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# Generating functions for generalized tribonacci and generalized tricobsthal polynomials

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**Abstract.** In this work, we consider generating functions which are generalized tribonacci polynomials $T_n(x)$ and generalized tricobsthal polynomials $\mathbf{J}_n(x)$ which are defined in [7]. We derive generating functions for $(m+n)-th$ order of generalized tribonacci polynomials and generalized tricobsthal polynomials for $m \geq 2$. Furthermore, we obtain various families of bilinear and bilateral generating functions and give their special cases for these polynomials. Also, we obtain the summation formula of generalized tribonacci polynomials and generalized tricobsthal polynomials.

**AMS Subject Classifications**: 11B83, 11C08, 33C45.

**Keywords**: Generalized tricobsthal, generalized tribonacci polynomials, bilinear and bilateral generating functions.

## Contents



## 1. Introduction

There are so many studies in the literature that concern about the special polynomials. In [10], they introduced generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials and various families of multilinear and multilateral generating functions for these polynomials are derived. In [11], authors derived various families of multilinear and multilateral generating functions for generalized bivariate Fibonacci and Lucas polynomials. In [13], Mansour and Shattuck investigated some properties of polynomials whose coefficients are generalized tribonacci numbers. Recently, Kocer and Gedikce [12], has obtained some properties of the trivariate Fibonacci and Lucas polynomials by using these properties they gave some results for the tribonacci numbers and tribonacci polynomials. Also different types of polynomials are studied in [14], [15].

In [7], authors defined new kinds of polynomials called as generalized tribonacci polynomials and generalized tricobsthal polynomials. For these classes of polynomials, they found various results including recurrence relations and Binet's formulas, which can be useful also related our problem. Because in our work, we give the families of bilinear and bilateral generating functions which are generalized tribonacci polynomials $T_n(x)$ and generalized tricobsthal polynomials $\mathbf{J}_n(x)$ and are give their special cases. In addition to, we

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formulate the summation formula for these polynomials. Furthermore we give the exponential generating functions for generalized tribonacci polynomials and generalized tricobsthal polynomials.

Tribonacci numbers [5] which are defined by,

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \text{for} \quad n \geq 4, \tag{1.1}$$

with initial conditions $T_1 = 1$, $T_2 = 1$ and $T_3 = 2$. In [5], they present tribonacci polynomials defined by recurrence relation

$$t_n(x) = x^2 t_{n-1}(x) + x t_{n-2}(x) + t_{n-3}(x) \quad \text{for} \quad n \geq 4,$$

with initial conditions

$$t_1(x) = 1,\ t_2(x) = x^2,\ t_3(x) = x^4 + x \tag{1.2}$$

and property $t_n(1) = T_n$.

**Definition 1.1.** *[7] Generalized tribonacci polynomials are defined by recurrence relation*

$$T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x) \quad \textit{for} \quad n \geq 4, \tag{1.3}$$

*with initial conditions*

$$\begin{aligned} T_1(x) &= a, \\ T_2(x) &= b_2 x^2 + b_1 x + b_0, \\ T_3(x) &= c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0, \end{aligned} \tag{1.4}$$

*where $b_2$, $c_1$, $c_4$ positive integers and others parametres are nonnegative integers as initial conditions for tribonacci polynomials.*

**Theorem 1.2.** *[7]The Binet formula for generalized tribonacci polynomials defined by (1.3) with initial conditions (1.4) is*

$$T_n(x) = C_{1,T}\alpha_T^{n-1} + C_{2,T}\beta_T^{n-1} + C_{3,T}\gamma_T^{n-1} \tag{1.5}$$

*where $n$ is positive integer,*

$$\begin{aligned} C_{1,T} &= \frac{T_3(x) - (\gamma_T + \beta_T)T_2(x) + \gamma_T\beta_T T_1(x)}{(\alpha_T - \gamma_T)(\alpha_T - \beta_T)}, \\ C_{2,T} &= \frac{T_3(x) - (\gamma_T + \alpha_T)T_2(x) + \gamma_T\alpha_T T_1(x)}{(\beta_T - \gamma_T)(\beta_T - \alpha_T)}, \\ C_{3,T} &= \frac{T_3(x) - (\alpha_T + \beta_T)T_2(x) + \alpha_T\beta_T T_1(x)}{(\gamma_T - \alpha_T)(\gamma_T - \beta_T)} \end{aligned}$$

