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Statistical extension some types of symmetrically continuity

PELDA EVIRGEN[1] AND MEHMET KÜÇÜKASLAN[*2]

[1] *Institute of Science, Department of Mathematics, Mersin University, Mersin, Turkey.*
[2] *Faculty of Science, Department of Mathematics, Mersin University, Mersin, Turkey.*

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Abstract. In this paper, the notions of symmetric continuity, weak continuity, and weak symmetric continuity were introduced in [P. Pongsriiam and T.Thongsiri, Weakly symmetrically continuous function, Chamchuri Journal of Mathematics, vol 8(2016),49-65] are generalized by using natural density defined on $\mathbb{N}$. Among the others, some basic properties of a generalized form of symmetrically continuity is investigated with several useful examples.

AMS Subject Classifications: 26A15, 40A35, 54C08

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Contents

1. Introduction

The conception of continuity is one of the essential notions of mathematical analysis. Let X be a nonempty subset of $\mathbb{R}$ and $\phi : X \to \mathbb{R}$ be a function. Continuity of the function ϕ at a point $\xi_0 \in X$ can be checked in two ways:

(I) For all $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\ \phi(\xi) - \phi(\xi_0)\ | < \epsilon$$

holds for all ξ which is satisfying $|\ \xi - \xi_0\ | < \delta$.

(II) If $\phi(\xi_n)$ tends to $\phi(\xi_0)$ when $n \to \infty$ holds for all sequence (ξ_n) tends to ξ_0 when $(n \to \infty)$.

The statement given in (I) is known as the Cauchy definition of continuity and (II) as the Heine definition of continuity. It is well known that definitions (I) and (II) are equivalent on the space, which has a countable basis.

It is more important to classify the discontinuity at that point rather than investigate the continuity of the function. There are three discontinuity types at a point: removable discontinuity, jump discontinuity, and infinite discontinuity. In 1958, Pervin and Levine [20] showed that a function with a removable discontinuity is

*Corresponding author. Email addresses: **peldaevirgennn@gmail.com** (Pelda Evirgen), **mkucukaslan@mersin.edu.tr** (Mehmet Küçükaslan)

continuous under certain conditions. In addition, in 1960, Halfer [12] proved, with minor modification, given by Pervin and Levine [20] that the continuity and the removable discontinuity are equivalent under certain conditions.

In recent years, a characterization of symmetrical continuous functions at points of removable discontinuity has been intensively studied. The symmetric continuity of functions emerged as an application of trigonometric series theory. Mazurkiewicz [15] first gave symmetric continuity of functions [15] in 1919. Afterward, many studies have been done in this direction [2, 4, 11, 13, 19, 21, 24, 25, 30]. Afterwards, many studies have been done on this direction [2, 4, 11, 13, 19, 21, 24, 25, 30].

Let X be a nonempty subset of $\mathbb{R}$. A function $\phi : X \to \mathbb{R}$ is called at a point $\xi_0 \in X$

(I) symmetrically continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(\xi_0 + \lambda) - \phi(\xi_0 - \lambda)| < \epsilon$$

holds, for every $|\lambda| < \delta$. This can be also checked as $\lim_{\lambda\to 0} \phi(\xi_0 + \lambda) - \phi(\xi_0 - \lambda) = 0$.

(II) weakly continuous if there are sequence $\xi_n \nearrow \xi_0$ and sequence $\eta_n \searrow \xi_0$ so that

$$\lim_{n\to\infty} \phi(\xi_n) = \lim_{n\to\infty} \phi(\eta_n) = \phi(\xi_0)$$

(III) weakly symmetrically continuous if there is a sequence $(\lambda_n) \subset \mathbb{R}^+$ with $(\lambda_n) \to 0$, $n \to \infty$ such that

$$\lim_{n\to\infty} (\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)) = 0.$$

In addition to symmetric continuity of functions, there are many studies on weak continuity [18, 22] and weak symmetric continuity of functions [23, 29]. To ensure coordination between published studies, we will stick to the notations used in the study [23]; SC for the set of symmetrically continuous functions, WC for the set of weakly continuous functions and WSC for the set of weakly symmetrically continuous functions.

With the help of the definition of natural density given below, these spaces will be expanded and larger spaces will be obtained. The smallness of a subset of natural numbers depends on its natural density. Natural density of a subset A of natural numbers is determined by (if limit exists)

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n} |\{k \in A : k \leq n\}|$$

where $|\{k \in A : k \leq n\}|$ denotes the number of elements of A.

Considering the definition of natural density, it can be say that a number sequence (ξ_k) is statistical convergent $\xi \in \mathbb{R}$ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n\colon |\ \xi_k - \xi\ | \geq \epsilon\}| = 0.$$

It is denoted by the symbol $st - \lim \xi_k = \xi$.

Statistical convergence was first defined by Fast [8] and Steinhaus [28] in 1951. Later, in 1959, Schoenberg [27] statistical convergence was reintroduced. In [9], Fridy gave specific results on statistical convergence. Last ten decades, in literature there are several studies in different directions on statistical convergence [1, 3, 5, 7, 10, 14, 16, 17, 26].

The aim of this paper by using natural density to give the statistical version of continuous function, weakly continuous function, weakly symmetrically continuous function, and strong weakly symmetrically continuous function. Then, investigate the relationship between these new type continuities regarding inclusion with some counterexamples.

Throughout this paper, we will consider X as a nonempty subset of $\mathbb{R}$.

Definition 1.1. *[6] The function $\phi : X \to \mathbb{R}$ is called to be statistical continuous at a point ξ_0 if for all sequence (ξ_n) in $\mathbb{R}$ such that $\lim_{n\to\infty} \xi_n = \xi_0$ implies that $\forall \epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0$$

holds.

Let

$$L_{\xi_0}(X) := \{(\xi_n) \subset X : (\xi_n) \text{ strictly increasing and } \lim_{n\to\infty} \xi_n = \xi_0\}$$

$$U_{\xi_0}(X) := \{(\eta_n) \subset X : (\eta_n) \text{ strictly decreasing and } \lim_{n\to\infty} \eta_n = \xi_0\}.$$

Definition 1.2. *The function $\phi : X \to \mathbb{R}$ is called to be statistical weakly continuous at a point ξ_0 if the undermentioned statements hold:*

1. *if $L_{\xi_0}(X) \neq \emptyset$, then there exists $(\xi_n) \in L_{\xi_0}(X)$ such that $\forall \epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0,$$

holds,

2. *if $U_{\xi_0}(X) \neq \emptyset$, then there exists $(\eta_n) \in U_{\xi_0}(X)$ such that $\forall \epsilon > 0$,*

$$\delta(\{n: |\phi(\eta_n) - \phi(\xi_0)| \geq \epsilon\}) = 0.$$

holds.

Let

$$S_{\xi_0}(X) := \{(\lambda_n) \subset \mathbb{R}^+ : \lim_{n\to\infty} \lambda_n = 0 \text{ and } \xi_0 + \lambda_n, \xi_0 - \lambda_n \in X\}.$$

Definition 1.3. *The function $\phi : X \to \mathbb{R}$ is said to be statistical weakly symmetrically continuous at ξ_0 if $S_{\xi_0}(X) \neq \emptyset$, then there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall \epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

holds.

