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On decomposition of intuitionistic fuzzy primary submodules

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Abstract
This article is in continuation to author's previous paper, on intuitionistic L-fuzzy primary and P-primary submodules, Malaya Journal of Mathematik, Vol. 8, no. 4, 2020, pp. 1417-1426. In this paper, we explore the decomposition of intuitionistic fuzzy submodule as the intersection of finite many intuitionistic fuzzy primary submodules. Many other forms of decomposition like irredundant and normal decomposition are also investigated.

Keywords
Intuitionistic fuzzy primary ideal (submodules), Residual quotient, Intuitionistic fuzzy primary decomposition, Irredundant and normal intuitionistic fuzzy primary decomposition.

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1. Introduction

Primary ideals play a central role in commutative ring theory. One of the natural generalizations of primary ideals which have attracted the interest of several authors is the notion of primary submodules (see for example [1], [13] and [14]). These have led to more information on the structure of the R-module M. A proper submodule Q of an R-module M is called a primary submodule provided that for any $s \in R$ and $m \in M$, $sm \in Q$ implies that $m \in Q$ or $s^n \in (Q:M)$ for some positive integer n. Note that every primary submodule is a prime submodule but converse need not be true. If Q is a primary submodule of M, then the radical of the ideal $(Q:M)$ is a prime ideal of R. If $P = \sqrt{(Q:M)}$, then Q is called a P-primary submodule of M.

The decomposition of an ideal (submodule) into primary ideal (primary submodule) is a traditional pillar of ideal (module) theory. It provides the algebraic foundation for decomposing an algebraic variety into its irreducible components. From another point of view primary decomposition provides a generalization of the factorization of an integer as a product of prime-powers. A submodule N of M has a primary decomposition if $N = \bigcap_{i=1}^{k} Q_i$ with $\sqrt{(Q_i:M)} = P_i$. If no $Q_j \supset \bigcap_{i=1, j\neq i}^{n} Q_i, \forall j$ and the ideals $P_i, 1 \leq j \leq k$ are all distinct, then the primary decomposition is named as minimal and the set $Ass(N) = \{P_1, P_2, ..., P_k\}$ is termed as the set of associated prime ideals of N.

In this paper, we study intuitionistic fuzzy primary decomposition, irredundant intuitionistic fuzzy primary decomposition and normal intuitionistic fuzzy primary decomposition of intuitionistic fuzzy submodule.

2. Preliminaries

In the entire article, R will be treated as a commutative ring with unity $1, 1 \neq 0, M$ a unitary R-module with θ its zero element.

Definition 2.1. *([2], [4]) Let $X \neq \emptyset$. An intuitionistic fuzzy set (IFS) A in X is a complex function $A = (f_A, g_A) : X \to [0,1] \times$*

$[0,1]$ *where* $f_A(x)$ *denote the degree of membership and* $g_A(x)$ *denote the degree of non-membership of element* $x \in X$ *to the set A satisfying the condition that* $0 \leq f_A(x) + g_A(x) \leq 1$ *for each* $x \in X$.

Remark 2.2.

(i) When $f_A(x) + g_A(x) = 1$, *i.e.,* $g_A(x) = 1 - f_A(x) = f_{A^c}(x)$. *Then A is called a fuzzy set.*

(ii) The class of IFSs of X is denoted by IFS(X).

For $A, B \in IFS(X)$ we utter $A \subseteq B$ iff $f_A(x) \leq f_B(x)$ and $g_A(x) \geq g_B(x), \forall x \in X$. Also, $A \subset B$ iff $A \subseteq B$ and $A \neq B$.

Definition 2.3 ([3], [4], [9]). *Let* $A \in IFS(R)$. *Then A is termed as an intuitionistic fuzzy ideal* (IFI) *of R if for all* $r, s \in R$, *the followings hold*

(i) $f_A(r-s) \geq f_A(r) \wedge f_A(s)$;

(ii) $f_A(rs) \geq f_A(r) \vee f_A(s)$;

(iii) $g_A(r-s) \leq g_A(r) \vee g_A(s)$;

(iv)$g_A(rs) \leq g_A(r) \wedge g_A(s)$.