*and $\alpha_T, \beta_T, \gamma_T$ are different solutions of characteristic equation $y^3 - x^2y^2 - xy - 1 = 0$ of (1.3).*

$$\begin{aligned} \alpha_T &= \frac{x^2}{3} - \frac{2^{1/3}(-3x - x^4)}{3\delta_T} + \frac{\delta_T}{3.2^{1/3}}, \qquad (1.6) \\ \beta_T &= \frac{x^2}{3} + \frac{(1 + i\sqrt{3})(-3x - x^4)}{3.2^{2/3}\delta_T} - \frac{(1 - i\sqrt{3})\delta_T}{6.2^{1/3}}, \\ \gamma_T &= \frac{x^2}{3} + \frac{(1 - i\sqrt{3})(-3x - x^4)}{3.2^{2/3}\delta_T} - \frac{(1 + i\sqrt{3})\delta_T}{6.2^{1/3}} \end{aligned}$$

*with*

$$\delta_T = \sqrt[3]{27 + 9x^3 + 2x^6 + 3\sqrt{3}\sqrt{27 + 14x^3 + 3x^6}}. \tag{1.7}$$

In [7], tricobsthal polynomials are defined by recurrence formula

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) + x^2 J_{n-3}(x) \quad \text{for} \quad n \geq 4,$$

with initial conditions:

$$J_1(x) = 1, \;\; J_2(x) = 1 \text{ and } J_3(x) = x + 1.$$

The choice of initial conditions is according to property $J_n(1) = t_n(1) = T_n$ is $n-th$ tribonacci number, by analogy to Jacobsthal and Fibonacci polynomials ( [5], [6]). Analogously they can define generalized tricobsthal polynomials:

**Definition 1.3.** *[7]Generalized tricobsthal polynomials are defined by recurrence relation*

$$\mathbf{J}_n(x) = \mathbf{J}_{n-1}(x) + x\mathbf{J}_{n-2}(x) + x^2\mathbf{J}_{n-3}(x) \quad \textit{for} \quad n \geq 4, \tag{1.8}$$

*with initial condition:*

$$\begin{aligned}\mathbf{J}_1(x) &= a\\ \mathbf{J}_2(x) &= b\\ \mathbf{J}_3(x) &= c_1 x + c_0\end{aligned} \tag{1.9}$$

*where parameters $c_1$ is positive integers and $a$, $b$, $c_0$ are non-negative integers.*

**Theorem 1.4.** *[7]The Binet formula for generalized tricobsthal polynomials defined by (1.8) with initial conditions (1.9) is*

$$\mathbf{J}_n(x) = C_{1,\mathbf{J}}\alpha_{\mathbf{J}}^{n-1} + C_{2,\mathbf{J}}\beta_{\mathbf{J}}^{n-1} + C_{3,\mathbf{J}}\gamma_{\mathbf{J}}^{n-1}, \tag{1.10}$$

*where $n$ is positive integer, $x \neq 0$ and*

$$\begin{aligned}C_{1,\mathbf{J}} &= \frac{\mathbf{J}_3(x) - (\gamma_{\mathbf{J}} + \beta_{\mathbf{J}})\mathbf{J}_2(x) + \gamma_{\mathbf{J}}\beta_{\mathbf{J}}\mathbf{J}_1(x)}{(\alpha_{\mathbf{J}} - \gamma_{\mathbf{J}})(\alpha_{\mathbf{J}} - \beta_{\mathbf{J}})},\\ C_{2,\mathbf{J}} &= \frac{\mathbf{J}_3(x) - (\gamma_{\mathbf{J}} + \alpha_{\mathbf{J}})\mathbf{J}_2(x) + \gamma_{\mathbf{J}}\alpha_{\mathbf{J}}\mathbf{J}_1(x)}{(\beta_{\mathbf{J}} - \gamma_{\mathbf{J}})(\beta_{\mathbf{J}} - \alpha_{\mathbf{J}})},\\ C_{3,\mathbf{J}} &= \frac{\mathbf{J}_3(x) - (\alpha_{\mathbf{J}} + \beta_{\mathbf{J}})\mathbf{J}_2(x) + \alpha_{\mathbf{J}}\beta_{\mathbf{J}}\mathbf{J}_1(x)}{(\gamma_{\mathbf{J}} - \alpha_{\mathbf{J}})(\gamma_{\mathbf{J}} - \beta_{\mathbf{J}})}\end{aligned}$$

