* $(\mathrm{A};*,1)$ *is called a commutative BE-algebra [1].*

Definition 2.5. *A BE-algebra* $(A;*,1)$ *is said to be self distributive if* $\mathrm{a}*(\mathrm{b}*\mathrm{c})=(\mathrm{a}*\mathrm{b})*(\mathrm{a}*\mathrm{c}), \forall \mathrm{a},\mathrm{b},\mathrm{c}\in\mathrm{A}$

Definition 2.6. *A BE- algebra* $(A;*,1)$ *is said to be transitive [10] if for any* $a,b,c\in A$,

$$b*c \leq (a*b)*(a*c).$$

3. Cartesian Product of BE-algebras

In this section we study the properties of Cartesian product of BE-algebras.

Theorem 3.1. *Let $(A;*,1)$ be a system consisting of a non-empty set A, a binary operation "$*$" and a distinct element 1. Let $B = A\times A = \{(a_1,a_2) : a_1,a_2 \in A\}$. For $u,v \in B$ with $u = (a_1,a_2), v = (b_1,b_2)$, we define an operation "$\odot$" in B as*

$$u\odot v = (a_1 * b_1, a_2 * b_2).$$

Then $(B,\odot,(1,1))$ is a BE-algebra iff $(A;*,1)$ is a BE-algebra.

Proof. Suppose that $(B,\odot,(1,1))$ be a BE-algebra. Let $a \in A$ and we choose $u = (a,1) \in B$. Then

(1) $u\odot u = (1,1) \Rightarrow (a*a, 1*1) = (1,1)$

$$\Rightarrow a*a = 1, \text{ since } 1*1 = 1.$$

(2) $u\odot (1,1) = (1,1) \Rightarrow (a*1, 1*1) = (1,1)$

$$\Rightarrow a*1 = 1.$$

(3) $(1,1)\odot u = u \Rightarrow (1*a, 1*1) = (a,1)$

$$\Rightarrow 1*a = a.$$

(4) Let $a, b, c \in A$ and we choose $u = (a,1), v = (b,1)$, and $w = (c,1)$. Then

$$\begin{aligned} &u\odot(v\odot w) = v\odot(u\odot w)\\ \Rightarrow&(a*(b*c), 1*(1*1)) = (b*(a*c), 1*(1*1))\\ \Rightarrow&a*(b*c) = b*(a*c).\end{aligned}$$

This proves that $(A;*,1)$ is a BE-algebra.
Conversely, suppose that $(A;*,1)$ is a BE-algebra. Let $u = (a_1,a_2) \in B$. Then

(1) $u\odot u = (a_1,a_2)\odot(a_1,a_2)$

$$\begin{aligned}&= (a_1*a_1, a_2*a_2)\\ &= (1,1).\end{aligned}$$

(2) $u\odot(1,1) = (a_1,a_2)\odot(1,1)$

$$\begin{aligned}&= (a_1*1, a_2*1)\\ &= (1,1).\end{aligned}$$

(3) $(1,1)\odot u = (1,1)\odot(a_1,a_2)$

$$\begin{aligned}&= (1*a_1, 1*a_2)\\ &= (a_1,a_2)\\ &= u.\end{aligned}$$

(4) Let $u = (a_1,a_2), v = (b_1,b_2)$, and $w = (c_1,c_2)$ be any three elements of B.

Then

$$\begin{aligned} u\odot(v\odot w) &= (a_1,a_2)\odot((b_1,b_2)\odot(c_1,c_2))\\ &= (a_1,a_2)\odot(b_1*c_1, b_2*c_2)\\ &= (a_1*(b_1*c_1), a_2*(b_2*c_2))\\ &= (b_1*(a_1*c_1), b_2*(a_2*c_2))\\ &= (b_1,b_2)\odot(a_1*c_1, a_2*c_2)\\ &= (b_1,b_2)\odot((a_1,a_2)\odot(c_1,c_2))\\ &= v\odot(u\odot w).\end{aligned}$$

Hence $(B,\odot,(1,1))$ be a BE-algebra. □

Corollary 3.2. *If $(A;*,1)$ and $(B;o,e)$ are two BE-algebras, then $C = A\times B$ is also a BE$-$ algebra under the operation defined as follows: For $u = (a_1,b_1)$ and $v = (a_2,b_2)$ in C,*

$$u\odot v = (a_1 * a_2, b_1 o b_2)$$

Here the distinct element of C is $(1,e)$.

Note 3.3. *The above result can be extended for finite number of BE-algebras.*

Theorem 3.4. *Let $(A;*,1)$ be a BE-algebra and let $B = A\times A$. Then*

(a) B is commutative iff A is commutative.

(b) B is self distributive iff A is self distributive.

Proof. (a) First suppose that B is commutative. Let a and b be arbitrary elements of A. We choose $u = (a,1)$ and $v = (b,1)$. since B is commutative, we have

$$(u\odot v)\odot v = (v\odot u)\odot u.$$

This gives $((a*b)*b,1) = ((b*a)*a,1)$, which in turns imply that

$$(a*b)*b = (b*a)*a.$$

Hence A is commutative. Conversely suppose that A is commutative. Let $u = (a_1,a_2)$ and $v = (b_1,b_2)$ be any two arbitrary elements of B. Then

$$\begin{aligned}(u\odot v)\odot v &= ((a_1,a_2)\odot(b_1,b_2))\odot(b_1,b_2)\\ &= (a_1*b_1, a_2*b_2)\odot(b_1*b_2)\\ &= ((a_1*b_1)*b_1, (a_2*b_2)*b_2)\\ &= ((b_1*a_1)*a_1, (b_2*a_2)*a_2)\\ &= ((b_1,b_2)\odot(a_1,a_2))\odot(a_1,a_2)\\ &= (v\odot u)\odot u\end{aligned}$$

