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Williams meet matrices base b on A-sets

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Abstract
In this study Williams meet matrices base $b=3$ on A-sets are considered to study the results based on structure theorem. The determinant and inverse of the matrix are also found. In addition the matrix is expressed in terms of the other two matrices.

Keywords
Meet matrices, Williams meet matrices base b, A-sets.

AMS Subject Classification
15A09, 03G10.

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Contents

1. Introduction

Williams number base b are numbers of the form $(b-1)b^n-1$, for integers $b\geq 2$ and $n\geq 1$. The meet matrices are defined on a set of positive integers $S=\{x_1,x_2,x_3,\ldots\ldots\ldots,x_n\}$. To define these matrices a complex valued function f from $P\to C$ is considered where P is a locally finite lattice $(P,\vee,\wedge)$. Also S is considered as the subset of P. The meet matrix is denoted as $(S)_f$ The elements of these matrices are denoted as $((S)_f)_{ij}=f(x_i\wedge x_j)$. Haukkanen [2] has shown the method to find the determinant and inverse of these meet matrices. The generalizations of GCD and LCM matrices are obtained in [1]. The conjecture of Beslin and Ligh on greatest common divisor matrices as Meet matrices is studied in [3]. In this study we are going to study the nature of Williams Meet matrices base b with the special case of $b=2$. Also find the determinant and inverse of Williams meet matrices base b on A-sets with the special case of $b=3$.

2. Preliminaries

In this section, we recall some definitions which are very useful to this work.

1. If S_1 and S_2 are nonempty subsets of P then $S_1\sqcap S_2=[x\wedge y/x\in S_l,y\in S_2,x\neq y\}$ where $\sqcap$ is a binary operation.

2. $S=\{x_1,x_2,\ldots\ldots\ldots\ldots x_n\}\subset P$ with $x_i<x_j\Rightarrow i<j$ and let $A=\{a_l,a_2,\ldots\ldots\ldots\ldots a_{n-1}\}$ be a multichain with $a_1\leq a_2\leq\ldots\ldots\ldots\leq a_{n-1}$ The set S is said to be an A -set if $\{x_k\}\sqcap\{x_{k+1},\ldots\ldots,x_n\}=\{a_k\}$ for all $k=1,2,\ldots\ldots,n-1$.

3. A matrix (S) of size $n\times n$, where $\left((S)_f\right)_{ij}=f\left(x_i\wedge x_j\right)$, is called the meet matrix on S with respect to f.

4. Williams number base b is a natural number of the form $(b-1)b^n-1$ for integers $b\geq 2$ and $n\geq 1$.

5. Williams meet matrix base b is defined as a matrix $(S)_f$ of size $n\times n$, where $\left((S)_f\right)_{ij}=(\mathrm{b}-1)\mathrm{b}^{\mathrm{x}_i\wedge\mathrm{x}_j}-1$.

3. Theorems on Williams number base $b=2$.

Theorem 3.1. *If b and n are natural numbers such that b^n-1 is prime, then $b=2$ or $n=1$.*

Proof. $\mathrm{b}\equiv 1(\bmod b-1)\Rightarrow b^n\equiv 1(\bmod \mathrm{b}-1)$. $\therefore b^n-1\equiv 0(\bmod \mathrm{n}-1)$ Hence $\mathrm{b}-1\mid b^n-1$. But b^n-1 is prime, hence $b-1=b^n-1$ or $b-1=\pm 1$. From the first value $b=b^n\Rightarrow$

$b=0,1$ (which is a contradiction, as neither 1 nor 0 is prime) or $n=1$. From the second value, $b=2$ or $b=0$. If $b=0$ we get $0^n-1=0-1=-1$ which is not prime. $\therefore b=2$. □

Theorem 3.2. *If $(b-1)b^n-1$ when $b=2$ is prime, then n is prime.*

Proof. If n is composite, then $n=pq$ with p and $q>1$,

$$\begin{aligned} \mathrm{b} &=2 \Rightarrow 2^n-1=2^{pq}-1=(2^p)^q-1 \\ &=(2^p-1)\left((2^p)^{q-1}+(2^p)^{q-2}+\ldots+2^p+1\right). \end{aligned}$$