Definition 1.4. *The function $\phi : X \to \mathbb{R}$ is said to be statistical strong weakly symmetrically continuous at the point ξ_0 if for all real valued sequence (λ_n) with $\xi_0 + \lambda_n, \xi_0 - \lambda_n \in X$ and $\lim_{n\to\infty} \lambda_n = 0$ such that $\forall \epsilon > 0$,*

$$\delta(\{n: |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0$$

holds.

Symbolically $\mathscr{C}^{st}$, $\mathscr{W}\mathscr{C}^{st}$, $\mathscr{W}\mathscr{S}\mathscr{C}^{st}$ and $\mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$ will be used for the set of statistical continuous functions, statistical weakly continuous functions, statistical weakly symmetrically continuous functions and statistical strong weakly symmetrically continuous functions, respectively.

Lemma 1.5. *Let $\phi : X \to \mathbb{R}$ be a function and $\xi_0 \in X$. The undermentioned statements are true:*

(i) $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ if and only if there exists such a set

$$T = \{t_1 < t_2 < ... < t_n < ...\}$$

that $\delta(T) = 1$ and $\lim_{n\to\infty}(\phi(\xi_0 + \lambda_{t_n}) - \phi(\xi_0 - \lambda_{t_n})) = 0$.

(ii) $\phi \in \mathscr{SWSC}^{st}$ *if and only if there exists such a set*

$$T = \{t_1 < t_2 < ... < t_n < ...\}$$

that $\delta(T) = 1$ *and* $\lim_{n\to\infty}(\phi(\xi_0 + \lambda_{t_n}) - \phi(\xi_0 - \lambda_{t_n})) = 0$.

(iii) $\phi \in \mathscr{WC}^{st}$ *if and only if there exists such a set*

$$T = \{t_1 < t_2 < ... < t_n < ...\}$$

that $\delta(T) = 1$ *and* $\lim_{n\to\infty}\phi(\xi_{t_n}) = \lim_{n\to\infty}\phi(\eta_{t_n}) = \phi(\xi_0)$.

Proof. We are going to bestow upon only the proof of (i). Statements (ii) and (iii) can be proved by following the same steps given in (i).

Assume that $S_{\xi_0}(X) \neq \emptyset$ and $\exists(\lambda_t) \in S_{\xi_0}(X)$ such that $\forall\epsilon > 0$,

$$\delta(\{t \in \mathbb{N}: |\phi(\xi_0 + \lambda_t) - \phi(\xi_0 - \lambda_t)| \geq \epsilon\}) = 0$$

holds. Put a set for $j = 1, 2, ...,$

$$T_j := \{n \in \mathbb{N}: |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| < \frac{1}{j}\}.$$

It is clear that

$$T_1 \supset T_2 \supset ... \supset T_j \supset T_{j+1} \supset ..., \tag{1.1}$$

satisfies and for all $j \in \mathbb{N}$

$$\delta(T_j) = 1. \tag{1.2}$$

Let an arbitrary element $s_1 \in T_1$. Considering (1.2) there exists $s_2 \in T_2$ satisfying $s_2 > s_1$ and for all $n \geq s_2$ we have $\frac{T_2(n)}{n} > \frac{1}{2}$. Further, according to (1.2) there exists $s_3 \in T_3$ with $s_3 > s_2$, such that for all $n \geq s_3$, we have

$$\frac{T_3(n)}{n} > \frac{2}{3}.$$

Thus, we obtain a sequence of positive integers

$$s_1 < s_2 < ... < s_j < s_{j+1} < ...$$

that $s_j \in T_j$ $(j = 1, 2, ...)$ and for all $n \geq s_j$

$$\frac{T_j(n)}{n} > \frac{j-1}{j} \tag{1.3}$$

holds.

Let us consider the set T as follows: Each natural number of the interval $(1, s_1)$ belongs to T further, any natural number of the interval (s_j, s_{j+1}) belongs to T if and only if it belongs to T_j $(j = 1, 2, ...)$. According to the equations (1.1) and (1.3) for each n, $s_j \leq n < s_{j+1}$ we get

$$\frac{T(n)}{n} \geq \frac{T_j(n)}{n} > \frac{j-1}{j}$$

From this calculation it is apparent that $\delta(T) = 1$. Let $\epsilon > 0$. There exists a natural number j such that $\frac{1}{j} < \epsilon$. Let $n \geq s_j$, $n \in T$. Then, there exists such a number $l \geq j$ that $s_l \leq n < s_{l+1}$. From the definition of T, we have $n \in T_l$.

Hence,

$$|\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|<\frac{1}{l}\leq\frac{1}{j}<\epsilon$$

Therefore,

$$|\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|<\epsilon$$

for each $n\in T$ with $n\geq s_j$, i.e.,

$$\lim_{t\to\infty}(\phi(\xi_0+\lambda_t)-\phi(\xi_0-\lambda_t))=0.$$

For to prove converse implication, assume that there exists a set $T=\{t_1<t_2<...<t_n<...\}\subset\mathbb{N}$ with $\delta(T)=1$ such that

$$\lim_{n\to\infty}(\phi(\xi_0+\lambda_{t_n})-\phi(\xi_0-\lambda_{t_n}))=0$$

is satisfied. So, for any $\epsilon>0$, it can choose a number $n_0\in\mathbb{N}$ that for each $n>n_0$ we have

$$|\phi(\xi_0+\lambda_{t_n})-\phi(\xi_0-\lambda_{t_n})|<\epsilon. \tag{1.4}$$

Put $A_\epsilon=\{n\in\mathbb{N}\colon|\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|\geq\epsilon\}$. Then, from (1.4) we get

$$A_\epsilon\subset\mathbb{N}-\{t_{n_0+1},t_{n_0+2},...\}.$$

Therefore $\delta(A_\epsilon)=0$ and this completed the proof. ■

Theorem 1.6. *Let $\phi:X\to\mathbb{R}$ be a function. If $\phi\in\mathscr{C}^{st}$ then $\phi\in\mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Let ϕ be statistical continuous at ξ_0. Then, for every sequence (ξ_n) in $\mathbb{R}$ for which $\xi_n\to\xi_0$ $(n\to\infty)$ implies that $\forall\epsilon>0$,

$$\delta(\{n\in\mathbb{N}\colon|\phi(\xi_n)-\phi(\xi_0)|\geq\epsilon\})=0. \tag{1.5}$$

Since (1.5) is provided for every sequence (ξ_n) in $\mathbb{R}$ which is convergent to ξ_0 then, we can choose $(\xi_n)=(\xi_0+\lambda_n)$ such that $(\lambda_n)\in\mathbb{R}^+$ and $\lambda_n\to0$. Therefore,

$$\delta(\{n\in\mathbb{N}\colon|\phi(\xi_0+\lambda_n)-\phi(\xi_0)|\geq\frac{\epsilon}{2}\})=0. \tag{1.6}$$

