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Cycle neighbor polynomial of some graph operations

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Abstract

The Cycle Neighbor Polynomial of a graph G is defined as, $CN^*[G,z] = \Sigma_{k=0}^{c(G)} c_k(G)z^k$, where $c_0(G)$ is the number of isolated vertices, $c_1(G)$ is the number of non isolated vertices which does not belong to any cycle of G, $c_2(G)$ is the number of bridges and $c_k(G)$ is the number of cycles of length k in G for $g(G) \leq k \leq c(G)$ with $g(G)$ and $c(G)$ are respectively the girth and circumference of G. This paper deals with the cycle neighbor polynomial of some graph operations, graph modifications and that of graphs derived from the given graph.

Keywords

Cycle neighbor polynomial, graph operations.

AMS Subject Classification

05C31, 05C76.

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Contents

1. Introduction

Many graph polynomials are introduced and studied in graph theory. Chromatic polyomial [13], Tutte polynomial[7], clique polynomial [10], etc., are some examples. These polynomials are studied because some of them are generating functions of some graph properties, some count the number of occurrences of certain graph features and some others make an attempt to find complete graph invariants and so on. In [3] Annie Sabitha Paul and Raji Pilakkat introduced one such polynomial called cycle neighbor polynomial of a graph. For a graph G its cycle neighbor polynomial is defined as $CN^*[G,z] = \Sigma_{k=0}^{c(G)} c_k(G)z^k$, where $c_0(G)$ is the number of isolated vertices, $c_1(G)$ is the number of non isolated vertices which does not belong to any cycle of G, $c_2(G)$ is the number of bridges and $c_k(G)$ is the number of cycles of length k in G for $3 \leq k \leq n$ with $c(G)$ as the circumference of G. It is a generating polynomial for the number of cycles of different lengths varying from $g(G)$ (girth of G) to $c(G)$ (circumference of G). It reveals many graph properties like girth [5], circumference [5], hamiltonicity [5], pacyclicity [4], Whether the graph is bipartite or not etc., of a graph.

Many graph modification problems concern destroying or creating cycles. In this paper we study the cycle neighbor polynomial of some graph operations, graph modifications and that of graphs derived from the given graph.

2. Cycle Neighbor Polynomial of Graph Operations

First we consider the corona [11] of two graphs G and H denoted by GoH. Let G be a connected graph of order $n \geq 2$ with k, $1 \leq k \leq n$ cycle neighbor free vertices which is the set of all vertices which do not belong to any cycle of G. Note that no cycle will be added or deleted from the induced subgraph G of GoH. We obtain $CN^*[GoH;z]$ when H is a path, cycle or a star graph in terms of cycle neighbor polynomial of G.

Theorem 2.1. *Let P_m be a path on $m \geq 1$ vertices. Then*

$$CN^*[GoP_m;z] = \begin{cases} CN^*[G;z] + nz^2 + nz, & \text{if } m = 1; \\ CN^*[G;z] + nz^3 - kz, & \text{if } m = 2; \\ CN^*[G;z] + n\{\Sigma_{k=3}^{m+1}\{m-(k-2)\}z^k\} - kz, & \text{if } m \geq 3; \end{cases}$$

Proof. When $m=1$, corresponding to the vertex in each of the n copies of P_1, n new edges and n new cycle neighbor free vertices will be introduced in GoP_1.
When $m=2$, corresponding to the edge in each of the n copies of P_2, n triangles will be introduced in GoP_2 and there are no cycle neighbor free vertices in GoP_2.
When $m \geq 3$, together with all cycles of different lengths in G, $(m-1)$ triangles, $(m-2)$ 4-cycles,..., one m-cycle will be formed in GoP_m with a vertex of G common to all these cycles in GoP_m. Hence every vertex of G belong to at least one cycle of GoP_m and we have $CN^*[GoP_m;z] = CN^*[G;z] + n\{\Sigma_{k=3}^{m+1}\{m-(k-2)\}z^k\} - kz$. □

