Existence results for generalized vector quasi-equilibrium problems

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Abstract
In this work, we extend Minty's type lemma for a class of generalized vector quasi-equilibrium problems in Hausdorff topological vector spaces and establish some results on existence of solutions both under compact and noncompact assumption by using 1- person game theorems.

Keywords
Escaping sequence, upper semicontinuity, hemicontinuous, P- convex, P- monotone mapping.

AMS Subject Classification
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Contents

1. Introduction and Formulation

Suppose K be a nonempty subset of real t.v.s X and Y be any real t.v.s. Suppose $C : K \to Y$ is a closed convex and solid cone in Y. Suppose $f : K \times K \to Y$ be a vector-valued bifunction, then vector equilibrium problem (for short, VEP) is the problem of finding $u_0 \in K$ such that

$$f(u_0, v) \notin -intC, \text{ for all } v \in K, \quad (1.1)$$

where $intC$ denotes the interior of C. Problem (1.1) is a mathematical model of many problems such as, Nash equilibria, variational inequalities, optimization problems, fixed point problems, vector saddle point problems and complementarity problems, see [2–4, 9, 12, 14, 21]. Also the solution for various kind of equilibrium problems have been considered and extensively studied by numerous authors, for detail, we refer to [2, 10, 12, 17, 20].

As a generalization of VEP many authors studied the quasi- version of VEP which include Nash equilibria, quasi-complementarity problems, vector optimization problems, vector quasi saddle point problems, vector quasi-variational inequality problems as a special cases, see for example [5, 6, 15, 16, 19, 22, 23]. For more generalized form of VEP, many authors studied generalized quasi-equilibrium problems and obtained existence results for solution in Hausdorff t.v.s under both compact and noncompact settings, see [1, 11, 13, 15, 16, 18].

Motivated and inspired by the aforesaid research work, we obtain the existence of solutions to generalized version of vector quasi equilibrium problems in compact and noncompact settings by using 1- person game theorems.

From now unless, otherwise stated:

Suppose K be a nonempty convex subset of Hausdorff t.v.s X and Y be an ordered Hausdorff t.v.s. Suppose the multifunction $C : K \to 2^Y$ is a closed convex and solid cone that is, $intC(u) \neq \phi$, for each $u \in K$. Define partial order relation $\leq_{C(u)}$ on Y as follows:

$$v \leq_{C(u)} w \Leftrightarrow w - v \in C(u).$$

If the ordering is strict, then $v <_{C(u)} w \Leftrightarrow w - v \in intC(u)$. Let $P := \bigcap_{u \in K} C(u)$.

Suppose the multifunction $A : K \to 2^K$ be given continuous and let $M, N : K \times K \to 2^Y$ be any two multifunctions, then the generalized vector quasi-equilibrium problem(for short, GVQEP) is the problem of finding $u_0 \in K$ such that $u_0 \in cl_X A(u_0)$ and

$$M(u_0, v) + N(u_0, v) \nsubseteq -int_Y C(u_0), \forall y \in A(u_0). \quad (1.2)$$

In section 2, we shall recall some results and elementary definitions which will be used in latter sections. Results on existence of solutions of problem (1.2) both under compact and noncompact assumptions are presented in section 3 and 4, respectively.

2. Preliminaries

Now we shall recall the following:

Definition 2.1. *We shall denote $dom f$, the domain of a function $f : X \to \overline{\mathbb{R}}$ and is defined as*

$$dom f = \{u \in X : f(u) \in \mathbb{R}\}.$$

Also f is said to be:

(i) *upper semicontinuous (u.s.c) at $u_0 \in dom f$ iff, $\exists$ a neighborhood $\mathcal{N}$ around u_0 satisfying*

$$f(u) \leq f(u_0) + \varepsilon, \quad \forall u \in \mathcal{N}, \varepsilon > 0,$$

(ii) *lower semicontinuous (l.s.c) at $u_0 \in dom f$ iff $\exists$ a neighborhood $\mathcal{N}$ around u_0 such that*

$$f(u) \geq f(u_0) - \varepsilon < \forall u \in , \varepsilon > 0.$$

Definition 2.2. *For any two topological spaces X and Y, the multifunction $T : X \to 2^Y$ is said to be:*