Definition 2.4 ([4], [9]). *Let* $A \in IFS(M)$. *Then A is termed as an intuitionistic fuzzy module* (IFM) *of M if for all* $m, n \in M, r \in R$, *the followings hold*

(i) $f_A(m-n) \geq f_A(m) \wedge f_A(n)$;

(ii) $f_A(rm) \geq f_A(m)$;

(iii) $f_A(\theta) = 1$;

(iv) $g_A(m-n) \leq g_A(m) \vee g_A(n)$;

(v)$g_A(rm) \leq g_A(m)$;

(vi)$g_A(\theta) = 0$.

We designate the set of all intuitionistic fuzzy R-modules of M by $IFM(M)$ and the set of all intuitionistic fuzzy ideals of R by $IFI(R)$. Notice that when $R = M$, then $A \in IFM(M)$ iff $f_A(\theta) = 1, g_A(\theta) = 0$ and $A \in IFI(R)$.

Definition 2.5. *([9]) For* $P, Q \in IFS(M)$ *and* $S \in IFS(R)$, *define the residual quotient* $(P:Q)$ *and* $(P:S)$ *as follows:*
$(P:Q) = \bigcup\{J : J \in IFS(R)$ *such that* $J \cdot Q \subseteq P\}$ *and*
$(P:S) = \bigcup\{K : K \in IFS(M)$ *such that* $S \cdot K \subseteq P\}$.

Theorem 2.6. *([9]) For* $P, Q \in IFS(M)$ *and* $S \in IFS(R)$. *Then we have*

(i) $(P:Q) \cdot Q \subseteq P$ *;*

(ii) $S \cdot (P:S) \subseteq P$ *;*

(iii) $S \cdot Q \subseteq P \Leftrightarrow S \subseteq (P:Q) \Leftrightarrow Q \subseteq (P:S)$.

Theorem 2.7. *([9]) For* $P_i (i \in J), Q \in IFS(M)$ *and* $S \in IFS(R)$. *Then we have*

(i) $(\bigcap_{i \in J} P_i : Q) = \bigcap_{i \in J} (P_i : Q)$;

(ii) $(\bigcap_{i \in J} P_i : S) = \bigcap_{i \in J} (P_i : S)$.

Theorem 2.8. *([9]) For* $P, Q \in IFS(M)$ *and* $S \in IFS(R)$

(i) If $P \in IFM(M)$, *then* $(P:Q) = \bigcup\{J : J \in IFI(R)$ *such that* $J \cdot Q \subseteq P\}$;

(ii) If $S \in IFI(R)$, *then* $(P:S) = \bigcup\{K : K \in IFS(M)$ *such that* $S \cdot K \subseteq P\}$.

Theorem 2.9. *([9]) For* $P, Q \in IFM(M)$ *and* $S \in IFI(R)$. *Then* $(P:Q) \in IFI(R)$ *and* $(P:S) \in IFM(M)$.

Theorem 2.10. *([9]) For* $P, Q_i \in IFS(M)$ *and* $S_i \in IFS(R), (i \in J)$ *. Then we have*

(i) $(P : \bigcup_{i \in J} Q_i) = \bigcap_{i \in J} (P : Q_i)$;

(ii) $(P : \bigcup_{i \in J} S_i) = \bigcap_{i \in J} (P : S_i)$.

Definition 2.11. *([3], [8]) For a non-constant* $C \in IFI(R), C$ *is called an intuitionistic fuzzy prime (respectively, primary) ideal of R if for any intuitionistic fuzzy points* $x_{(p,q)}, y_{(r,s)} \in IFP(R)$, $x_{(p,q)} y_{(r,s)} \in C$ *implies that either* $x_{(p,q)} \in C$ *or* $y_{(r,s)} \in C$ *(or respectively, either* $x_{(p,q)} \in C$ *or* $y^n_{(r,s)} \in C$, *for some* $n \in \mathbb{N}$*).*