Theorem 2.2. *Let C_m be a cycle on $m \geq 3$ vertices. Then,* $CN^*[GoC_m;z] = CN^*[G;z] + n\{\Sigma_{l=3}^{m+1} z^l + z^m\} - kz$

Proof. In GoC_m at each of the n vertices of G, there is a wheel graph on $m+1$ vertices with the central vertex as the vertex of G. For a wheel graph $W_{m+1} \cong C_m + K_1$, $m \geq 3$, $CN^*[W_{m+1}, z] = m\sum_{k=3}^{m+1} z^k + z^m$ [3]. Hence it follows that $CN^*[GoC_m;z] = CN^*[G;z] + n\{m\{\Sigma_{l=3}^{m+1} z^l\} + z^m\} - kz$. □

Theorem 2.3. *Let $S_{m+1} \cong K_{m,1}$, $m \geq 3$. Then,* $CN^*[GoS_{m+1};z] = CN^*[G;z] + n\{mz^3 + \binom{m}{2}z^4\} - kz$

Proof. Let H be a subgraph of GoS_{m+1} induced by a vertex of G and a copy of S_{m+1}, $m \geq 3$. Then there are two vertices say u and v of degree $m+1$ and m vertices of degree two in H. Let $V(H) = A \cup B$, where $A = \{u, v\}$ and $B = \{v_1, v_2, ..., v_m\}$, the set of all vertices of degree two in H. Since v_i, $1 \leq i \leq m$ is adjacent to u and v only, there are exactly m triangles in H. There are $(m-1)$ 4-cycles through each vertex v in H. Infact there are $\frac{m(m-1)}{2}$ 4-cycles in H. Finally note that the maximum length of any cycle in H is four. Since $V(H) = A \cup B$ and no two vertices in B are adjacent, the sequence of vertices which form any cycle in H will be either an alternating sequence of vertices from A and B respectively or a sequence of the form v_i, u, v, $1 \leq i \leq 3$. In the first case, since there are only two vertices in A, any alternating vertex sequence from A and B without repetition contain a maximum of four vertices. In the second case, $\{v_i, u, v\}$ induces a triangle in H. Hence there are no cycles of length greater than four in H. □

"Subdivision graph [19] $S(G)$ of a graph G is obtained by subdividing each edge of G exactly once by a new vertex". In the next result, we compare $CN^*[G;z]$ and $CN^*[S(G);z]$ of a graph G.

Theorem 2.4. *Let G be a connected graph of order $n \geq 2$ with $CN^*[G;z] = \Sigma_{k=1}^{c(G)} c_k z^k$ and let $S(G)$ be the subdivision graph of G. Then $CN^*[S(G);z] = (c_1+c_2)z + 2c_2z^2 + \Sigma_{k=3}^{c(G)} c_k z^{2k}$.*

Proof. The number of edges in G will be doubled in its subdivision graph $S(G)$ by the introduction of a new vertex on every edge of G. Hence corresponding to every bridge in G, there is a cycle neighbor free vertex in $S(G)$. Also the number of bridges and lengths of every cycle in G will be doubled in $S(G)$. □

It follows from $CN^*[S(G);z]$ that the subdivision graph of every simple graph G is bipartite. The fact that $g(S(G)) = 2g(G)$ and $c(S(G)) = 2c(G)$ is immediate from $CN^*[G;z]$, where $g(G)$ and $c(G)$ are respectively the girth and circumference of G.

"Square of a graph G [12] is obtained by adding edges in G, which connect pairs of vertices of G at a distance two apart". It is denoted by G^2. Next we obtain $CN^*[G^2;z]$, when G is a path or a star graph.

Theorem 2.5. *Let P_n be a path on $n \geq 3$ vertices. Then,* $CN^*[P_n^2;z] = \Sigma_{k=3}^{n}\{n-(k-1)\}z^k$.