(i) *u.s.c on X if for each $u \in X$ and for each open set $\mathcal{N}_2$ in Y containing $T(u)$, $\exists$ an open neighborhood $\mathcal{N}_1$ of u in X such that $T(v) \subseteq \mathcal{N}_2, \forall v \in \mathcal{N}_1$,*

(ii) *l.s.c on X if for each $u \in X$ and each open set $\mathcal{N}_2$ in Y with $T(u) \cap \mathcal{N}_2 \neq \phi$, $\exists$ an open neighborhood $\mathcal{N}_1$ of u in X such that $T(v) \cap \mathcal{N}_2 \neq \phi, \forall v \in \mathcal{N}_1$.*

Definition 2.3. *Suppose $M, N : K \to 2^K$ be the multifunction, then for each $u \in K$, we define the multifunctions clM, coM, $M \cap N : K \to 2^X$ as $(clM)(u) = clM(u)$, $(coM)(u) = coM(u)$ and $(M \cap N)(u) = M(u) \cap N(u)$.*

Definition 2.4. *The graph of the multifunction $N : K \to 2^Y$ is denoted by $Gr(N)$ and is defined by*

$$Gr(N) = \{(u, v) \in X \times Y : u \in X, v \in N(u)\}.$$

Then inverse of N, denoted by N^{-1} is a multifunction $N^{-1} : Ran(N) \to X$ and is defined by

$$u \in N^{-1}(v) \iff v \in N(u).$$

Definition 2.5. *Suppose K be a convex subset of t.v.s X and $P \subset Y$ be a convex cone. Suppose $M : K \times K \to 2^Y$ and $N : K \to 2^Y$ be two multifunctions, then*

(i) *M is P-monotone if,*

$$M(u, v) + M(v, u) \subset -P, \forall u, v \in K.$$

(ii) *N is P-convex if,*

$$N(\alpha u + (1 - \alpha)v)) \subset \alpha N(u) + (1 - \alpha)N(v) - P,$$

$$\forall u, v \in K \text{ and } \alpha \in [0, 1].$$

Definition 2.6. *[15] Suppose K be a subset of t.v.s X such that $K = \bigcup_{p=1}^{\infty} K_p$, where $\{K_p\}_{p \in \mathbb{N}}$ is expanding sequence of nonempty compact sets in the perception that $K_p \subseteq K_{p+1}, \forall p \in \mathbb{N}$. Then a sequence $\{u_p\}_{p \in \mathbb{N}}$ in K is said to be escaping sequence from K (relative to $\{K_p\}_{p \in \mathbb{N}}$) if for each l, $\exists M$ such that $l \geq M$, $u_p \notin K_p$.*

Following 1-person game theorems are particular case of [7, Theorem 2] and [8, Theorem 2], respectively.

Theorem 2.7. *Supose X be a Hausdorff t.v.s and $K \subset X$ is compact and convex. Suppose $P, A, cl_X A : K \to 2^K$ are multifunctions such that $A(u)$ is convex set, for each $u \in K$. Both $A^{-1}(v)$ and $P^{-1}(v)$ are open in K, for each $v \in K$. $cl_X A$ is u.s.c, $u \notin coP(u)$, for each $u \in K$. Then $\exists u_0 \in K$ such that $u_0 \in cl_X A(u_0)$ and $A(u_0) \cap P(u_0) = \phi$.*

Theorem 2.8. *Suppose X locally convex Hausdorff t.v.s and $K \subset X$, and $D \subset K$. Suppose that $A, P : K \to 2^D$ and $cl_X A : K \to 2^K$ are multifunctions such that $A(u)$ is a nonempty convex set, for each $u \in K$. Both $A^{-1}(v)$ and $P^{-1}(v)$ are open sets in K, for each $v \in D$. $cl_X A$ is u.s.c, for each $u \in K$, $u \notin coP(u)$. Then $\exists u_0 \in K$ such that $u_0 \in cl_X A(u_0)$ and $A(u_0) \cap P(u_0) = \phi$.*

3. Solutions in Compact Setting

First of all, we prove Minty's type lemma to obtain the solutions of GVQEP in compact setting. Let $P := \bigcap_{u \in K} C(u)$.