Similarly, we can choose $(\xi_n)=(\xi_0-\lambda_n)$ such that $(\lambda_n)\in\mathbb{R}^+$ where $\lambda_n\to0$ and equation (1.5) implies that

$$\delta(\{n\in\mathbb{N}\colon|\phi(\xi_0-\lambda_n)-\phi(\xi_0)|\geq\frac{\epsilon}{2}\})=0. \tag{1.7}$$

So, $S_{\xi_0}(X)\neq\emptyset$ and from (1.6) and (1.7) we have

$$\begin{aligned}&\{n\in\mathbb{N}\colon|\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|\geq\epsilon\}\subseteq\\&\subseteq\{n\in\mathbb{N}\colon|\phi(\xi_0+\lambda_n)-\phi(\xi_0)|\geq\frac{\epsilon}{2}\}\cup\{n\in\mathbb{N}\colon|\phi(\xi_0-\lambda_n)-\phi(\xi_0)|\geq\frac{\epsilon}{2}\}\end{aligned}$$

and related inequality

$$\begin{aligned}&\delta(\{n\in\mathbb{N}\colon|\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|\geq\epsilon\})\leq\\&\leq\delta(\{n\in\mathbb{N}\colon|\phi(\xi_0+\lambda_n)-\phi(\xi_0)|\geq\frac{\epsilon}{2}\})+\delta(\{n\in\mathbb{N}\colon|\phi(\xi_0-\lambda_n)-\phi(\xi_0)|\geq\frac{\epsilon}{2}\})\end{aligned}$$

holds. This implies that

$$\delta(\{n\in\mathbb{N}\colon|\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|\geq\epsilon\})=0.$$

Hence, ϕ is statistical weakly symmetrically continuous at ξ_0. ■

Theorem 1.7. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{C}^{st}$, then $\phi \in \mathscr{W}\mathscr{C}^{st}$.*

Proof. If $\phi \in \mathscr{C}^{st}$ then, for every real valued sequence (ξ_n) in X for which $\xi_n \to \xi_0$ $(n \to \infty)$ implies that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0.$$

If $L_{\xi_0}(X)$ and $U_{\xi_0}(X)$ are not empty, then there are $(\xi_n) \in L_{\xi_0}(X)$ and $(\eta_n) \in U_{\xi_0}(X)$ such that $\xi_n \to \xi_0$ and $\eta_n \to \xi_0$ holds. Since ϕ statistical continuous, then

$$\delta(\{n\colon |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0$$

and

$$\delta(\{n\colon |\phi(\eta_n) - \phi(\xi_0)| \geq \epsilon\}) = 0$$

are satisfied. This prove our assertion. ■

Theorem 1.8. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{C}^{st}$, then $\phi \in \mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Let ϕ be a statistical continuous function at ξ_0. Then, for every sequence (ξ_n) in $\mathbb{R}$ for which $\xi_n \to \xi_0$ $(n \to \infty)$ implies that $\forall \epsilon > 0$,

$$\delta(\{n\colon |\phi(\xi_n) - \phi(\xi_0)| \geq \epsilon\}) = 0. \tag{1.8}$$

If we choose $(\xi_n) = (\xi_0 + \lambda_n)$ for $\lambda_n \to 0$ when $n \to \infty$, then, (1.8) implies that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) = 0. \tag{1.9}$$

Similarly, if we choose $(\xi_n) = (\xi_0 - \lambda_n)$ for $\lambda_n \to 0$ when $n \to \infty$, $\forall \epsilon > 0$, from (1.8) we have

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) = 0. \tag{1.10}$$

Therefore, $\forall \epsilon > 0$ we have

$$\begin{aligned} &\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq \\ &\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\} \cup \{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\} \end{aligned}$$

and from (1.9), (1.10) following inequality

$$\begin{aligned} &\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) \leq \\ &\leq \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) + \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 - \lambda_n) - \phi(\xi_0)| \geq \frac{\epsilon}{2}\}) \end{aligned}$$

holds. Hence,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Hence, ϕ is statistical strong weakly symmetrically continuous at ξ_0. ■

Theorem 1.9. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$, then $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Suggesting that ϕ is statistical strong weakly symmetrically continuous at ξ_0. Then, for sequence $\forall(\lambda_n) \in \mathbb{R}$ with $\xi_0 + \lambda_n, \xi_0 - \lambda_n \in X$ satisfying $\lim_{n\to\infty} \lambda_n = 0$ such that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

We can choose a subsequence (λ_{n_k}) of (λ_n) such that $(\lambda_{n_k}) \in \mathbb{R}^+$ with $\xi_0 + \lambda_{n_k}, \xi_0 - \lambda_{n_k} \in X$ satisfying $\lambda_{n_k} \to 0$ $(n_k \to \infty)$.
Therefore, $\forall \epsilon > 0$

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_{n_k}) - \phi(\xi_0 - \lambda_{n_k})| \geq \epsilon\}) = 0.$$

Thus, ϕ is statistical weakly symmetrically continuous at ξ_0. ■

Following examples are related to Theorem 1.6, Theorem 1.7, Theorem 1.8 and Theorem 1.9.

Example 1.10. *Let* $K = \{\frac{1}{n^3} : n \in \mathbb{Z} - \{0\}\} \cup \{0\}$ *a set and define a* $\phi : \mathbb{R} \to \{-1, 0, 1\}$ *by*

$$\phi(\xi) := \begin{cases} 0, & \xi \in K, \\ 2, & \xi > 0 \wedge \xi \notin K, \\ -2, & \xi < 0 \wedge \xi \notin K. \end{cases}$$

If we consider (λ_n) *as*

$$\lambda_n := \begin{cases} \frac{1}{n^3}, & n \neq k^3, \\ \frac{1}{n^3+1}, & n = k^3, \end{cases}$$

then, it is clear that $(\lambda_n) \in U_0(\mathbb{R})$*,* $(-\lambda_n) \in L_0(\mathbb{R})$ *and*

$$|\phi(\lambda_n) - \phi(0)| = \begin{cases} 0, \ n \neq k^3, \\ 2, \ n = k^3, \end{cases}$$

holds. This implies that for all $\epsilon > 0$*,*

$$\{n \in \mathbb{N} \colon |\phi(\lambda_n) - \phi(0)| \geq \epsilon\} \subseteq \{k^3 : k \in \mathbb{N}\}.$$

Therefore, $\delta(\{n \in \mathbb{N} \colon |\phi(\lambda_n) - \phi(0)| \geq \epsilon\}) = 0$*. Similarly, we have*

$$|\phi(-\lambda_n) - \phi(0)| = \begin{cases} 0, \ n \neq k^3, \\ 2, \ n = k^3, \end{cases}$$

and $\delta(\{n \in \mathbb{N} \colon |\phi(-\lambda_n) - \phi(0)| \geq \epsilon\}) = 0$*. Therefore,* ϕ *is statistical weakly continuous at* 0*. Now, let us consider following sequence*

$$\lambda_t := \begin{cases} \frac{1}{t^3}, \ t \neq k^2, \\ \frac{1}{t^2}, \ t = k^2. \end{cases}$$