Proof. Let the vertices of P_n be labelled as $v_1, v_2, ..., v_n$. Then for $1 \leq i \leq n-2$, each v_i is adjacent to v_{i+2} in P_n^2. Hence it follows that for $1 \leq i \leq n-2$, the graph induced by the set $\{v_i, v_{i+1}, v_{i+2}\}$ is a triangle in P_n^2 and no triangles are induced by $\{v_i, v_j, v_k\}$ if v_i, v_j, v_k does not form a set of consecutive vertices of $V(P_n)$. Therefore, there are exactly $n-2$ triangles. Also, since every vertex belongs to at least one triangle, there are no bridges or cycle neighbor free vertices in P_n^2. In general, for $1 \leq i \leq k-1$,
$v_iv_{i+1}v_{i+3}v_{i+5}...v_{i+(k-2)}v_{i+(k-1)}v_{i+(k-3)}v_{i+(k-5)}...v_{i+2}v_i$, is a k-cycle for odd k in P_n^2 and
$v_iv_{i+1}v_{i+3}v_{i+5}...v_{i+(k-1)}v_{i+(k-2)}v_{i+(k-4)}v_{i+(k-6)}...v_{i+2}v_i$ is a k-cycle for even k where $3 \leq k \leq n$. Hence in P_n^2, there are $(n-3)$ 4-cycles, $(n-4)$ 5-cycles,..., $(n-(k-1))$ k-cycles,..., one n-cycle without duplication. Hence the proof. □

It follows from $CN^*[P_n^2;z]$ that P_n^2 is hamiltonian. Moreover, P_n^2 is pancyclic for $n \geq 3$ since it contains cycles of every length from 3 to n.

Theorem 2.6. *Let G be a graph of diameter two. If order of G is n, Then,* $CN^*[G^2;z] = \frac{n!}{2}[\frac{z^3}{3(n-3)!} + \frac{z^4}{4(n-4)!} + ... + \frac{z^{n-2}}{(n-2)2!} + \frac{z^{n-1}}{(n-1)} + \frac{z^n}{n}]$.

Proof. Since $diam(G) = 2$, $d(v_i, v_j) \leq 2$, for every $v_i, v_j \in V(G)$. Hence in G^2, v_i is adjacent to v_j, for every i, j, $1 \leq i, j \leq n$, $i \neq j$. Therefore, $G^2 \cong K_n$. Hence the result follows from the expression for cycle neighbor polynomial of complete graphs[3]. □

Corollary 2.7 is a direct consequence of Theorem 2.6 Since $diam(S_{m+1}) = 2$ for $S_{m+1} \cong K_{m,1}$.

Corollary 2.7. $CN^*[S_{m+1}^2;z] = \frac{(m+1)!}{2}[\frac{z^3}{3(m-2)!} + \frac{z^4}{4(m-3)!} + ... + \frac{z^{m-1}}{(m-1)2!} + \frac{z^m}{m} + \frac{z^{m+1}}{m+1}]$

In general, power of a graph G^k, $k = 2, 3, 4, ...$ is obtained by adding edges in G which connect pairs of vertices v_i, v_j if $d(v_i, v_j) \leq k$. Then $G^k \cong K_n$, therefore we have;

Theorem 2.8. *Let G be a graph of diameter k, $k = 2, 3, 4, ...$. If order of G is n, Then,* $CN^*[G^k;z] = CN^*[K_n, z]$.

From the expression for $CN^*[G^k;z]$ of G^k, $k=2,3,4,...$ it is clear that G^k is pancyclic if $diam(G)=k$.

"The splitting graph $S^{'}(G)$ [15] of a graph G is obtained by adding new vertices $v^{'}$ to G, corresponding to each vertex v of G and then joining the vertex $v^{'}$ to all vertices of G adjacent to v in G". Now we find $CN^*[S^{'}(G);z]$ when G is a path or a star graph.