Lemma 3.1. *Suppose $\phi \neq K \subseteq X$ is closed compact and convex set. Consider the multifunctions $A : K \to 2^K, M, N : K \times K \to 2^Y$ with nonempty convex values. Assume that*

(i) *$M(u, u) = \{0\}, N(u, u) = \{0\}, \forall u \in K$,*

(ii) *M is P-monotone,*

(iii) *$Gr(W)$ is closed in $X \times Y$, where $W : K \to 2^Y$ is multifunction given by $W(u) := Y \setminus -int_Y C(u)$,*

(iv) *the map $\alpha \in [0, 1] \mapsto M(\alpha v + (1 - \alpha)u,)$ is u.s.c at $\alpha = 0^+, \forall u, v \in K$,*

(v) *$M(u, .), N(u, .) : K \to 2^Y$ be P-convex, $\forall u \in K$.*

Then the following are equivalent:

(I) *$\exists u_0 \in K$ such that $u_0 \in cl_K A(u_0)$ and*

$$M(v, u_0) - N(u_0, h) \nsubseteq int_Y C(u_0), \forall v \in A(u_0).$$

(II) *$\exists u_0 \in K$ such that $u_0 \in cl_K A(u_0)$ and*

$$M(u_0, v) + N(u_0, h) \nsubseteq -int_Y C(u_0), \forall v \in A(u_0).$$

Proof. Suppose $\exists u_0 \in K$ such that $u_0 \in cl_K A(u_0)$ and

$$M(v, u_0) - N(u_0, v) \not\subseteq int_Y C(u_0), \forall v \in A(u_0).$$

We set $u_\alpha := \alpha v + (1 - \alpha)u_0, \ \alpha \in [0, 1]$.
Clearly $u_\alpha \in A(u_0), \forall \alpha \in [0, 1]$ and so, we have

$$u_0 \in A(u_0) \quad \text{and} \quad M(u_\alpha, u_0) - N(u_0, u_\alpha) \not\subseteq int_Y C(u_0).$$

Using P−convexity of $M(u, .), \ \forall u \in K$, we have

$$\alpha M(u_\alpha, v) + (1 - \alpha)M(u_\alpha, u_0) \subseteq M(u_\alpha, u_\alpha) + P \subseteq C(u_0). \tag{3.1}$$

Now using P−convexity of $N(u, .), \ \forall u \in K$, we have

$$\begin{aligned}
\alpha N(u_0, v) &\subseteq \alpha N(u_0, v) + (1 - \alpha)N(u_0, u_0) \\
&\subseteq N(u_0, u_\alpha) + P \\
&\subseteq N(u_0, u_\alpha) + C(u_0). \tag{3.2}
\end{aligned}$$

From (3.1) and (3.2), we have

$$\begin{aligned}
\alpha M(u_\alpha, v) + \alpha(1 - \alpha)N(u_0, v) &\subseteq -(1 - \alpha)M(u_\alpha, u_0) \\
&\quad + (1 - \alpha)N(u_0, u_\alpha) + C(u_0). \tag{3.3}
\end{aligned}$$

We claim that

$$M(u_\alpha, v) + (1 - \alpha)N(u_0, v) \not\subseteq -int_Y C(u_0), \forall \alpha \in (0, 1]. \tag{3.4}$$

Suppose that (3.4) is not true, then $\exists \alpha_0 \in (0, 1]$ such that

$$M(u_{\alpha_0}, v) + (1 - \alpha_0)N(u_0, v) \subseteq -int_Y C(u_0). \tag{3.5}$$

By (3.3), we have

$$\begin{aligned}
&\alpha_0[M(u_{\alpha_0}, v) + (1 - \alpha_0)N(u_0, v)] \\
&\subseteq -(1 - \alpha_0)[M(u_{\alpha_0}, u_0) - N(u_0, u_{\alpha_0})] + C(u_0) \\
&\Rightarrow (1 - \alpha_0)[M(u_{\alpha_0}, u_0) - N(u_0, u_{\alpha_0})] \subseteq \\
&- \alpha_0[M(u_{\alpha_0}, v) + (1 - \alpha_0)N(u_0, v)] + C(u_0) \\
&\subseteq int_Y C(u_0) + C(u_0) \subset int_Y C(u_0),
\end{aligned}$$