It is clear that $(\lambda_t) \in S_0(\mathbb{R})$ *and*

$$|\phi(0 + \lambda_t) - \phi(0 - \lambda_t)| = \begin{cases} 0, \ t \neq k^2, \\ 4, \ t = k^2. \end{cases}$$

So, for any $\epsilon > 0$ *we have*

$$\{t \in \mathbb{N} \colon |\phi(0 + \lambda_t) - \phi(0 - \lambda_t)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

and this inclusion implies that

$$\delta(\{t \in \mathbb{N} \colon |\phi(\lambda_t) - \phi(-\lambda_t)| \geq \epsilon\}) = 0.$$

Therefore, ϕ *is statistical weakly symmetrically continuous at* 0*.*
Now, let define

$$\lambda_m := \begin{cases} \frac{1}{m^3}, \ m \neq 3k - 1, \\ \frac{1}{m^2}, \ m = 3k - 1. \end{cases}$$

such that $\lambda_m \to 0$ $(m \to \infty)$*. Then,*

$$|\phi(0 + \lambda_m) - \phi(0 - \lambda_m)| = \begin{cases} 0, \ m \neq 3k - 1, \\ 4, \ m = 3k - 1. \end{cases}$$

Let $S \subset \mathbb{N}$ be a finite set and for any $\epsilon > 0$, we have

$$\{m \in \mathbb{N}: |\phi(0+\lambda_m) - \phi(0-\lambda_m)| \geq \epsilon\} \supseteq \{3k-1 : k \in \mathbb{N}\} \backslash S$$

and

$$\delta(\{m \in \mathbb{N}: |\phi(\lambda_m) - \phi(-\lambda_m)| \geq \epsilon\}) \geq \frac{1}{3}.$$

Hence, ϕ is not statistical strong weakly symmetrically continuous at 0.
Also, ϕ is not statistical continuous at 0. Because $\lambda_m \to 0$ $(m \to \infty)$ for $\forall m \in \mathbb{N}$, we have

$$|\phi(\lambda_m) - \phi(0)| = \begin{cases} 0, \, m \neq 3k-1, \\ 2, \, m = 3k-1. \end{cases}$$

There exists $S \subset \mathbb{N}$ finite set and for $\forall \epsilon > 0$ such that

$$\{m \in \mathbb{N}: |\phi(\lambda_m) - \phi(0)| \geq \epsilon\} \supseteq \{3k-1 : k \in \mathbb{N}\} \backslash S$$

satisfies. So,

$$\delta(\{m \in \mathbb{N}: |\phi(\lambda_m)| \geq \epsilon\}) \geq \frac{1}{3} \neq 0.$$

Example 1.11. *Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$, $L = \{\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$, $M = \{-\frac{1}{n} : n \in \mathbb{N}\}$, $P = \{-\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$ and $X = K \cup L \cup M \cup P \cup \{0\}$. Define a function $\phi : \mathbb{X} \to \mathbb{R}$ by*

$$\phi(\xi) := \begin{cases} 1, \, \xi \in K \cup P \cup \{0\}, \\ \xi, \quad \xi \in L \cup M. \end{cases}$$

For all sequence $(\lambda_n) \in S_0(X)$, we have

$$|\phi(0+\lambda_n) - \phi(0-\lambda_n)| = \begin{cases} \left|1 + \frac{1}{n}\right|, & (\lambda_n) \in K, \\ \left|\frac{\sqrt{2}}{n+\sqrt{n}} - 1\right|, & (\lambda_n) \in L. \end{cases}$$

So, for any $\epsilon > 0$, there exists finite set $S \subset \mathbb{N}$ such that

$$\{n \in \mathbb{N}: |\phi(0+\lambda_n) - \phi(0-\lambda_n)| \geq \epsilon\} = \begin{cases} \mathbb{N}, & (\lambda_n) \in K, \\ \mathbb{N} - S, & (\lambda_n) \in L, \end{cases}$$

is true. Hence, we have

$$\delta(\{n \in \mathbb{N}: |\phi(0+\lambda_n) - \phi(0-\lambda_n)| \geq \epsilon\}) > 0.$$

Thus, ϕ is not statistical weakly symmetrically continuous at 0. Also, it is known from Theorem 1.9 that the function ϕ is not statistical strong weakly symmetrically continuous at 0. Let $\eta_t \in U_0(X)$ and $\xi_m \in L_0(X)$ as follows

$$\eta_t := \begin{cases} \frac{1}{t}, & t \neq k^2, \\ \frac{\sqrt{2}}{t+\sqrt{t}}, & t = k^2, \end{cases} \quad \text{and} \quad \xi_m := \begin{cases} -\frac{\sqrt{2}}{m+\sqrt{m}}, & m \neq k^2, \\ -\frac{1}{m}, & m = k^2, \end{cases}$$

respectively. Then, we have

$$|\phi(\eta_t) - \phi(0)| = \begin{cases} 0, & t \neq k^2, \\ \left|\frac{\sqrt{2}}{t+\sqrt{t}} - 1\right|, & t = k^2, \end{cases}$$

and

$$\{t \in \mathbb{N}\colon |\phi(\eta_t) - \phi(0)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

is satisfied for all $\epsilon > 0$. *Hence,*

$$\delta(\{t \in \mathbb{N}\colon |\phi(\eta_t) - \phi(0)| \geq \epsilon\}) = 0.$$

Similarly,

$$|\phi(\xi_m) - \phi(0)| = \begin{cases} 0, & m \neq k^2, \\ \left|\frac{1}{m} + 1\right|, & m = k^2, \end{cases}$$

and

$$\{m \in \mathbb{N}\colon |\phi(\xi_m) - \phi(0)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

implies that

$$\delta(\{m \in \mathbb{N}\colon |\phi(\xi_m) - \phi(0)| \geq \epsilon\}) = 0.$$

Therefore, ϕ *is statistical weakly continuous at* 0.