Theorem 2.9. *Let P_n be a path on $n \geq 2$ vertices. Then*

$$CN^*[S^{'}(P_n);z] = \begin{cases} 3z^2+4z, & \text{if } n=2; \\ \Sigma_{k=3}^{n-1}\{n-(k-1)\}z^{2k}, & \text{if } n \geq 3; \end{cases}$$

Proof. Let the vertices of P_n be labelled as $v_1,v_2,...,v_n$, with v_1 and v_2 as the pendant vertices. Let $v_i^{'}$ be the vertex in $S^{'}(P_n)$ corresponding to v_i, $1 \leq i \leq n$. Then $v_1^{'}$ and $v_n^{'}$ are the pendant vertices of $S^{'}(P_n)$. For $1 \leq i \leq n-k$,
$v_iv_{i+1}^{'}v_{i+2}v_{i+3}^{'}v_{i+4}...v_{i+(k-2)}^{'}v_{i+(k-1)}v_{i+k}v_{i+(k-1)}^{'}...v_{i+2}v_{i+1}v_i$
is a $2k$-cycle in $S^{'}(P_n)$, when k is odd and
$v_iv_{i+1}^{'}v_{i+2}v_{i+3}^{'}v_{i+4}...v_{i+(k-1)}^{'}v_{i+k}v_{i+(k-1)}v_{i+(k-2)}^{'}...v_{i+2}^{'}v_{i+1}v_i$
is a $2k$-cycle in $S^{'}(P_n)$, when k is even. Hence there are $n-2$ 4-cycles, $(n-3)$ 6-cycles,...,two $2(n-2)$-cycles and one $2(n-1)$ cycle in $S^{'}(P_n)$. Hence the proof. □

Theorem 2.10. *Let $S_{m+1}^{'}$ be the splitting graph of $S_{m+1} \cong K_{m,1}$, $m \geq 2$. Then $CN^*[S_{m+1}^{'};z] = \binom{m}{2}z^4+mz^2+mz$*

Proof. Let $V(S_{m+1}^{'}) = A \cup B$ with $A=\{v,v^{'}\}$, where v and $v^{'}$ are the central vertex of S_{m+1} and its corresponding vertex in $S_{m+1}^{'}$ respectively and $B=\{u_1,u_2,...,u_m,u_1^{'},u_2^{'},...,u_m^{'}\}$ where u_i and $u_i^{'}$, $1 \leq i \leq m$ are the pendant vertices of S_{m+1} and its corresponding vertex in $S_{m+1}^{'}$ respectively. Then A and B are independent sets. Hence the vertices of any cycle in $S_{m+1}^{'}$ is an alternating sequence of vertices from A and B. Since $|A|=2$, the length of any cycle in $S_{m+1}^{'}$ is four and there are $(m-1)$ 4-cycles through u_1, $(m-2)$ 4-cycles through u_2 without repetition and so on. Hence there are $\binom{m}{2}$ 4-cycles in $S_{m+1}^{'}$. Also there are m pendant vertices and m pendant edges corresponding to $u_i^{'}$, $1 \leq i \leq m$. Hence the proof. □

Since there are no odd cycles in both $P_n^{'}$ and $S_{m+1}^{'}$, it follows that the splitting graph of a path as well as that of a star graph are bipartite.

"Duplication of a vertex v of a graph G is the graph $G^{'}$ obtained by adding a vertex $v^{'}$ in G with $N(v^{'})=N(v)$". Here we consider $CN^*[G^{'};z]$ of G, when G is a path, cycle or a star graph.

Since the duplication of a pendant vertex of a path P_n, $n \geq 2$ adds a new vertex in $P_n^{'}$ which is adjacent to a single vertex of P_n, we have;

Proposition 2.11. *Let $P_n^{'}$ be the graph obtained by the duplication of a pendant vertex of P_n, $n \geq 2$. Then* $CN^*[P_n^{'};z] = CN^*[P_n;z]+z^2+z$.