a contradiction to (3.5).
Let $k(\alpha) := M(u_\alpha, v) + (1 - \alpha)N(u_0, v)$, then $k(\alpha) \in Y \setminus -int_Y C(u_0), \forall \alpha \in (0, 1]$. Using hypothesis (iii) and hemicontinuity of $k(\alpha) := G(u_\alpha, v) + (1 - \alpha)N(u_0, v)$, it follows that $k(0) \in Y \setminus -int_Y C(u_0)$ as $\alpha \to 0^+$ i.e. $\exists u_0 \in K$ such that $u_0 \in cl_K A(u_0)$ and

$$M(u_0, v) + N(u_0, v) \not\subseteq -int_Y C(u_0), \forall v \in A(u_0).$$

Conversely, suppose that (II) holds i.e. $\exists u_0 \in K$ such that $u_0 \in cl_K A(u_0)$ and

$$M(u_0, v) + N(x_0, v) \not\subseteq -int_Y C(u_0), \forall v \in A(u_0).$$

If possible, let us assume that (I) does not hold. Then $\exists v_0 \in A(u_0)$ such that

$$M(v_0, u_0) - N(u_0, v_0) \subseteq int_Y C(u_0).$$

Since M is P-monotone,

$$M(v_0, u_0) \subseteq -M(u_0, v_0) - P.$$

Now

$$\begin{aligned}
M(v_0, u_0) - N(u_0, v_0) &\subseteq -M(u_0, v_0) - N(u_0, v_0) - P \\
\Rightarrow -[M(u_0, v_0) + N(u_0, v_0)] &\subseteq M(v_0, u_0) - N(u_0, v_0) + P \\
&\subseteq int_Y C(u_0) + C(u_0) \\
&\subseteq int_Y C(u_0), \tag{3.6}
\end{aligned}$$

a contradiction to (II). Therefore $\exists u_0 \in K$ such that $u_0 \in cl_K A(u_0)$ and

$$M(v, u_0) - N(u_0, v) \not\subseteq -int_Y C(u_0), \forall v \in A(u_0).$$

$\square$

Theorem 3.2. *Suppose the multifunctions $M, N : K \times K \to 2^Y, C : K \to 2^Y$ and $A : K \to 2^K$ satisfying the following conditions:*

(i) $M(u, u) = \{0\}, N(u, u) = \{0\}, \forall u \in K$,

(ii) M is $P-$ monotone,

(iii) $M(u, .)$ and $N(u, .)$ be $P-$ convex, $\forall u \in K$,

(iv) *the function $W : K \to 2^Y$ defined by $W(u) = Y \setminus (-int_Y C(u))$, for all $u \in K$, has closed graph in $X \times Y$,*

(v) $A^{-1}(v)$ *is open $\forall v \in K$, and $A(u) \neq \phi$ is convex $\forall u \in K$. Also $cl_X A : K \to 2^K$ is u.s.c.*

(vi) $M(u, .)$ *is u.s.c and $N(., v)$ is l.s.c, $\forall u, v \in K$,*

(vii) *the function $\alpha \in [0, 1] \mapsto M(\alpha v + (1 - \alpha)u, v)$ is u.s.c at $\alpha = 0^+, \forall u, v \in K$.*

Then $\exists u_0 \in K$ such that $u_0 \in cl_X A(u_0)$ and

$$M(u_0, v) + N(u_0, v) \not\subseteq -int_Y C(u_0), \forall v \in A(x_0).$$

Proof. Consider the multifunction $P : K \to 2^K$ defined by

$$P(u) = \{v \in K : M(v, u) - N(u, v) \subseteq int_Y C(u)\}, \ \forall u \in K.$$

Firstly, let us prove that $u \notin coP(u), \forall u \in K$. If possible, let us assume that $\exists u_0 \in K$ such that $u_0 \in coP(u_0)$. This implies that u_0 can be expressed as

$$u_0 = \sum_{i=1}^{n} \alpha_i v_i, \quad \text{with} \quad \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1,$$

where $\{v_1, v_2, ..., v_n\}$ be a finite subset of K. Since for each $i, \ v_i \in P(u_0)$, we have

$$M(v_i, u_0) - N(u_0, v_i) \subseteq int_Y C(u_0).$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i[M(v_i, u_0) - N(u_0, v_i)] \subseteq int_Y C(u_0). \quad (3.7)$$

Since $N(u, .)$ is P-convex, it follows that

$$\sum_{i=1}^{n} \alpha_i N(u_0, v_i) \subseteq N(u_0, u_0) + P \subseteq C(u_0)$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i N(u_0, v_i) \subseteq C(u_0). \quad (3.8)$$