Example 1.12. *Let* $K = \{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\}$, $L = \{\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$, $M = \{-\frac{\sqrt{2}}{n+\sqrt{n}} : n \in \mathbb{N}\}$ *and* $X = K \cup L \cup M \cup \{0\}$. *Define the function* $\phi : X \to \mathbb{R}$ *by*

$$\phi(\xi) := \begin{cases} 1, & \xi \in K, \\ \xi, & \xi \in X - K. \end{cases}$$

Let $(\lambda_n) \in S_0(X)$ *as*

$$\lambda_n := \begin{cases} \frac{1}{n}, & n \neq k^2, \\ \frac{\sqrt{2}}{n+\sqrt{n}}, & n = k^2. \end{cases}$$

So, we have

$$|\phi(0 + \lambda_n) - \phi(0 - \lambda_n)| = \begin{cases} 0, & n \neq k^2, \\ \left|\frac{2\sqrt{2}}{n+\sqrt{n}}\right|, & n = k^2, \end{cases}$$

and for every $\epsilon > 0$,

$$\{n \in \mathbb{N}\colon |\phi(0 + \lambda_n) - \phi(0 - \lambda_n)| \geq \epsilon\} \subseteq \{k^2 : k \in \mathbb{N}\}$$

imply that

$$\delta(\{n \in \mathbb{N}\colon |\phi(\lambda_n) - \phi(-\lambda_n)| \geq \epsilon\}) = 0.$$

Therefore, ϕ *is statistical weakly symmetrically continuous at* 0. *For all* $(\eta_m) \in U_0(X)$,

$$|\phi(\eta_m) - \phi(0)| = \begin{cases} 1, & \eta_m \in K, \\ \frac{\sqrt{2}}{m+\sqrt{m}}, & \eta_m \in L. \end{cases}$$

Hence, for $\forall \epsilon > 0$, *there exists* $S \subset \mathbb{N}$ *finite set such that*

$$\{m \in \mathbb{N}\colon |\phi(\eta_m) - \phi(0)| \geq \epsilon\} = \begin{cases} \mathbb{N}, & \eta_m \in K, \\ \mathbb{N} \backslash S, & \eta_m \in L. \end{cases}$$

Therefore,

$$\delta(\{m \in \mathbb{N}\colon |\phi(\eta_m) - \phi(0)| \geq \epsilon\}) > 0.$$

Thus, ϕ *is not statistical weakly continuous at* 0.

As a summary of the Theorems and Examples given above, we can provide the following inclusions:

(i) $\mathscr{SWSC}^{st} \subseteq \mathscr{WSC}^{st}$ and $\mathscr{WSC}^{st} \nsubseteq \mathscr{SWSC}^{st}$

(ii) $\mathscr{SWSC}^{st} \nsubseteq \mathscr{WC}^{st}$ and $\mathscr{WC}^{st} \nsubseteq \mathscr{SWSC}^{st}$

(iii) $\mathscr{WC}^{st} \nsubseteq \mathscr{WSC}^{st}$ and $\mathscr{WSC}^{st} \nsubseteq \mathscr{WC}^{st}$

(iv) $\mathscr{SWSC}^{st} \nsubseteq \mathscr{C}^{st}$, $\mathscr{WSC}^{st} \nsubseteq \mathscr{C}^{st}$ and $\mathscr{WC}^{st} \nsubseteq \mathscr{C}^{st}$

(v) $\mathscr{C}^{st} \subseteq \mathscr{SWSC}^{st}$, $\mathscr{C}^{st} \subseteq \mathscr{WSC}^{st}$ and $\mathscr{C}^{st} \subseteq \mathscr{WC}^{st}$

2. Some algebraic properties of new continuities

This section examines the algebraic properties of the set of $\mathscr{WSC}^{st}$. The results concluded that the set $\mathscr{WSC}^{st}$ does not form a linear space over real numbers.

Theorem 2.1. *Let $\phi : X \to \mathbb{R}$ be a function. If $\phi \in \mathscr{WSC}^{st}$ and $c \in \mathbb{R}$ then, $|\phi|$, $c\phi \in \mathscr{WSC}^{st}$.*

Proof. Suppose that $S_{\xi_0}(X) \neq \emptyset$. Then, there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that

$$\delta(\{n\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0$$

holds for all $\epsilon > 0$. So, the following inclusion

$$\{n \in \mathbb{N}\colon ||\phi|\,(\xi_0 + \lambda_n) - |\phi|\,(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$

$$\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}$$

implies that

$$\delta(\{n \in \mathbb{N}\colon ||\phi|\,(\xi_0 + \lambda_n) - |\phi|\,(\xi_0 - \lambda_n)| \geq \epsilon\}) \leq$$
$$\leq \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\})$$

is true. Then,

$$\delta(\{n \in \mathbb{N}\colon ||\phi|\,(\xi_0 + \lambda_n) - |\phi|\,(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Therefore, $|\phi|$ is statistical weakly symmetrically continuous at ξ_0.

Additionally, $c \in \mathbb{R}$ and $\forall \epsilon > 0$ the following inclusion

$$\{n \in \mathbb{N}\colon |(c\phi)(\xi_0 + \lambda_n) - (c\phi)(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$

$$\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{|c|}\}$$

and related inequality

$$\delta(\{n \in \mathbb{N}\colon |(c\phi)(\xi_0 + \lambda_n) - (c\phi)(\xi_0 - \lambda_n)| \geq \epsilon\}) \leq$$
$$\leq \delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{|c|}\})$$

hold.

So, we have

$$\delta(\{n \in \mathbb{N}\colon |(c\phi)(\xi_0 + \lambda_n) - (c\phi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Hence, $c\phi$ is statistical weakly symmetrically continuous at ξ_0. ■

Theorem 2.2. *Let $\phi : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$ be functions. If $\phi \in \mathscr{WSC}^{st}$ and $\psi \in \mathscr{SWSC}^{st}$ then, $\phi + \psi$, $\phi - \psi$, $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\} \in \mathscr{WSC}^{st}$.*

Proof. Suppose that ϕ is statistical weakly symmetrically continuous function at the point ξ_0 and ψ is statistical strong weakly symmetrically continuous function at the point ξ_0. Then, $S_{\xi_0}(X) \neq \emptyset$ implies that there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

From Theorem 1.9, ψ is statistical weakly symmetrically continuous function at the point ξ_0. Then,

$$\delta(\{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

holds. Therefore, following equality

$$\{n \in \mathbb{N}\colon |(\phi + \psi)(\xi_0 + \lambda_n) - (\phi + \psi)(\xi_0 - \lambda_n)| \geq \epsilon\} =$$
$$= \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\} \cup$$
$$\cup \{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}$$

implies that

$$\delta(\{n \in \mathbb{N}\colon |(\phi + \psi)(\xi_0 + \lambda_n) - (\phi + \psi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Similarly, we have

$$\delta(\{n \in \mathbb{N}\colon |(\phi - \psi)(\xi_0 + \lambda_n) - (\phi - \psi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Consequently, $\phi + \psi$ and $\phi - \psi$ are statistical weakly symmetrically continuous at the point ξ_0.