Theorem 2.12. *Let $P_n^{'}$ be the graph obtained by the duplication of a non pendant vertex of P_n, $n \geq 2$ then* $CN^*[P_n^{'};z] = CN^*[P_n;z]+z^4-2z^2-3z$.

Proof. The subgraph of $P_n^{'}$ induced by the duplication of a non pendant vertex vertex of P_n, its corresponding vertex and their neighbors is a 4-cycle in $P_n^{'}$ and consequently, the number of cycle neighbor free vertices of P_n will be reduced by three and number of bridgrs of P_n will be reduced by two in $P_n^{'}$. □

Theorem 2.13. *Let $C_n^{'}$ be the graph obtained by the duplication of a vertex of the cycle C_n, $n \geq 3$ then* $CN^*[C_n^{'};z] = CN^*[C_n;z]+z^n+z^4$.

Proof. Let v be any vertex of C_n and $v^{'}$ be the duplication of v in $C_n^{'}$. Then $\{v,v^{'}\} \cup N(v)$ induces a 4-cycle and $\{v^{'}\} \cup V(C_n) \setminus \{v\}$ induces an n-cycle in $C_n^{'}$. Therefore there are two n-cycles and a 4-cycle in $C_n^{'}$. □

Theorem 2.14. *Let $S_{m+1}^{'}$ be the graph obtained by the duplication of the central vertex of $S_{m+1} \cong K_{m,1}$, $m \geq 2$. Then* $CN^*[S_{m+1}^{'};z] = \binom{m}{2}z^4$.

Proof. Let $V(S_{m+1}^{'}) = A \cup B$ with $A=\{v,v^{'}\}$, where v and $v^{'}$ are the central vertex of S_{m+1} and duplication of v in $S_{m+1}^{'}$ respectively and $B=\{u_1,u_2,...,u_m\}$, where u_i, $1 \leq i \leq m$ are the pendant vertices of S_{m+1}. Since A and B form a partition of $V(S_{m+1}^{'})$ and since A and B are independent sets, as in the case of splitting graph of S_{m+1} there are $\binom{m}{2}$ 4-cycles in $S_{m+1}^{'}$. Also since both v and $v^{'}$ are adjacent to all the vertices of B, There are no cycle neighbor free vertices or bridges in $S_{m+1}^{'}$. Hence the result. □

It follows that the graph obtained by the duplication of the central vertex of $S_{m+1} \cong K_{m,1}$, $m \geq 2$ is bipartite. It is obvious that if $G^{'}$ is the graph obtained by the duplication of any one of the pendant vertices of S_{m+1}, then $CN^*[G^{'};z] = CN^*[S_{m+1};z]+z^2+z$

"Duplication of of a vertex $w \in V(G)$ of a graph G by an edge [18] produces a new graph $G^{'}$ by adding an edge $e^{'}=u^{'}v^{'}$ to G such that $N(v^{'})=\{w,u^{'}\}$ and $N(u^{'})=\{w,v^{'}\}$". In the next result we obtain $CN^*[G^{'};z]$ of a graph G.

Theorem 2.15. *Let G be a connected graph of order $n \geq 2$ which contains k, $0 \leq k \leq n$ cycle neighbor free vertices and let $G^{'}$ be the graph obtained by the duplication of a vertex $w \in V(G)$ by an edge. Then*

$$CN^*[G^{'};z] = \begin{cases} CN^*[G;z]+z^3-z, \\ \quad \text{if } w \text{ is a cycle neighbor free vertex} \\ CN^*[G;z]+z^3, \\ \quad \text{otherwise} \end{cases}$$

Proof. Duplication of of a vertex $w \in V(G)$ of a graph G by an edge $e=uv$ produces a triangle $wuvw$ in $G^{'}$. Therefore, the number of cycle neighbor free vertices will be reduced by one if w is a cycle neighbor free vertex. □

Let $G^{'}$ be the graph obtained by duplication of each vertex of G by edges, then clearly $G^{'} \cong GoP_2$ hence it follows from Theorem [2.1] that $CN^*[G^{'};z] = CN^*[G;z] + nz^3 - kz$, where k is the number of cycle neighbor free vertices in G.