*and $\alpha_{\mathbf{J}}, \beta_{\mathbf{J}}, \gamma_{\mathbf{J}}$ are different solutions of characteristic equation $y^3 - y^2 - xy - x^2 = 0$ of (1.8).*

$$\begin{aligned}\alpha_{\mathbf{J}} &= \frac{8(3x+1)}{3\sqrt[3]{4}\delta_{\mathbf{J}}} + \frac{\delta_{\mathbf{J}}}{3\sqrt[3]{2}} + \frac{1}{3},\\ \beta_{\mathbf{J}} &= \frac{-(1+i\sqrt{3})(3x+1)}{3\sqrt[3]{4}\delta_{\mathbf{J}}} - \frac{(1-i\sqrt{3})\delta_{\mathbf{J}}}{6.\sqrt[3]{2}} + \frac{1}{3},\\ \gamma_{\mathbf{J}} &= \frac{-(1-i\sqrt{3})(3x+1)}{3.\sqrt[3]{4}\delta_{\mathbf{J}}} - \frac{(1+i\sqrt{3})\delta_{\mathbf{J}}}{6.\sqrt[3]{2}} + \frac{1}{3}\end{aligned} \tag{1.11}$$

*and*

$$\delta_{\mathbf{J}} = \sqrt[3]{27x^2 + 3\sqrt{3}\sqrt{27x^4 + 14x^3 + 3x^2} + 9x + 2}.$$

**Theorem 1.5.** *[7] Generating function for generalized tribonacci polynomials is given by formula*

$$\mathcal{G}_T(y) = \frac{T_1(x) + y(T_2(x) - x^2T_1(x)) + y^2(T_3(x) - x^2T_2(x) - xT_1(x))}{1 - yx^2 - y^2x - y^3} \tag{1.12}$$

*and for generalized tricobsthal polynomials by*

$$\mathcal{G}_{\mathbf{J}}(y) = \frac{\mathbf{J}_1(x) + y(\mathbf{J}_2(x) - \mathbf{J}_1(x)) + y^2(\mathbf{J}_3(x) - \mathbf{J}_2(x) - x\mathbf{J}_1(x))}{1 - y - xy^2 - x^2y^3}. \tag{1.13}$$

**Definition 1.6.** *Generalized tribonacci polynomials and generalized tricobsthal polynomials are defined for generating function by respectively:*

$$\sum_{n=0}^{\infty} T_{n+1}(x)t^n = \mathcal{G}_T(t) \tag{1.14}$$

$$\sum_{n=0}^{\infty} \mathbf{J}_{n+1}(x)t^n = \mathcal{G}_{\mathbf{J}}(t) \tag{1.15}$$

*where $\mathcal{G}_T(t)$ in (1.12) and $\mathcal{G}_{\mathbf{J}}(t)$ in (1.13) .*

Note that for generalized tribonacci polynomials and generalized tricobsthal polynomials are

$$\alpha_T + \beta_T + \gamma_T = x^2 \tag{1.16}$$

$$\alpha_T\beta_T\gamma_T = 1 \tag{1.17}$$

$$\alpha_{\mathbf{J}} + \beta_{\mathbf{J}} + \gamma_{\mathbf{J}} = \mathcal{T}$$

$$\alpha_{\mathbf{J}}\beta_{\mathbf{J}}\gamma_{\mathbf{J}} = \mathcal{K}$$

with

$$\begin{aligned}
\mathcal{T} &= 1 + \frac{2(3x+1)}{\sqrt[3]{4}\delta_{\mathbf{J}}} \\
\mathcal{K} &= \frac{1}{864\delta_{\mathbf{J}}}\left(\sqrt[3]{4}\delta_{\mathbf{J}}^2 + 2\delta_{\mathbf{J}} + 8\sqrt[3]{2}(1+3x)\right) \\
&\quad \times \left(4\delta_{\mathbf{J}} + 2\sqrt[3]{2}i(i+\sqrt{3})(1+3x) - \delta_{\mathbf{J}}^2\sqrt[3]{4}(1+i\sqrt{3})\right) \\
&\quad \times \left(4\delta_{\mathbf{J}} - 2\sqrt[3]{2}i(-i+\sqrt{3})(1+3x) - \delta_{\mathbf{J}}^2\sqrt[3]{4}(1-i\sqrt{3})\right).
\end{aligned}$$

## 2. Bilinear and Bilateral Generating Functions

In this section we will consider the families of bilinear and bilateral generating functions for generalized tribonacci polynomials $T_n(x)$ and generalized tricobsthal polynomials $\mathbf{J}_n(x)$ which are generated by (1.14), (1.15) and given explicitly by (1.12), (1.13) using the similar method considered in [1], [2], [3], [4], [8].