Now, the following inequality

$$|\max\{\phi, \psi\}(\xi_0 + \lambda_n) - \max\{\phi, \psi\}(\xi_0 - \lambda_n)| \leq$$
$$\leq \frac{|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)|}{2} + \frac{|\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)|}{2} +$$
$$+ \frac{||\phi - \psi|(\xi_0 + \lambda_n) - |\phi - \psi|(\xi_0 - \lambda_n)|}{2} \leq$$
$$\leq \frac{|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)|}{2} + \frac{|\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)|}{2} +$$
$$+ \frac{|(\phi - \psi)(\xi_0 + \lambda_n) - (\phi - \psi)(\xi_0 - \lambda_n)|}{2} \leq$$
$$\leq |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| + |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)|$$

implies that

$$\{n \in \mathbb{N}\colon |\max\{\phi, \psi\}(\xi_0 + \lambda_n) - \max\{\phi, \psi\}(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$
$$\subseteq \{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}$$
$$\cup \{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}$$

holds. So, we have

$$\delta(\{n \in \mathbb{N}\colon |\max\{\phi, \psi\}(\xi_0 + \lambda_n) - \max\{\phi, \psi\}(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Similarly, the following inequality

$$|\min\{\phi, \psi\}(\xi_0 + \lambda_n) - \min\{\phi, \psi\}(\xi_0 - \lambda_n)| \leq$$

$$\leq |\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|+|\psi(\xi_0+\lambda_n)-\psi(\xi_0-\lambda_n)|$$

implies that

$$\{n\in\mathbb{N}\colon |\min\{\phi,\psi\}(\xi_0+\lambda_n)-\min\{\phi,\psi\}(\xi_0-\lambda_n)|\geq\epsilon\}\subseteq$$
$$\subseteq\{n\in\mathbb{N}\colon |\phi(\xi_0+\lambda_n)-\phi(\xi_0-\lambda_n)|\geq\frac{\epsilon}{2}\}$$
$$\cup\{n\in\mathbb{N}\colon |\psi(\xi_0+\lambda_n)-\psi(\xi_0-\lambda_n)|\geq\frac{\epsilon}{2}\}$$

holds. Hence,

$$\delta(\{n\in\mathbb{N}\colon |\min\{\phi,\psi\}(\xi_0+\lambda_n)-\min\{\phi,\psi\}(\xi_0-\lambda_n)|\geq\epsilon\})=0.$$

Thus, the functions $\max\{\phi,\psi\}$ and $\min\{\phi,\psi\}$ are statistical weakly symmetrically continuous at ξ_0. ■

Example 2.3. *(Exp.3.3. in [23]) Let $A=\{\frac{1}{n}:n\in\mathbb{Z}-\{0\}\}$ and $B=\{\frac{\sqrt{2}}{n}:n\in\mathbb{Z}-\{0\}\}$. Consider the functions $\phi,\psi:\mathbb{R}\to\mathbb{R}$ as follows:*

$$\phi(\xi)=\begin{cases}\xi, & \xi\in A\cup\{0\},\\ -1, & \xi>0\wedge\xi\notin A,\\ 1, & \xi<0\wedge\xi\notin A,\end{cases}\quad \text{and}\quad \psi(\xi)=\begin{cases}\xi, & \xi\in B\cup\{0\},\\ -2, & \xi>0\wedge\xi\notin B,\\ 2, & \xi<0\wedge\xi\notin B.\end{cases}$$

The functions ϕ and ψ are weakly symmetrically continuous at 0 (see in [23]). By Lemma 1.5 the functions ϕ and ψ are also statistical weakly symmetrically continuous at 0.

$$(\phi+\psi)(\xi)=\begin{cases}-3, & \xi>0\wedge\xi\notin A\cup B,\\ 3, & \xi<0\wedge\xi\notin A\cup B,\\ \xi-2, & \xi>0\wedge\xi\in A,\\ \xi+2, & \xi<0\wedge\xi\in A,\\ \xi-1, & \xi>0\wedge\xi\in B,\\ \xi+1, & \xi<0\wedge\xi\in B,\\ 0, & \xi=0,\end{cases}$$

$$(\phi-\psi)(\xi)=\begin{cases}1, & \xi>0\wedge\xi\notin A\cup B,\\ -1, & \xi<0\wedge\xi\notin A\cup B,\\ \xi+2, & \xi>0\wedge\xi\in A,\\ \xi-2, & \xi<0\wedge\xi\in A,\\ -\xi-1, & \xi>0\wedge\xi\in B,\\ -\xi+1, & \xi<0\wedge\xi\in B,\\ 0, & \xi=0,\end{cases}$$

$$\max\{\phi,\psi\}(\xi)=\begin{cases}-1, & \xi>0\wedge\xi\notin A\cup B,\\ 2, & \xi<0\wedge\xi\notin A\cup B,\\ \xi, & \xi>0\wedge\xi\in A\cup B,\\ 2, & \xi<0\wedge\xi\in A,\\ 1, & \xi<0\wedge\xi\in B,\\ 0, & \xi=0,\end{cases}$$

$$\min\{\phi,\psi\}(\xi)=\begin{cases}-2, & \xi>0\wedge\xi\notin A\cup B,\\ 1, & \xi<0\wedge\xi\notin A\cup B,\\ -2, & \xi>0\wedge\xi\in,\\ \xi, & \xi<0\wedge\xi\in A\cup B,\\ -1, & \xi>0\wedge\xi\in B,\\ 0, & \xi=0.\end{cases}$$

For $\forall(\lambda_n)\in S_0(\mathbb{R})$ *and* $\forall\epsilon>0$*,*

$$|(\phi+\psi)(0+\lambda_n)-(\phi+\psi)(0-\lambda_n)|=\begin{cases}6, & \lambda_n\notin A\cup B,\\ |2\lambda_n-4|, & \lambda_n\in A,\\ |2\lambda_n-2|, & \lambda_n\in B.\end{cases}$$

There exists a finite subset of natural numbers S *such that*

$$\{n\in\mathbb{N}\text{: }|(\phi+\psi)(\lambda_n)-(\phi+\psi)(-\lambda_n)|\geq\epsilon\}=\begin{cases}\mathbb{N}, & \lambda_n\notin A\cup B,\\ \mathbb{N}\backslash S, & \lambda_n\in A\cup B,\end{cases}$$

Hence,

$$\delta(\{n\in\mathbb{N}\text{: }|(\phi+\psi)(\lambda_n)-(\phi+\psi)(-\lambda_n)|\geq\epsilon\})>0.$$

Therefore $(\phi+\psi)$ *is not statistical weakly symmetrically continuous at* 0*. Similarly, for* $\forall n\in\mathbb{N}$*,*

$$|(\phi-\psi)(0+\lambda_n)-(\phi-\psi)(0-\lambda_n)|=\begin{cases}2, & \lambda_n\notin A\cup B,\\ |2\lambda_n+4|, & \lambda_n\in A,\\ |-2\lambda_n-2|, & \lambda_n\in B,\end{cases}$$

$$|\max\{\phi,\psi\}(0+\lambda_n)-\max\{\phi,\psi\}(0-\lambda_n)|=\begin{cases}3, & \lambda_n\notin A\cup B,\\ |\lambda_n-2|, & \lambda_n\in A,\\ |\lambda_n-1|, & \lambda_n\in B,\end{cases}$$

$$|\min\{\phi,\psi\}(0+\lambda_n)-\min\{\phi,\psi\}(0-\lambda_n)|=\begin{cases}3, & \lambda_n\notin A\cup B,\\ |-\lambda_n-2|, & \lambda_n\in A,\\ |-\lambda_n-1|, & \lambda_n\in B,\end{cases}$$