"The middle graph $M(G)$ (also known as the semi total (line) graph $T_1(G)$ [14]) of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if they are adjacent edges of G or one is a vertex and the other is an edge incident with it." Next we obtain $CN^*[M(G);z]$ when $G \cong P_n$.

Theorem 2.16. *Let $M(P_n)$ be the middle graph of path P_n, $n \geq 2$. Then,*

$$CN^*[M(P_n);z] = \begin{cases} 2z^2 + 3z, & \text{if } n = 2 \\ (n-2)z^3 + 2z^2 + 2z, & \text{if } n > 2 \end{cases}$$

Proof. Let $V(M(P_n)) = A \cup B$ where $A = \{v_1, v_2, ..., v_n\}$, is the set of vertices of P_n and $B = \{u_1, u_2, ..., u_{n-1}\}$ be the vertices of $M(P_n)$ corresponding to the edges of P_n. The subgraph of $M(P_n)$ induced by B is P_{n-1} and for $2 \leq i \leq n-1$, v_i is adjacent to u_{i-1} and u_i. Hence the graph induced by $\{u_{i-1}, v_i, u_i\}$ is a triangle for $2 \leq i \leq n-1$. Also u_1v_1 and v_nu_{n-1} are bridges of $M(P_n)$. Hence the result. □

If $V(G)$ of a graph G can be partitioned into an independent set and an acyclic set, then G is said to be a near-bipartite graph [1]. From the proof of Theorem 2.16, it is clear that middle graph of path P_n, $n \geq 2$ is near bipartite.

Theorem 2.17. *Let $M(S_{m+1})$ be the middle graph of $S_{m+1} \cong K_{m,1}$, $m \geq 3$. Then,*
$CN^*[M(S_{m+1});z] = \frac{(m+1)!}{2}[\frac{z^3}{3(m-2)!} + \frac{z^4}{4(m-3)!} + ... + \frac{z^{m-1}}{(m-1)2!} + \frac{z^m}{m} + \frac{z^{m+1}}{m+1}] + mz^2 + mz.$

Proof. Let $V(M(S_{m+1})) = \{v, v_1, v_2, ..., v_m, u_1, u_2, ..., u_m\}$, where $\{v, v_1, v_2, ..., v_m\}$ is $V(S_{m+1})$ with v as the central vertex and $\{u_1, u_2, ..., u_m\}$ corresponds to the edges of S_{m+1}. Since every edge in S_{m+1} are adjacent and are incident with v, the subgraph of $M(S_{m+1})$ induced by $\{v, u_1, u_2, ..., u_m\}$ is K_{m+1} and in $M(S_{m+1})$, $|N(v_i)| = 1$ for $1 \leq i \leq m$. Hence the result follows from the expression for $CN^*[K_{m+1}]$ [3]. □

"A split graph [8] is a graph whose vertices can be partitioned into two subsets, such that one subset induces a clique, and the other induces an independent set." " A graph is called a cograph or complement reducible graph [6] if it contains no induced P_4 and a graph is called trivially perfect [9] if it is a cograph and contains no induced C_4." It is obvious from $CN^*[M(S_{m+1});z]$ of $S_{m+1} \cong K_{m,1}$ that the middle graph of every star graph is a split graph. Also since the graph induced by any four vertices of $M(S_{m+1})$ contains a triangle, middle graph of every star graph is a cograph and is trivially perfect too.

"The semi total (point) graph $T_2(G)$ [14]) of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if they are adjacent vertices of G or one is a vertex and the other is an edge incident with it."