Using the polynomials mentioned above, we derived the following results:

**Theorem 2.1.** *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of $r$ complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order $\mu$, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; t) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)t^k$$

*where $a_k \neq 0$ , $\mu, \psi \in \mathbb{C}$ and*

$$\theta_{n,p,\mu,\psi}(x; y_1, ..., y_s; \xi) := \sum_{k=0}^{[n/p]} a_k T_{n+1-pk}(x)\Omega_{\mu+\psi k}(y_1, ..., y_r)\xi^k.$$

*Then, for $n, p \in \mathbb{N}$; we have*

$$\sum_{n=0}^{\infty} \theta_{n,p,\mu,\psi}(x; y_1, ..., y_s; \frac{\eta}{t^p})t^n = \mathcal{G}_T(t)\Lambda_{\mu,\psi}(y_1, ..., y_r; \eta). \tag{2.1}$$

**Proof**. For convenience, let $S$ denote the first member of the assertion (2.1) of Theorem 2.1. Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kT_{n+1-pk}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\frac{\eta^k}{t^{pk}}t^n.$$

Replacing $n$ by $n+pk$ and then using relation (1.14) we may write

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}T_{n+1}(x)a_k\,\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^kt^n\\&=\left(\sum_{n=0}^{\infty}T_{n+1}(x)t^n\right)\left(\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\right)\\&=\mathcal{G}_{\mathbf{T}}(t)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\end{aligned}$$

which completes the proof. ■

By using a similar idea, we also get the next result immediately.

**Theorem 2.2.** *Let*

$$\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\xi\right):=\sum_{k=0}^{[n/p]}a_k\mathbf{J}_{n-pk+1}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k. \tag{2.2}$$

*If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k$$

*then, for every nonnegative integer* $\mu$, *we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n=\mathcal{G}_{\mathbf{J}}(t)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta). \tag{2.3}$$

**Proof**. If we denote the left-hand side of (2.3) by $T$ and use (2.2), then we obtain

$$T=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\mathbf{J}_{n-pk+1}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^kt^{n-pk}.$$

Replacing $n$ by $n+pk$,

$$\begin{aligned}T&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,\mathbf{J}_{n+1}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^kt^n\\&=\sum_{n=0}^{\infty}\mathbf{J}_{n+1}(x)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\\&=\mathcal{G}_{\mathbf{J}}(t)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\end{aligned}$$

which completes the proof. ■

We derive generating functions for the $(m+n)-th$ order of generalized tribonacci polynomials and generalized tricobsthal polynomials for $m\geq 2$.

**Theorem 2.3.** *The following generating functions holds true for generalized tribonacci polynomials and generalized tricobsthal polynomials defined by (1.3) and (1.8) respectively:*

$$g_{T,m}(x,t) = \frac{T_m(x) + t\left(T_{m+1}(x) - x^2 T_m(x)\right) + t^2 T_{m-1}(x)}{1 - tx^2 - xt^2 - t^3}, \quad m \geq 2 \tag{2.4}$$

$$g_{\mathbf{J},m}(x,t) = \frac{\mathbf{J}_m(x) + t\left(\mathbf{J}_{m+1}(x) - \mathcal{T}\mathbf{J}_m(x)\right) + t^2 \mathcal{K}\mathbf{J}_{m-1}(x)}{1 - tx^2 - xt^2 - t^3}, \quad m \geq 2 \tag{2.5}$$

*where*

$$\sum_{n=0}^{\infty} T_{n+m}(x) t^n = g_{T,m}(x,t), \tag{2.6}$$

$$\sum_{n=0}^{\infty} \mathbf{J}_{n+m}(x) t^n = g_{\mathbf{J},m}(x,t). \tag{2.7}$$