Definition 2.12. *([12]) Let A be an intuitionistic fuzzy submodule of B. Then A is called an intuitionistic fuzzy prime (respectively, primary) submodule of B, if* $r_{(s,t)} \in IFP(R), x_{(p,q)} \in IFP(M)$ $(r \in R, x \in M, s, t, p, q \in (0,1))$, $r_{(s,t)} x_{(p,q)} \in A$ *implies that either* $x_{(p,q)} \in A$ *or* $r^n_{(s,t)} B \subseteq A$, *for some* $n \in \mathbb{N}$ *(or respectively, either* $x_{(p,q)} \in A$ *or* $r_{(s,t)} B \subseteq A$*).*

In particular, taking $B = \chi_M$, *if for* $r_{(s,t)} \in IFP(R), x_{(p,q)} \in IFP(M)$ *we have* $r_{(s,t)} x_{(p,q)} \in A$ *implies that either* $x_{(p,q)} \in A$ *or* $r^n_{(s,t)} \chi_M \subseteq A$, *for some* $n \in \mathbb{N}$, *then A is called an intuitionistic fuzzy prime (respectively, primary) submodule of M (or respectively, either* $x_{(p,q)} \in A$ *or* $r_{(s,t)} \chi_M \subseteq A$*).*

The following theorem says that intuitionistic fuzzy primary submodule and intuitionistic fuzzy primary ideals coincide when R is considered to be a module over itself.

Theorem 2.13. *([12]) If* $M = R$, *then* $B \in IFM(M)$, *is an intuitionistic fuzzy primary submodule of M iff* $B \in IFI(R)$ *is an intuitionistic fuzzy primary ideal.*

Theorem 2.14. *([12]) (a) Let N be a primary submodule of M and* $p, q \in (0,1)$ *such that* $p + q < 1$. *If A is an IFS of M defined by*

$$f_A(x) = \begin{cases} 1, & \text{if } x \in N \\ p, & \text{if otherwise} \end{cases} ; \quad g_A(x) = \begin{cases} 0, & \text{if } x \in N \\ q, & \text{otherwise.} \end{cases}$$

for all $x \in M$. *Then A is an intuitionistic fuzzy primary submodule of M.*
(b)Conversely, any intuitionistic fuzzy primary submodule can be obtained as in (a).

Corollary 2.15. *([12]) Let A be an intuitionistic fuzzy primary submodules of M. Then*

$$A_* = \{m \in M : f_A(m) = f_A(\theta) \text{ and } g_A(m) = g_A(\theta)\}$$

is a primary submodule of M.

Theorem 2.16. *([12]) If $B \in IFM(M)$, A a intuitionistic primary submodule of M. Then*

(i) if $B \subseteq A$, then $(A : B) = \chi_R$ and

(ii) if $B \nsubseteq A$, then $\sqrt{(A : B)} = \sqrt{(A : \chi_M)}$.

Theorem 2.17. *([12]) If $A \in IFM(M)$, $C \in IFI(R)$ and A be an intuitionistic fuzzy primary submodule of M. Then*

(i) $C \nsubseteq \sqrt{(A : \chi_M)} \Rightarrow (A : C) = A$;

(ii) $C \subseteq (A : \chi_M) \Rightarrow (A : C) = \chi_M$.

Theorem 2.18. *([12]) If A is an intuitionistic fuzzy primary submodule of M, then $\sqrt{(A : \chi_M)}$ is an intuitionistic fuzzy prime ideal of R.*

Definition 2.19. *([12]) Let A be an intuitionistic fuzzy primary submodule of M and $P = \sqrt{(A : \chi_M)}$. Then A is said to be an intuitionistic fuzzy P-primary submodule of M.*

Theorem 2.20. *([12]) Let A be an intuitionistic fuzzy P-primary submodule of M and $B \in IFM(M)$. If $(A : B) \neq \chi_R$, then $(A : B)$ is an intuitionistic fuzzy P-primary ideal of R.*

Theorem 2.21. *([12]) Let $A \in IFM(M)$ and $C \in IFI(R)$. If A is an intuitionistic fuzzy P-primary submodule of M and $(A : C) \neq \chi_M$, then $(A : C)$ is an intuitionistic fuzzy P-primary submodule of M.*