Proposition 2.18. *Let P_n be a path on $n \geq 2$ vertices. Then $CN^*[T_2(P_n);z] = (n-2)z^3$*

Proof. Let $V(T_2(P_n)) = A \cup B$, where $A = \{v_1, v_2, ..., v_n\} = V(P_n)$ and $B = \{u_1, u_2, ..., u_{n-1}\}$ be the vertices of $T_2(P_n)$ corresponding to the edges of P_n. Then for $1 \leq i \leq n-1$, $\langle\{v-i, u_i, v_{i+1}\}\rangle$ is a triangle in $T_2(P_n)$ and every vertex of $T_2(P_n)$ is in at least one triangle. Also since u_i, u_j, $1 \leq i, j \leq n-1$, $i \neq j$ are non adjacent and $N(u_i) = \{v_i, v_i + 1\}$ for $1 \leq i \leq n-1$ there are no cycles of length greater than three. □

Proposition 2.19. $CN^*[T_2(S_n);z] = (n-2)z^3$*, where $S_n \cong K_{n-1,1}$ and $n \geq 3$.*

Proof. Let $V(T_2(S_n)) = A \cup B$, where $A = \{v_1, v_2, ..., v_n\} = V(S_n)$ with v_n as the central vertex and $B = \{u_1, u_2, ..., u_{n-1}\}$ be the vertices of $T_2(P_n)$ corresponding to the edges of S_n such that u_i is incident with v_i and v_n of S_n for $1 \leq i \leq n-1$. Then for $1 \leq i \leq n-1$, $\langle\{v_i, u_i, v_n\}\rangle$ is a triangle in $T_2(S_n)$ and every vertex of $T_2(S_n)$ is in at least one triangle. Also since u_i, u_j, $1 \leq i, j \leq n-1$, $i \neq j$ are non adjacent and $N(u_i) = \{v_i, v_n\}$ for $1 \leq i \leq n-1$ there are no cycles of length greater than three as in the case of $T_2(P_n)$. □

As in the case of middlle graph of P_n, the semi total (point) graph of P_n is also near bipartite. And from the proof of expression for $CN^*[T_2(S_n);z]$, it is clear that $T_2(S_n)$, $S_n \cong K_{n-1,1}$ is totally perfect. Also it is trivial from the expressions of $CN^*[T_2(P_n);z]$ and $CN^*[T_2(S_n);z]$ that $T_2(P_n)$ and $T_2(S_n)$ are *cyn**-equivalent [2].

"The total graph $T(G)$ of a graph G is a graph whose vertex set is $V(T(G)) = V(G) \cup E(G)$ and two distinct vertices x and y of $T(G)$ are adjacent if x and y are adjacent vertices of G or adjacent edges of G or x is a vertex incident with edge y." Now we find $CN^*[T(G);z]$ when $G \cong P_n$.

Theorem 2.20. *Let $T(P_n)$ be the total graph of path P_n, $n \geq 2$. Then, $CN^*[T(P_n);z] = \Sigma_{k=3}^{2n-1}(2n-k)z^k$.*

Proof. Let $V(T(P_n)) = \{v, v_1, v_2, ..., v_n, u_1, u_2, ..., u_{n-1}\}$, where $\{v, v_1, v_2, ..., v_n\} = V(T(P_n))$ and u_i, $1 \leq i \leq n-1$ are the vertices of $T(P_n)$ corresponding to the edges of P_n. Then for $2 \leq i \leq n-1$ and $2 \leq j \leq n-2$, $|N(v_i)| = |N(u_j)| = 4$, $|N(u_1)| = |N(u_{n-1}| = 3$ and $|N(v_1)| = |N(v_n)| = 2$. Let $3 \leq k \leq n$. When k is odd, for, $1 \leq i \leq n - \lfloor \frac{k}{2} \rfloor$,
$v_iv_{i+1}v_{i+2}...v_{i+\frac{k-1}{2}}u_{i+\frac{k-3}{2}}u_{i+\frac{k-5}{2}}...u_iv_i$ is a k-cycle and for for, $1 \leq i \leq n - \lceil \frac{k}{2} \rceil$,
$u_iu_{i+1}u_{i+2}...u_{i+\frac{k+1}{2}}v_{i+\frac{k+1}{2}}v_{i+\frac{k-1}{2}}...v_{i+1}u_i$ is a k-cycle. And when k is even, for, $1 \leq i \leq n - \frac{k}{2}$,
$v_iv_{i+1}v_{i+2}...v_{i+\frac{k}{2}}u_{i+\frac{k}{2}}u_{i+\frac{k-2}{2}}...u_iv_i$ and
$u_iu_{i+1}u_{i+2}...u_{i+\frac{k-2}{2}}v_{i+\frac{k}{2}}v_{i+\frac{k-2}{2}}...v_iu_i$ is a k-cycle in $T(P_n)$.
Hence in both cases, there are $n - \frac{k-1}{2} + n - \frac{k+1}{2} = n - \frac{k}{2} + n - \frac{k}{2} = 2n - k$ k-cycles in in $T(P_n)$. Hence the proof. □