**Proof**. From Binet formulas for generalized tribonacci polynomials and equation (1.16) and (1.17), we obtained

$$\begin{aligned}
\sum_{n=0}^{\infty} T_{n+m}(x) t^n &= \sum_{n=0}^{\infty} \left(C_{1,T}\alpha_T^{n+m-1} + C_{2,T}\beta_T^{n+m-1} + C_{3,T}\gamma_T^{n+m-1}\right) t^n \\
&= \left(\alpha_T^{m-1} C_{1,T} \sum_{n=0}^{\infty} \alpha_T^n t^n\right) + \left(\beta_T^{m-1} C_{2,T} \sum_{n=0}^{\infty} \beta_T^n t^n\right) \\
&\quad + \left(\gamma_T^{m-1} C_{3,T} \sum_{n=0}^{\infty} \gamma_T^n t^n\right) \\
&= \frac{\alpha_T^{m-1} C_{1,T}}{1 - \alpha_T t} + \frac{\alpha\beta_T^{m-1} C_{2,T}}{1 - \beta_T t} + \frac{\gamma_T^{m-1} C_{3,T}}{1 - \gamma_T t} \\
&= \frac{\left\{\begin{array}{c} (C_{1,T}\alpha_T^{m-1} + C_{2,T}\beta_T^{m-1} + C_{3,T}\gamma_T^{m-1}) \\ -t(C_{1,T}\alpha_T^{m-1}(x^2 - \alpha_T) + C_{2,T}\beta_T^{m-1}(x^2 - \beta_T) + C_{3,T}\gamma_T^{m-1}(x^2 - \gamma_T)) \\ +t^2(C_{1,T}\alpha_T^{m-1}\beta_T\gamma_T + C_{2,T}\beta_T^{m-1}\alpha_T\gamma_T + C_{3,T}\gamma_T^{m-1}\alpha_T\beta_T) \end{array}\right\}}{\left\{\begin{array}{c} 1 - t(\alpha_T + \beta_T + \gamma_T) + t^2(\alpha_T\beta_T + \alpha_T\gamma_T + \beta_T\gamma_T) \\ -t^3(\alpha_T\beta_T\gamma_T) \end{array}\right\}} \\
&= \frac{T_m(x) + t\left(T_{m+1}(x) - x^2 T_m(x)\right) + t^2 T_{m-1}(x)}{1 - tx^2 - xt^2 - t^3}.
\end{aligned}$$

The other cases for generalized tricobsthal polynomials can be done similarly. ■

For $T_{n+m}(x)$ and $\mathbf{J}_{n+m}(x)$, similar theorems will be found.

**Theorem 2.4.** *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of $r$ complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order $\mu$, let*

$$\Lambda_{m,\mu,\psi}(x; y_1, ..., y_r; t) := \sum_{k=0}^{\infty} a_k T_{m+pk}(x) \Omega_{\mu+\psi k}(y_1, ..., y_r) t^k$$

*where $a_k \neq 0$, $\mu, \psi \in \mathbb{C}$ and*

$$\theta_{\mu,\psi}(y_1, ..., y_r; \xi) := \sum_{k=0}^{[n/p]} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \xi^k.$$

*Then, for* $n, m \in \mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty} T_{n+m}(x)\theta_{\mu,\psi}(y_1,...,y_r;z)t^n = \Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p). \tag{2.8}$$

**Proof**. For convenience, let $H$ denote the first member of the assertion (2.8) of Theorem 2.4. Then,

$$H = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k T_{n+m}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)z^k t^n.$$

Replacing $n$ by $n+pk$ and then using relation (1.14) we may write

$$\begin{aligned} H &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} T_{n+m+pk}(x)a_k\,\Omega_{\mu+\psi k}(y_1,...,y_r)z^k t^{n+pk} \\ &= \left(\sum_{k=0}^{\infty} a_k\left(\sum_{n=0}^{\infty} T_{n+m+pk}(x)t^n\right)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k\right) \\ &= \sum_{k=0}^{\infty} a_k g_{T,m+pk}(x,t)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k \\ &= \Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p). \end{aligned}$$

which completes the proof. ■

**Theorem 2.5.** *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* $r$ *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* $\mu$*, let*

$$\Lambda_{m,\mu,\psi}(x;y_1,...,y_r;t) := \sum_{k=0}^{\infty} a_k \mathbf{J}_{m+pk}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)t^k$$

*where* $a_k \neq 0$, $\mu,\psi\in\mathbb{C}$ *and*

$$\theta_{\mu,\psi}(y_1,...,y_r;\xi) := \sum_{k=0}^{[n/p]} a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

*Then, for* $n, m \in \mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty} \mathbf{J}_{n+m}(x)\theta_{\mu,\psi}(y_1,...,y_r;z)t^n = \Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p). \tag{2.9}$$