Theorem 2.22. *([12]) If A is an intuitionistic fuzzy primary submodule of M, then $IFrad_{\chi_M}(A) = \sqrt{(A : \chi_M)} = \bigcap\{C \in IFSpec(R) | (A : \chi_M) \subseteq C\}$ is intuitionistic fuzzy primary ideal of R if and only if $rad_M(A_*) = \bigcap\{P \in Spec(R) | (A_* : M) \subseteq P\}$ is primary ideal of R.*

3. Decomposition of intuitionistic fuzzy primary submodules

Definition 3.1. *Let $A \in IFM(M)$. A decomposition of A as a finite intersection, $A = \cap_{i=1}^{n} A_i$ of intuitionistic fuzzy primary submodules A_i of M is called an intuitionistic fuzzy primary decomposition of A and the set of intuitionistic fuzzy prime ideals $\{\sqrt{(A_i : \chi_M)} | i = 1, 2, ..., n\}$ is called the set of associated intuitionistic fuzzy prime ideals of A.*

An intuitionistic fuzzy primary decomposition $A = \cap_{i=1}^{n} A_i$ is called irredundant of no A_i contains $\bigcap_{j=1, j\neq i}^{n} A_j$ and an irredundant intuitionistic fuzzy primary decomposition of A is called normal if distinct A_i have distinct associated intuitionistic fuzzy prime ideals.

Definition 3.2. *An intuitionistic fuzzy primary submodule A_i in the normal prime decomposition $A = \cap_{i=1}^{n} A_i$ is called isolated if the associated intuitionistic fuzzy prime ideal $\sqrt{(A_i : \chi_M)}$ is minimal in the set of associated intuitionistic fuzzy prime ideals of A.*

Theorem 3.3. *If A_i , $(i = 1, 2, 3....., n)$ be intuitionistic fuzzy P-primary submodules of M. Then $\bigcap_{i=1}^{n} A_i$ is also intuitionistic fuzzy P-primary submodule of M.*

Proof. Let $A = \bigcap_{i=1}^{n} A_i$, where $A_1, A_2, ..., A_n$ be intuitionistic fuzzy P-primary submodules of M, then $\sqrt{(A_1 : \chi_M)} = \sqrt{(A_2 : \chi_M)} = ... = \sqrt{(A_n : \chi_M)} = P$. Let $r_{(s,t)} \in IFI(R)$ and $x_{(p,q)} \in IFM(M)$ such that $r_{(s,t)} x_{(p,q)} \in A = \bigcap_{i=1}^{n} A_i$ and $r_{(s,t)} \notin \sqrt{(A : \chi_M)}$. Since $\sqrt{(A : \chi_M)} = \sqrt{(\bigcap_{i=1}^{n} A_i : \chi_M)} = \bigcap_{i=1}^{n} \sqrt{(A_i : \chi_M)}$, by using Theorem (4.6) of [6] and Theorem (3.4) of [9]. Thus we get $r_{(s,t)} x_{(p,q)} \in A_i$ and $r_{(s,t)} \notin \sqrt{(A_i : \chi_M)}$, then since each A_i are intuitionistic fuzzy P-primary submodules of M, we have $x_{(p,q)} \in A_i, \forall i = 1, 2, ..., n$, so $x_{(p,q)} \in \bigcap_{i=1}^{n} A_i = A$. It remain to show that $\sqrt{(A : \chi_M)} = P$.

If $r_{(s,t)} \in P$ then there exists $n_i \in \mathbb{N}$ such that $r_{(s,t)}^{n_i} \chi_M \subseteq A_i$, $\forall i \in \{1, 2, ..., n\}$. Let $k = \sum_{i=1}^{n} n_i$, then $r_{(s,t)}^{k} \chi_M \subseteq A_i$, $\forall i \in \{1, 2, ..., n\}$. So we have $r_{(s,t)}^{k} \chi_M \subseteq \bigcap_{i=1}^{n} A_i = A$. Thus $r_{(s,t)} \in \sqrt{(A : \chi_M)}$. So we have $P \subseteq \sqrt{(A : \chi_M)}$....(1)
Conversely, if $r_{(s,t)} \in \sqrt{(A : \chi_M)}$, then
$r_{(s,t)} \in \bigcap_{i=1}^{n} \sqrt{(A_i : \chi_M)} = P$, so $\sqrt{(A : \chi_M)} \subseteq P$......(2).
From (1) and (2) we get $\sqrt{(A : \chi_M)} = P$. This complete the result. □