It is obvious from the expression for $CN^*[T(P_n);z]$ that total graph of path P_n, is pancyclic for $n \geq 2$.

Derived graph of a simple graph G denoted by $G^\dagger$ was introduced by Jog et al in their paper [16]. "For a simple graph $G(V,E)$, its derived graph $G^\dagger$ is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if their distance in G is two."

It is clear from the definition of derived graph $G^\dagger$ of G that for a path P_n, $n \geq 2$, $P_2^\dagger \cong \overline{K_2}$, $P_n^\dagger \cong P_{\lfloor \frac{n}{2} \rfloor} \cup P_{\lceil \frac{n}{2} \rceil}$, when n is odd and $P_n^\dagger \cong P_{\frac{n}{2}} \cup P_{\frac{n}{2}}$ when n is even. Hence we have;

Proposition 2.21. *Let $P_n^\dagger$ be the derived graph of path P_n, $n \geq 2$. Then,*

$$CN^*[P_n^\dagger;z] = \begin{cases} 2, & \text{if } n = 2; \\ z^2 + 2z + 1, & \text{if } n = 3; \\ (n-2)z^2 + nz, & \text{if } n \geq 3; \end{cases}$$

Since number of edges in $P_n^\dagger$ is less than $n-1$, it is clear from $CN^*[P_n^\dagger;z]$ that $P_n^\dagger$ is disconnected.

Similarly for a cycle C_n, $n \geq 3$, $C_3^\dagger \cong \overline{K_3}$, $C_4^\dagger \cong P_2 \cup P_2$, $C_n^\dagger \cong C_n$ if n is odd and $n \geq 5$ and $C_n^\dagger \cong C_{\frac{n}{2}} \cup C_{\frac{n}{2}}$ if n is even and $n \geq 6$ we have the result;

Proposition 2.22. *Let $C_n^\dagger$ be the derived graph of cycle c_n, $n \geq 3$. Then,*

$$CN^*[C_n^\dagger;z] = \begin{cases} 3, & \text{if } n = 3; \\ 2z^2 + 4z + 1, & \text{if } n = 4; \\ z^n, & \text{if } n \geq 5 \text{ and } n \text{ is odd}; \\ 2z^{\frac{n}{2}}, & \text{if } n \geq 6 \text{ and } n \text{ is even}; \end{cases}$$

$CN^*[C_n^\dagger;z] = 2z^{\frac{n}{2}}$ for $n \geq 6$ and n is even, implies that $C_n^\dagger$ is disconnected, otherwise the two cycles in $C_n^\dagger$ will have a vertex in common and hence n cannot be even.

For star graph $S_{m+1} \cong K_{m,1}$, $S_{m+1}^\dagger \cong K_m \cup K_1$. Hence we have, $CN^*[S_{m+1}^\dagger;z] = CN^*[K_m;z] + 1$. It is obvious from $CN^*[S_{m+1}^\dagger;z]$ that $S_{m+1}^\dagger$ is disconnected and it is a split graph.

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