**Proof**. For convenience, let $S$ denote the first member of the assertion (2.9) of Theorem 2.5. Then,

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k \mathbf{J}_{n+m}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)z^k t^n.$$

Replacing $n$ by $n+pk$ and then using relation (2.4) we may write

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbf{J}_{n+m+pk}(x)a_k\,\Omega_{\mu+\psi k}(y_1,...,y_r)z^k t^{n+pk} \\ &= \left(\sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\mathbf{J}_{n+m+pk}(x)t^n\right)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k\right) \\ &= \sum_{k=0}^{\infty}a_k g_{\mathbf{J},m+pk}(x,t)\Omega_{\mu+\psi k}(y_1,...,y_r)(zt^p)^k \\ &= \Lambda_{m,\mu,\psi}(x,t;y_1,...,y_r;zt^p). \end{aligned}$$

which completes the proof. ■

## 3. Special Cases

We formulate the sum of the first $n$ terms of generalized tribonacci polynomials and generalized tricobsthal polynomials respectively.

**Theorem 3.1.** *The sum of the first $n-$terms of generalized tribonacci polynomials and generalized tricobsthal polynomials are given by*

$$\begin{aligned} \sum_{j=1}^{n}T_j(x) &= \frac{\left\{\begin{array}{c} T_{n+3}(x)+(1-x^2)T_{n+2}(x)+(1-x^2-x)T_{n+1}(x) \\ -(1-x^2-x)T_1(x)+(x^2-1)T_2(x)-T_3(x) \end{array}\right\}}{x^2+x}, \\ \sum_{j=0}^{n}\mathbf{J}_j(x) &= \frac{\mathbf{J}_{n+3}(x)-x\mathbf{J}_{n+1}(x)-\mathbf{J}_3(x)+x\mathbf{J}_1(x)}{x^2+x} \end{aligned}$$

*respectively.*

**Proof**. Note that, applying $T_n(x)=x^2T_{n-1}(x)+xT_{n-2}(x)+T_{n-3}(x)$, we deduce that

$$\begin{aligned} &n=4\Rightarrow T_4(x)=x^2T_3(x)+xT_2(x)+T_1(x) \\ &n=5\Rightarrow T_5(x)=x^2T_4(x)+xT_3(x)+T_2(x) \\ &\cdots \\ &n=n+2\Rightarrow T_{n+2}(x)=x^2T_{n+1}(x)+xT_n(x)+T_{n-1}(x) \\ &n=n+3\Rightarrow T_{n+3}(x)=x^2T_{n+2}(x)+xT_{n+1}(x)+T_n(x). \end{aligned} \tag{3.1}$$

If we sum of both sides of (3.1), then we obtain

$$\begin{aligned} T_4(x)+T_5(x)+\cdots+T_{n+3}(x) &= xT_2(x) \\ &\quad+\left[(x^2+x)\sum_{j=3}^{n+1}T_j(x)\right]+x^2T_{n+2}(x) \\ &\quad+\sum_{j=1}^{n}T_j(x). \end{aligned} \tag{3.2}$$

If we make necessary regulations, (3.2) becomes

$$(x^2+x)\sum_{j=1}^{n}T_j(x)=\left\{\begin{array}{c}(1-x^2)T_{n+2}(x)+T_{n+3}(x)-(x^2+x)T_3(x)\\ -xT_2(x)+(1-x^2-x)T_{n+1}(x)\\ -(1-x^2-x)(T_1(x)+T_1(x)+T_1(x))\end{array}\right\}.$$

Therefore

$$\sum_{j=1}^{n}T_j(x)=\frac{\left\{\begin{array}{c}T_{n+3}(x)+(1-x^2)T_{n+2}(x)+(1-x^2-x)T_{n+1}(x)\\ -(1-x^2-x)T_1(x)+(x^2-1)T_2(x)-T_3(x)\end{array}\right\}}{x^2+x}$$

as we claimed. The other cases for generalized tricobsthal polynomials can be done similarly. ■