Hence B is commutative.
(b) First suppose that B is self distributive. Let a, b and c be arbitrary elements of A. We choose $u = (a,1), v = (b,1)$ and $w = (c,1)$. since B is self distributive, we have

$$u\odot(v\odot w) = (u\odot v)\odot(u\odot w).$$

This gives $(a,1)\odot(b*c,1)=(a*b,1)\odot(a*c,1)$, which in turns imply that

$$a*(b*c)=(a*b)*(a*c).$$

Hence A is self distributive.
Conversely suppose that A is self distributive. Let $\mathrm{u}=(\mathrm{a}_1,\mathrm{a}_2)$, $\mathrm{v}=(\mathrm{b}_1,\mathrm{b}_2)$ and $\mathrm{w}=(\mathrm{c}_1,\mathrm{c}_2)$ be any three arbitrary elements of B. Then

$$\begin{aligned}\mathrm{u}\odot(\mathrm{v}\odot\mathrm{w}) &= (\mathrm{a}_1,\mathrm{a}_2)\odot((\mathrm{b}_1,\mathrm{b}_2)\odot(\mathrm{c}_1,\mathrm{c}_2))\\ &= (\mathrm{a}_1*\mathrm{a}_2)\odot(\mathrm{b}_1*\mathrm{c}_1,\mathrm{b}_2*\mathrm{c}_2)\\ &= (\mathrm{a}_1*(\mathrm{b}_1*\mathrm{c}_1),\mathrm{a}_2*(\mathrm{b}_2*\mathrm{c}_2))\\ &= ((\mathrm{a}_1*\mathrm{b}_1)*(\mathrm{a}_1*\mathrm{c}_1),((\mathrm{a}_2*\mathrm{b}_2)*(\mathrm{a}_2*\mathrm{c}_2))\\ &= ((\mathrm{a}_1*\mathrm{b}_1),(\mathrm{a}_2*\mathrm{b}_2))\odot((\mathrm{a}_1*\mathrm{c}_1),(\mathrm{a}_2*\mathrm{c}_2))\\ &= (u\odot v)\odot(u\odot w).\end{aligned}$$

Hence B is self distributive. □

Theorem 3.5. *Let* $(\mathrm{A};*,1)$ *be* aBE *-algebra and let* $\mathrm{B}=\mathrm{A}\times\mathrm{A}$. *Then* B *is transitive iff* A *is transitive.*

Proof. Let $(A;*,1)$ be a BE-algebra and $u=(a_1,b_1)$, $v=(a_2,b_2)$, and $w=(a_3,b_3)$ be any three arbitrary elements of B. Then

$$\begin{aligned}&(\mathrm{v}\odot\mathrm{w})\odot((\mathrm{u}\odot\mathrm{v})\odot(\mathrm{u}\odot\mathrm{w}))\\ &=(\mathrm{v}\odot\mathrm{w})\odot((\mathrm{a}_1*\mathrm{a}_2,\mathrm{b}_1*\mathrm{b}_2)\odot(\mathrm{a}_1*\mathrm{a}_3,\mathrm{b}_1*\mathrm{b}_3))\\ &=(\mathrm{v}\odot\mathrm{w})\odot(((\mathrm{a}_1*\mathrm{a}_2)*(\mathrm{a}_1*\mathrm{a}_3)),((\mathrm{b}_1*\mathrm{b}_2)*(\mathrm{b}_1*\mathrm{b}_3)))\\ &=(\mathrm{a}_2*\mathrm{a}_3,\mathrm{b}_2*\mathrm{b}_3)\odot(((\mathrm{a}_1*\mathrm{a}_2)*(\mathrm{a}_1*\mathrm{a}_3)),((\mathrm{b}_1*\mathrm{b}_2)*(\mathrm{b}_1*\mathrm{b}_3)))\\ &=((\mathrm{a}_2*\mathrm{a}_3)*((\mathrm{a}_1*\mathrm{a}_2)*(\mathrm{a}_1*\mathrm{a}_3)),(\mathrm{b}_2*\mathrm{b}_3)*((\mathrm{b}_1*\mathrm{b}_2)*(\mathrm{b}_1*\mathrm{b}_3)))\\ &=(1,1).\end{aligned}$$

Therefore

$$\mathrm{v}\odot\mathrm{w}\leq(\mathrm{u}\odot\mathrm{v})\odot(\mathrm{u}\odot\mathrm{w}).$$

So B is transitive. Conversely assume that B be transitive. Let a, b and c be three arbitrary elements of A. We consider the elements $u=(a,1), v=(b,1)$ and $w=(c,1)$ of B. Since B is transitive, we have,

$$\begin{aligned}&\mathrm{v}\odot\mathrm{w}\leq(\mathrm{u}\odot\mathrm{v})\odot(\mathrm{u}\odot\mathrm{w})\\ &\Rightarrow(\mathrm{v}\odot\mathrm{w})\odot((\mathrm{u}\odot\mathrm{v})\odot(\mathrm{u}\odot\mathrm{w}))=(1,1)\\ &(\mathrm{b}*\mathrm{c})*((\mathrm{a}*\mathrm{b})*(\mathrm{a}*\mathrm{c})),1)=(1,1)\\ &\Rightarrow(b*c)*((a*b)*(a*c))=1\\ &\Rightarrow b*c\leq(a*b)*(a*c)\end{aligned}$$

Hence A is transitive. □

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