For $\forall\epsilon>0$*,*

$$\delta(\{n\in\mathbb{N}\text{: }|(\phi-\psi)(\lambda_n)-(\phi-\psi)(-\lambda_n)|\geq\epsilon\})>0,$$

$$\delta(\{n\in\mathbb{N}\text{: }|\max\{\phi,\psi\}(\lambda_n)-\max\{\phi,\psi\}(-\lambda_n)|\geq\epsilon\})>0,$$

$$\delta(\{n\in\mathbb{N}\text{: }|\min\{\phi,\psi\}(\lambda_n)-\min\{\phi,\psi\}(-\lambda_n)|\geq\epsilon\})>0.$$

Hence, the functions $\phi-\psi$*,* $\max\{\phi,\psi\}$ *and* $\min\{\phi,\psi\}$ *are not statistical weakly symmetrically continuous at* 0*.*

Theorem 2.4. *Let* $\phi: X\to\mathbb{R}$ *be a statistical weakly symmetrically continuous function at the point* ξ_0 *and let* $\psi: X\to\mathbb{R}$ *be a statistical strong weakly symmetrically continuous function at the point* ξ_0*. If* ϕ *and* ψ *are locally bounded at* ξ_0*, then* $\phi\psi$ *is statistical weakly symmetrically continuous at* ξ_0*.*

Proof. Suppose that ϕ is statistical weakly symmetrically continuous and ψ is statistical strong weakly symmetrically continuous at the point ξ_0. Then, $S_{\xi_0}(X) \neq \emptyset$ implies that there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall\epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}) = 0$$

holds. Also from Theorem 1.9, ψ is statistical weakly symmetrically continuous function at the point ξ_0. Then,

$$\delta(\{n \in \mathbb{N}\colon |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}) = 0.$$

holds for $\forall\epsilon > 0$.

Because of ϕ and ψ are locally bounded at ξ_0, there exists $K, M > 0$ and $\delta > 0$ such that $|\phi(\xi)| \leq M$ and $|\psi(\xi)| \leq K$ for all $\xi \in (\xi_0 - \delta, \xi_0 + \delta) \cap X$.

Since $(\lambda_n) \in S_{\xi_0}(X)$, we can pick $N \in \mathbb{N}$ such that $\xi_0 + \lambda_n, \xi_0 - \lambda_n \in (\xi_0 - \delta, \xi_0 + \delta) \cap X$ for $\forall n \geq N$ such that

$$|(\phi\psi)(\xi_0 + \lambda_n) - (\phi\psi)(\xi_0 - \lambda_n)| =$$

$$= |\phi(\xi_0 + \lambda_n)\psi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)\psi(\xi_0 - \lambda_n)| \leq$$

$$\leq |\phi(\xi_0 + \lambda_n)|\, |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| +$$

$$+ |\psi(\xi_0 - \lambda_n)|\, |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \leq$$

$$\leq M\, |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| + K\, |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)|$$

holds. So, following inclusion

$$\{n \in \mathbb{N}\colon |(\phi\psi)(\xi_0 + \lambda_n) - (\phi\psi)(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq$$

$$\subseteq \{n \in \mathbb{N}\colon M.\, |\psi(\xi_0 + \lambda_n) - \psi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\} \cup$$

$$\cup \{n \in \mathbb{N}\colon K.\, |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \frac{\epsilon}{2}\}$$

implies that

$$\delta(\{n \in \mathbb{N}\colon |(\phi\psi)(\xi_0 + \lambda_n) - (\phi\psi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Therefore, $\phi\psi$ is statistical weakly symmetrically continuous at ξ_0. ■

The following example shows that if $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ and $\psi \in \mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}$ but at least one of ϕ or ψ is not locally bounded, then $\phi\psi \notin \mathscr{W}\mathscr{S}\mathscr{C}^{st}$.

Example 2.5. *Consider the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\phi(\xi) = \xi \quad \textit{and} \quad \psi(\xi) = \begin{cases} \frac{1}{\ln(|\xi|+1)}, & \xi \notin [-\frac{1}{e}, \frac{1}{e}] \\ 0, & otherwise, \end{cases}$$

For every $(\lambda_n) \in \mathbb{R}$ with $\lambda_n \to 0$, we have for every $\epsilon > 0$

$$\delta(\{n \in \mathbb{N}\colon |\phi(\lambda_n) - \phi(-\lambda_n)| \geq \epsilon\}) = 0$$

and

$$\delta(\{n \in \mathbb{N}\colon |\psi(\lambda_n) - \psi(-\lambda_n)| \geq \epsilon\}) = 0.$$

Then, ϕ and ψ are statistical strong weakly symmetrically continuous at 0. The function ϕ is locally bounded at 0 however ψ is not. By Theorem 1.9, the function ϕ is statistical weakly symmetrically continuous at 0. Additionally,

$$(\phi\psi)(\xi) = \begin{cases} \frac{\xi}{\ln(|\xi|+1)}, & \xi \notin [-\frac{1}{e}, \frac{1}{e}] \\ 0, & otherwise. \end{cases}$$

Hence, $\forall \lambda_n \in S_0(\mathbb{R})$ and $\forall \epsilon > 0$ we have

$$|(\phi\psi)(0+\lambda_n) - (\phi\psi)(0-\lambda_n)| = \frac{2\lambda_n}{\ln(\lambda_n+1)}$$

$$\{n \in \mathbb{N}\text{: } |(\phi\psi)(0+\lambda_n) - (\phi\psi)(0-\lambda_n)| \geq \epsilon\} = \mathbb{N}$$

$$\delta(\{n \in \mathbb{N}\text{: } |(\phi\psi)(\lambda_n) - (\phi\psi)(-\lambda_n)| \geq \epsilon\}) > 0.$$

Hence, $\phi\psi$ is not statistical weakly symmetrically continuous at 0.

Theorem 2.6. *Let $\phi : X \to \mathbb{R}$ be a statistical weakly symmetrically continuous function at ξ_0. Suppose that $\phi(\xi) \neq 0$ for $\forall \xi \in X$ and $\frac{1}{\phi}$ is locally bounded at ξ_0. Then, $\frac{1}{\phi}$ is statistical weakly symmetrically continuous at ξ_0.*

Proof. Suppose that ϕ be a statistical weakly symmetrically continuous at a point ξ_0 and let $\phi(\xi) \neq 0$ for $\forall \xi \in X$ and $\frac{1}{\phi}$ is locally bounded at ξ_0. Let $S_{\xi_0}(X) \neq \emptyset$ then, there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall \epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\text{: } |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \geq \epsilon\}) = 0$$

and $\exists \delta, M > 0$ such that $\left|\frac{1}{\phi(\xi)}\right| \leq M$, for $\forall \xi \in (\xi_0-\delta, \xi_0+\delta) \cap X$. Since $(\lambda_n) \in S_{\xi_0}(X)$, then we can pick $N \in \mathbb{N}$ such that $\xi_0+\lambda_n, \xi_0-\lambda_n \in (\xi_0-\delta, \xi_0+\delta) \cap X$ for $\forall n \geq N$.