**Theorem 3.2.** *The exponential generating function of generalized tribonacci polynomials and generalized tricobsthal polynomials are given by*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{T_n(x)}{n!}t^n &= \frac{\beta_T\gamma_T C_{1,T}e^{\alpha_T t}+\alpha_T\gamma_T C_{2,T}e^{\beta_T t}+\alpha_T\beta_T C_{3,T}e^{\gamma_T t}}{\alpha_T\beta_T\gamma_T},\\ \sum_{n=0}^{\infty}\frac{\mathbf{J}_n(x)}{n!}t^n &= \frac{\beta_{\mathbf{J}}\gamma_{\mathbf{J}} C_{1,\mathbf{J}}e^{\alpha_{\mathbf{J}} t}+\alpha_{\mathbf{J}}\gamma_{\mathbf{J}} C_{2,\mathbf{J}}e^{\beta_{\mathbf{J}} t}+\alpha_{\mathbf{J}}\beta_{\mathbf{J}} C_{3,\mathbf{J}}e^{\gamma_{\mathbf{J}} t}}{\alpha_{\mathbf{J}}\beta_{\mathbf{J}}\gamma_{\mathbf{J}}}\end{aligned}$$

*respectively.*

**Proof**. Assuming that the exponential generating function of the generalized tribonacci polynomials, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{T_n(x)}{n!}t^n &= \sum_{n=0}^{\infty}\left(C_{1,T}\alpha_T^{n-1}+C_{2,T}\beta_T^{n-1}+C_{3,T}\gamma_T^{n-1}\right)\frac{t^n}{n!}\\ &= \frac{C_{1,T}}{\alpha_T}\sum_{n=0}^{\infty}\frac{(\alpha_T t)^n}{n!}+\frac{C_{2,T}}{\beta_T}\sum_{n=0}^{\infty}\frac{(\beta_T t)^n}{n!}+\frac{C_{3,T}}{\gamma_T}\sum_{n=0}^{\infty}\frac{(\gamma_T t)^n}{n!}\\ &= \frac{\beta_T\gamma_T C_{1,T}e^{\alpha_T t}+\alpha_T\gamma_T C_{2,T}e^{\beta_T t}+\alpha_T\beta_T C_{3,T}e^{\gamma_T t}}{\alpha_T\beta_T\gamma_T}.\end{aligned}$$

The other cases for generalized tricobsthal polynomials can be done similarly. ■

We can give many applications of our teorems obtained in the previous section with help of appropriate choices of the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems.

If we set

$$s=1\ \text{ and }\Omega_{\mu+\psi k}(y)=g^{(s)}_{\mu+\psi k}(\lambda,y)$$

in Theorem 2.1. Recall that, by $g_n^{(s)}(\lambda,x)$ we denote the generalized Cesáro polynomials (see, e.g. [3]) generated by

$$\sum_{n=0}^{\infty}g_n^{(s)}(\lambda,x)t^n=(1-t)^{-s-1}(1-xt)^{-\lambda} \tag{3.3}$$

where $|t|<\min\left\{1,|x|^{-1}\right\}$. Then, from Teorem 2.1, we get a family of the bilateral generating functions for the generalized Cesáro polynomials and the generalized tribonacci polynomials.

**Corollary 3.3.** *If*

$$\Lambda_{\mu,\psi}(\lambda,y;\zeta):=\sum_{k=0}^{\infty}a_k g_{\mu+\psi k}^{(s)}(\lambda,y)\zeta^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k T_{n+1-pk}(x)g_{\mu+\psi k}^{(s)}(\lambda,y)\eta^k t^{n-pk}=\mathcal{G}_T(t)\Lambda_{\mu,\psi}(\lambda,y;\eta)$$

**Remark 3.4.** *Using the generating relation (1.14) for generalized tribonacci polynomials and* $a_k=1$, $\mu=0$, $\psi=1$ *in Corollary 3.3, we find that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k T_{n+1-pk}(x)g_{k}^{(s)}(\lambda,y)\eta^k t^{n-pk}=\mathcal{G}_T(t)(1-\eta)^{-s-1}(1-y\eta)^{-\lambda}.$$

We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r\,)=\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)$$

in Theorem 2.2, where the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ [1], generated by

$$\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)z^n=(1-x_1z)^{-\alpha}\,e^{(x_2+...+x_r)z} \tag{3.4}$$

where $|z|<|x_1|^{-1}$.

The following results which provides a class of bilateral generating functions for generalized tribonacci polynomials and the family of multivariable polynomials given explicitly by (3.4).