Also, $M(u, .)$ is P-monotone in the first argument and P-convex in second argument, we have

$$\sum_{i=1}^{n} \alpha_i M(v_i, u_0) \subseteq \sum_{j,i=1}^{n} \alpha_i \alpha_j G(v_i, v_j) - P$$

$$\subseteq \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j (M(v_i, v_j) + M(v_j, v_i))) - P$$

$$\subseteq -P - P \subseteq -P \subseteq -C(u_0). \quad (3.9)$$

From (3.8) and (3.9), we have

$$\sum_{i=1}^{n} \alpha_i[M(v_i, u_0) - N(u_0, v_i)] \subseteq -C(u_0). \quad (3.10)$$

Thus, from (3.7) and (3.10), it follows that

$$\sum_{i=1}^{n} \alpha_i[M(v_i, u_0) - N(u_0, v_i)] \subseteq int_Y C(u_0) \cap \{-C(u_0)\} = \phi,$$

which is not possible. Thus, $u \in coP(u), \forall u \in K$.

Now our aim is to show $P^{-1}(v), \forall v \in K$, is open in K. For this, we need to show that $[P^{-1}(v)]' = K \setminus P^{-1}(v)$ is closed in K, we obtain

$$P^{-1}(v) = \{u \in K : M(v, u) - N(u, v) \subseteq int_Y C(u)\}$$

and

$$[P^{-1}(v)]' = \{u \in K : M(v, u) - N(u, v) \not\subseteq int_Y C(u)\}$$

Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $[P^{-1}(v)]'$ such that $u_i \to u$. Then we have $M(v, u_i) - N(u_i, v) \not\subseteq -int_Y C(u_i)$, for each $v \in K$ i.e. $M(v, u_i) - N(u_i, y) \subseteq W(u_i), \forall i \in \mathbb{N}$. Using hypothesis (iv) and upper semicontinuity of $f(., v) = G(v, .) - N(., v)$, it follows that $f(\xi, v) \subseteq W(\xi)$ i.e $f(\xi, v) \not\subseteq -int_Y C(\xi)$. Hence $\xi \in [P^{-1}(v)]'$. This proves that $P^{-1}(v)$ is open in K. Therefore, all the hypotheses of Theorem 2.7 are fulfilled. Thus $\exists u_0 \in K$ such that $u_0 \in cl_X A(u_0)$ and

$$M(v, u_0) - N(u_0, v) \not\subseteq int_Y C(u_0), \forall v \in A(u_0).$$

Hence, by above Lemma 3.1, $\exists u_0 \in K$ such that $u_0 \in cl_K A(u_0)$ and

$$M(u_0, v) + N(x_0, v) \not\subseteq -int_Y C(u_0), \forall v \in A(u_0).$$

This complete the proof. $\quad \square$

4. Solutions in noncompact Setting.

In this section, we prove existence of solution in noncompact setting.

Theorem 4.1. *Suppose* $K = \bigcup_{p=1}^{\infty} K_p$, *where* $\{K_p\}_{p \in \mathbb{N}}$ *is an expanding sequence of nonempty compact and convex subsets of* K. *Let* $M, N : K \times K \to 2^Y$, $C : K \to 2^Y A : K \to 2^K$ *are multifunctions satisfying the conditions (i)-(iv) of Theorem 3.2 and*

$(v)^0$ $M(u, .), N(., v), \forall u, v \in K$ *are continuous,*

$(vi)^0$ $A(u) \forall u \in K$ *is nonempty convex and* $A^{-1}(v)$ *is open in* K *for each* $v \in K$. *Also* $cl_X A : K \to 2^K$ *is u.s.c,*

$(vii)^0$ *for any sequence* $\{u_p\}_{p \in \mathbb{N}}$ *in* K *with* $u_p \in K_p, \forall p \in \mathbb{N}$, *which is escaping from* K *relative to* $\{K_p\}_{p \in \mathbb{N}}$ $\exists l \in \mathbb{N}$ *and* $v_l \in K_p \cap A(u_l)$ *such that for each* $u_l \in cl_K A(u_l)$,

$$M(u_l, v_l) + N(u_l, v_l) \subseteq -int_Y C(u_l).$$

Then $\exists u_0 \in K$ *such that* $u_0 \in cl_X A(u_0)$ *and*

$$M(u_0, v) + N(u_0, v) \not\subseteq -int_Y C(u_0), \forall v \in A(u_0).$$

Proof. Since for each $p \in \mathbb{N}$, K_p is compact and convex set in X, Theorem 3.2 shows that for all $p \in \mathbb{N}, \exists u_p \in K_p$ such that $u_p \in cl_K A(u_p)$ and

$$M(u_p, v) + N(u_p, v) \not\subseteq -int_Y C(u_p), \forall v \in A(u_p). \quad (4.1)$$