So, following inequality

$$\begin{aligned} \left|\frac{1}{\phi(\xi_0+\lambda_n)} - \frac{1}{\phi(\xi_0-\lambda_n)}\right| &= |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \left|\frac{1}{\phi(\xi_0+\lambda_n)\phi(\xi_0-\lambda_n)}\right| \\ &\leq M^2 |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \end{aligned}$$

and related inclusion

$$\begin{aligned} \{n \in \mathbb{N}\text{: } \left|\frac{1}{\phi(\xi_0+\lambda_n)} - \frac{1}{\phi(\xi_0-\lambda_n)}\right| \geq \epsilon\} \subseteq \\ \subseteq \{n \in \mathbb{N} : M^2 |\phi(\xi_0+\lambda_n) - \phi(\xi_0-\lambda_n)| \geq \epsilon\} \end{aligned}$$

holds. Then,

$$\delta(\{n \in \mathbb{N}\text{: } \left|\frac{1}{\phi(\xi_0+\lambda_n)} - \frac{1}{\phi(\xi_0-\lambda_n)}\right| \geq \epsilon\}) = 0.$$

Therefore, $\frac{1}{\phi}$ is statistical weakly symmetrically continuous at ξ_0. ■

Theorem 2.7. *Let $\phi : X \to \mathbb{R}$ be a statistical weakly symmetrically continuous function at a point ξ_0 and locally bounded at ξ_0. Let $\psi : X \to \mathbb{R}$ be a statistical strong weakly symmetrically continuous function at a point ξ_0. If $\psi(\xi) \neq 0$ for all $\xi \in X$ and $\frac{1}{\psi}$ is locally bounded at ξ_0 then, $\frac{\phi}{\psi}$ is statistical weakly symmetrically continuous at ξ_0.*

Proof. It is omitted because of similarity with Theorem 2.6. ■

Theorem 2.8. *Let $\phi : X \to Y$ and $\psi : Y \to \mathbb{R}$. Suppose that $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ and ψ be a uniformly continuous on Y. Then, $\psi \circ \phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$.*

Proof. Suppose that ϕ is statistical weakly symmetrically continuous at ξ_0 and ψ is uniformly continuous on Y. Then, $S_{\xi_0}(X) \neq \emptyset$ implies that there exists a sequence $(\lambda_n) \in S_{\xi_0}(X)$ such that $\forall\epsilon > 0$,

$$\delta(\{n \in \mathbb{N}\colon |\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0$$

and $\forall\epsilon > 0$, $\exists\delta \equiv \delta(\epsilon) > 0 \ni |\zeta_0 - \zeta_1| \leq \delta$ implies that for $\forall\zeta_0, \zeta_1 \in Y$

$$|\psi(\zeta_0) - \psi(\zeta_1)| < \epsilon \tag{2.1}$$

There is $N \in \mathbb{N}$ such that for all $n \geq N$

$$|\phi(\xi_0 + \lambda_n) - \phi(\xi_0 - \lambda_n)| < \delta. \tag{2.2}$$

By equation (2.1) and (2.2),

$$|(\psi \circ \phi)(\xi_0 + \lambda_n) - (\psi \circ \phi)(\xi_0 - \lambda_n)| = |\psi(\phi(\xi_0 + \lambda_n)) - \psi(\phi(\xi_0 + \lambda_n))| < \epsilon$$

So, we have below inclusion

$$\{n \in \mathbb{N}\colon |(\psi \circ \phi)(\xi_0 + \lambda_n) - (\psi \circ \phi)(\xi_0 - \lambda_n)| \geq \epsilon\} \subseteq \{1, 2, ...N\}$$

and

$$\delta(\{n \in \mathbb{N}\colon |(\psi \circ \phi)(\xi_0 + \lambda_n) - (\psi \circ \phi)(\xi_0 - \lambda_n)| \geq \epsilon\}) = 0.$$

Consequently, $\psi \circ \phi$ is statistical weakly symmetrically continuous at ξ_0. ■

The following example shows that when $\phi \in \mathscr{W}\mathscr{S}\mathscr{C}^{st}$ but ψ is not uniformly continuous on the domain, it will be $\psi \circ \phi \notin \mathscr{W}\mathscr{S}\mathscr{C}^{st}$

Example 2.9. *Define* $\phi, \psi : \mathbb{R} \to \mathbb{R}$ *by*

$$\psi(\xi) = \begin{cases} \frac{1}{\xi}, & \xi \neq 0, \\ 0, & \xi = 0. \end{cases} \quad \textit{and} \quad \phi(\xi) = \xi \cos \xi$$

The function ϕ is statistical weakly symmetrically continuous at 0 and ψ is not uniformly continuous on $\mathbb{R}$.

$$(\psi \circ \phi)(\xi) = \begin{cases} \frac{1}{\xi \cos \xi}, & \xi \neq 0 \wedge \xi \neq (k\pi + \frac{\pi}{2}), \\ 0, & otherwise, \end{cases}$$

for all $k \in \mathbb{Z}$. For $\forall(\lambda_n) \in S_0(\mathbb{R})$ and $\epsilon > 0$,

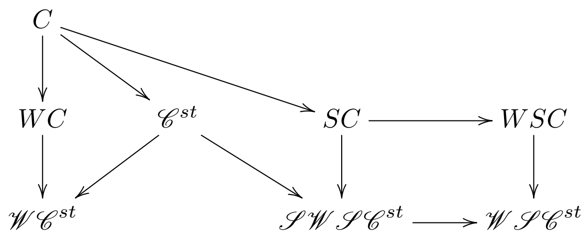
$$|(\psi \circ \phi)(0 + \lambda_n) - (\psi \circ \phi)(0 - \lambda_n)| = \frac{2}{\lambda_n \cos(\lambda_n)}$$

$$\{n \in \mathbb{N}\colon |(\psi \circ \phi)(\lambda_n) - (\psi \circ \phi)(-\lambda_n)| \geq \epsilon\} = \mathbb{N}$$
$$\delta(\{n \in \mathbb{N}\colon |(\psi \circ \phi)(\lambda_n) - (\psi \circ \phi)(-\lambda_n)| \geq \epsilon\}) = 1 > 0.$$

Hence, $\psi \circ \phi$ is not statistical weakly symmetrically continuous at 0.

3. Conclusion and some Remarks

P. Pongsriim-T. Thongsiri in [23] classified functions with removable discontinuity, and SC, WC and WSC classes were created. In this study, functions with removable discontinuities were subjected to a new classification with the help of natural density, and the following inclusions diagram was obtained. (Note that $E \to D$ means that $E \subseteq D$)



As a continuation of this study, the first question that comes to mind is to make a similar extension by taking a different kinds of densities instead of natural density, for example, logarithmic density, uniform density, and density produced by a regular matrix, generalized density, etc.

Maybe the other problem is determining whether there is any class of functions between X and Y where $X \in \{SC, WC, WSC\}$ and $Y \in \{\mathscr{S}\mathscr{W}\mathscr{S}\mathscr{C}^{st}, \mathscr{W}\mathscr{C}^{st}, \mathscr{W}\mathscr{S}\mathscr{C}^{st}\}$.

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