**Corollary 3.5.** *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)\zeta^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\mathbf{J}_{n+1-pk}(x)\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)\eta^k t^{n-pk}=\mathcal{G}_{\mathbf{J}}(t)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.5}$$

*provided that each member of (3.5) exists.*

**Remark 3.6.** *Using the generating relation (3.4) for the multivariable polynomials and getting* $a_k=1$, $\mu=0$, $\psi=1$ *in Corollary 3.1, we find that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\mathbf{J}_{n+1-pk}(x)\Phi_{k}^{(\alpha)}(y_1,...,y_r)\eta^k t^{n-pk}=\mathcal{G}_{\mathbf{J}}(t)\,(1-y_1\eta)^{-\alpha}\,e^{(y_2+...+y_r)\eta},$$

$$\left(|\eta|<\left\{|y_1|^{-1}\right\}\right).$$

If we set $s=1$

$$\Omega_{\mu+\psi k}(y)=\mathbf{J}_{\mu+\psi k-1}(y)$$

in Theorem 2.2.Then, from Teorem 2.2, we get a family of the bilinear generating functions for generalized tricobsthal polynomials given explicitly by (1.8).

**Corollary 3.7.** *If*

$$\Lambda_{\mu,\psi}(y;\zeta):=\sum_{k=0}^{\infty}a_k\mathbf{J}_{\mu+\psi k-1}(y)\zeta^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\mathbf{J}_{n+1-pk}(x)\mathbf{J}_{\mu+\psi k-1}(y)\eta^k t^{n-pk}=\mathcal{G}(t)\Lambda_{\mu,\psi}(y;\eta) \tag{3.6}$$

*provided that each member of (3.6) exists.*

**Remark 3.8.** *Using the generating relation (1.15) for generalized tricobsthal polynomials and getting $a_k=1$, $\mu=0,\psi=1$ in Corollary 3.2, we find that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\mathbf{J}_{n+1-pk}(x)\mathbf{J}_{k-1}(y)\eta^k t^{n-pk}=\mathcal{G}_{\mathbf{J}}(t)\mathbf{g}_{\mathbf{J}}(\eta)$$

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)$$

in Theorem 2.4. Recall that, by $h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ we denote the multivariable Lagrange-Hermite polynomials [8] generated by

$$\sum_{n=0}^{\infty}h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)t^n=\prod_{j=1}^{r}\left\{\left(1-x_jt^j\right)^{-\alpha_j}\right\} \tag{3.7}$$

where $|t|<\min\left\{|x_1|^{-1},...,|x_r|^{-1/r}\right\}$. Then, from Teorem 2.4, we obtain the following result which is aclass of bilateral generating functions for the multivariable Lagrange-Hermite polynomials and generalized tribonacci polynomials.

**Corollary 3.9.** *If*

$$\Lambda_{m,\mu,\psi}(x;y_1,...,y_r;t):=\sum_{k=0}^{\infty}a_kT_{m+pk}(x)h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)t^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

*and*

$$\theta_{\mu,\psi}(y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_kh_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)\xi^k$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kT_{n+m}(x)h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)z^kt^n=\Lambda_{m,\mu,\psi}(x.t;y_1,...,y_r;zt^p).$$

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)$$

in Theorem 2.5. Recall that, by $g_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ we denote the Chan-Chyan-Srivastava polynomials [9] generated by

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)t^n=\prod_{j=1}^{r}\left\{(1-x_jt)^{-\alpha_j}\right\} \tag{3.8}$$

where $|t|<\min\left\{|x_1|^{-1},...,|x_r|^{-1}\right\}$. Then, from Teorem 2.5, we obtain the following result which is aclass of bilateral generating functions for the Chan-Chyan-Srivastava polynomials and generalized tricobsthal polynomials.

**Corollary 3.10.** *If*

$$\Lambda_{m,\mu,\psi}(x;y_1,...,y_r;t):=\sum_{k=0}^{\infty} a_k\mathbf{J}_{m+pk}(x)g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)t^k$$
$$(a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})$$

*and*

$$\theta_{\mu,\psi}(y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]} a_k g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)\xi^k$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k\mathbf{J}_{n+m}(x)g_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(y_1,...,y_r)z^kt^n=\Lambda_{m,\mu,\psi}(x.t;y_1,...,y_r;zt^p).$$

Notice that, for every suitable choice of the coefficients $a_k$ $(k\in\mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r\in\mathbb{N}$, are expressed as an appropriate product of several simpler relatively functions, the assertions of Theorem 2.1, 2.2, 2.4 and Theorem 2.5 can be applied to yield many different families of multilinear and multilateral generating functions for generalized tribonacci polynomials and generalized tricobsthal polynomials.

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