Suppose that the sequence $\{u_p\}_{p \in \mathbb{N}}$ is escaping from K relative to $\{K_p\}_{p \in \mathbb{N}}$. Then by assumption $(vi)^0$, $\exists l \in \mathbb{N}$ and $w_l \in K_l \cap A(u_l)$ such that for each $u_l \in cl_K A(u_l)$, $M(u_l, w_l) + N(u_l, w_l) \subseteq -int_Y C(u_l)$, which is a contradiction to (4.1).

Hence $\{u_p\}_{p \in \mathbb{N}}$ is not an escaping sequence from K relative to $\{K_p\}_{p \in \mathbb{N}}$. Thus using similar argument to those used by Qun [23, $Theorem\ 2$] , $\exists r \in \mathbb{N}$ and $u_0 \in K_r$ such that $u_p \to u_0$ and $M(v, u_0) - N(u_0, v) \subseteq W(u_0)$. Since $cl_K A : K \to 2^K$ is u.s.c with compact values. Hence, by Lemma 3.1, $\exists u_0 \in K$ such that

$$u_0 \in cl_K A(u_0) \text{ and } M(u_0, v) + N(u_0, v) \not\subseteq -int_Y C(u_0), \forall v \in A(u_0).$$

This complete the proof. $\quad \square$

Theorem 4.2. *Suppose* K *be a nonempty convex subset of a locally convex Hausdorff t.v.s* X, *and* $D \subset K$ *be a nonempty compact set. Suppose* Y *be an ordered Hausdorff t.v.s. Consider the multifunctions* $M, N : K \times K \to 2^Y$, $C : K \to 2^Y$ *and* $A : K \to 2^D$. *Assume that the conditions* $(i) - (iv)$ *of above theorem are satisfied and*

$(v)^0$ $M(u, .), N(., v), \forall u, v \in K$ *are continuous,*

$(vi)^0$ $A(u)$ *is nonempty convex for each* $u \in K$ *and* $A^{-1}(v)$ *is open in* K *for each* $v \in K$. *Also,* $cl_K A : K \to 2^D$ *is u.s.c.*

Then $\exists u_0 \in K$ *such that*

$$u_0 \in cl_K A(u_0) \ and \ \ M(u_0, v) + N(u_0, v) \nsubseteq -int_Y C(u_0),$$

for all $u \in A(u_0)$.

Proof. Consider the multifunction $P : K \to 2^D$ defined by

$$P(u) = \{v \in D : M(v, u) - N(u, v) \subseteq int_Y C(u)\}, \forall u \in K.$$

Then by applying the similar argument which we have applied in proving Theorem 3.2, it follows that $u \notin coP(u)$, for each $u \in K$ and $P^{-1}(v)$ is open in D, Therefore, all the hypothesis of Theorem 2.8 are satisfied. Thus $\exists u_0 \in K$ such that

$$u_0 \in cl_K A(u_0) \ \text{and} \ A(u_0) \cap P(u_0) = \phi.$$

Hence $\exists u_0 \in K$ such that

$$u_0 \in cl_K A(u_0) \ \ \text{and} \ \ M(v, u_0) - N(u_0, v) \nsubseteq int_Y C(u_0),$$

for all $v \in A(u_0)$.
Therefore by Lemma 3.1, $\exists u_0 \in K$ such that

$$u_0 \in cl_K A(u_0) \ \text{and} \ G(u_0, v) + N(u_0, v) \nsubseteq -int_Y C(u_0),$$

for all $v \in A(u_0)$.
The complete the proof. □

5. Conclusion

In this study, we have considered generalized vector quasi-equilibrium problems (GVQEP) in Hausdorff topological vector spaces. We have extended Minty's type lemma to establish existence of solutions of GVQEP both under compact and noncompact setting by using 1- person game theorems.

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