$\therefore 2^n-1$ is composite. By contrapositive, if 2^n-1 is prime then n is prime. Hence the proof. □

Theorem 3.3. *Every prime p that divides $(b-1)b^n-1$ when $b=2$ must be 1 plus a multiple of* 2n *if* n *is an odd prime. The result is true even when 2^n-1 is prime.*

Proof. Using Fermat's little theorem, p is a factor of $2^{p-1}-1$. Now p is a factor of 2^n-1, for all positive integers a, p is also a factor of $2^{na}-1$. As n is prime and p is not a factor of $2^1-1, n$ is also the smallest positive integer y such that p is a factor of 2^y-1. For all positive integers y, p is a factor of 2^y-1 if and only if n is a factor of y. As p is a factor of $2^{p-1}-1, n$ is a factor of $p-1$ so $p\equiv 1(\bmod n)$. Now p is a factor of 2^n-1, which is odd $\Rightarrow p$ is odd $\therefore p\equiv 1(\bmod 2n)$. Hence the proof. □

4. Results on Williams meet matrices base b on A-sets

1. Williams meet matrices base b on A-sets satisfies structure theorem [5].
2. The determinant of this matrix is obtained by using the recursion formula.
3. The inverse of this matrix is also found by using these results [5].

We prove these results by considering the following example. We consider $b=3$. Then the Williams number base becomes $(b-1)b^n-1=2\cdot 3^n-1$.
The Williams meet matrix base b is $((S)_f)_{ij}=2.3^{\mathrm{x}_i\wedge \mathrm{x}_j}-1$. Consider $\mathrm{S}=\{1,2,3\}$ then $\mathrm{A}=\{1,1\}$.

$$(S)=\begin{bmatrix} 5 & 5 & 5 \\ 5 & 17 & 5 \\ 5 & 5 & 53 \end{bmatrix}$$

$$\begin{aligned} f(x) &=2\cdot 3^x-1=f_1(x), \\ f_2(x) &=f_1(x)-\frac{(f_1(a_1))^2}{(f_1(x_1))}=2\cdot 3^x-6, \\ f_3(x) &=f_2(x)-\frac{(f_2(a_2))^2}{(f_2(x_2))}=2\cdot 3^x-6 \\ f_1(x_1) &=5, f_2(x_2)=12, f_3(x_3)=48. \end{aligned}$$

$$\therefore \det(\mathrm{S})_f=f_1(x_1)f_2(x_2)f_3(x_3)=2880.$$

Hence determinant of Williams meet matrix base b $=3$ on A-sets is calculated.
Now $(S)_f$ is expressed as $(\mathrm{S})_f=\mathrm{M}^{\mathrm{T}}DM$

$$M=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D=\operatorname{diag}(5,12,48)$$

by using $(M)_{ij}=\frac{f_i(a_i)}{f_i(x_i)}$ for all i $<$ j and

$$D=\operatorname{diag}(f_1(x_1), f_2(x_2), \ldots\ldots, f_n(x_n))$$

from structure theorem [5]. By usual calculations it can be proved that $\mathrm{M}^{\mathrm{T}}\mathrm{DM}=(\mathrm{S})_f$. The value of $(S_f)^{-1}$ is also calculated from $(S_f)^{-1}=N\Delta N^{\mathrm{T}}$ using result from [5].

$$\mathrm{N}=\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\Delta=\operatorname{diag}\left(\frac{1}{5},\frac{1}{12},\frac{1}{48}\right)\Rightarrow (S_f)^{-1}=\begin{bmatrix} \frac{73}{240} & -\frac{1}{12} & -\frac{1}{48} \\ -\frac{1}{12} & \frac{1}{12} & 0 \\ -\frac{1}{48} & 0 & \frac{1}{48} \end{bmatrix}.$$

Thus the results of meet matrices on A-sets [5] are verified for Williams meet matrices base b $=3$ on A-sets.

5. Conclusion

In this paper we have proved some theorems on Williams number base $b=2$. The Williams meet matrices base $b=3$ on A-sets are considered to prove results from structure theorem. Other Williams meet matrices with different value of base b on A-sets may be considered for future study. Also some other type matrices and join matrices can also be included in the study.

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