Theorem 3.4. *Let $A \in IFM(M)$ and $A = \cap_{i=1}^{n} A_i$ be an irredundant intuitionistic fuzzy primary decomposition of A, where A_i are intuitionistic fuzzy P_i-primary submodules of M. Then an intuitionistic fuzzy prime ideal $C \in \{P_i = \sqrt{(A_i : \chi_M)} | i = 1, 2, ..., n\}$ if and only if there exist $B \in IFM(M)$ such that $\sqrt{(A : B)} = C$. Hence the set of intuitionistic fuzzy prime ideals $\{P_i = \sqrt{(A_i : \chi_M)} | i = 1, 2, ..., n\}$ is independent of the particular irredundant intuitionistic fuzzy primary decomposition of A.*

Proof. Let $A \in IFM(M)$ and $A = \cap_{i=1}^{n} A_i$ be an irredundant intuitionistic fuzzy primary decomposition of A, where A_i are intuitionistic fuzzy P_i-primary submodules of M. Now for any $B \in IFM(M)$, we have

$$(A : B) = (\bigcap_{i=1}^{n} A_i : B) = \bigcap_{i=1}^{n} (A_i : B).$$

Then by Theorem (2.16) and Theorem (2.20), we have $(A_i : B) = \chi_R$ if $B \subseteq A_i$ and $(A_i : B)$ is intuitionistic fuzzy P_i-primary ideal of R if $B \nsubseteq A_i$. Hence $(A_i : B) \in IFI(R)$. Thus we have

$$\sqrt{(A : B)} = \bigcap_{i=1}^{n} \sqrt{(A_i : B)} = \bigcap_{j=1}^{m} \sqrt{(A_{s_j} : \chi_M)},$$

where the intersection is taken over those s_j such that $B \nsubseteq A_{s_j}$.

Now suppose that $\sqrt{(A : B)} = C$ is an intuitionistic fuzzy prime ideal of R. Then by Theorem (3.18) of [5], we get

$$C = \sqrt{(A : B)} = \bigcap_{j=1}^{m} \sqrt{(A_{s_j} : \chi_M)} \supseteq \sqrt{(A_{s_1} : \chi_M)} \sqrt{(A_{s_2} : \chi_M)} ..... \sqrt{(A_{s_m} : \chi_M)}$$

and so $C \supseteq \sqrt{(A_{s_j} : \chi_M)}$ for some s_j. Also $\sqrt{(A : B)} \subseteq \sqrt{(A_{s_j} : \chi_M)}$. It follows that $C = \sqrt{(A_{s_j} : \chi_M)}$.

Next consider any one of the associated intuitionistic fuzzy prime ideal $\sqrt{(A_i : \chi_M)}$ of $A = \cap_{i=1}^{n} A_i$. Let $B = \bigcap_{j=1, j\neq i}^{n} A_j$. Then we have

$$(A : B) = ((\bigcap_{k=1}^{n} A_k) : (\bigcap_{j=1,j\neq i}^{n} A_j)) = \bigcap_{k=1}^{n}(A_k : \bigcap_{j=1,j\neq i}^{n} A_j),$$ by Theorem (2.7)

As $\bigcap_{j=1,j\neq i}^{n} A_j \subseteq A_j, \forall j, j \neq i$ implies $(A_j : \bigcap_{j=1,j\neq i}^{n} A_j) = \chi_R$, by Theorem (2.16)
By the irredundancy of the set of A_i, we have $\bigcap_{j=1,j\neq i}^{n} A_j \nsubseteq A_i$. Thus by Theorem (2.16) we get

$$\sqrt{(A_i : \bigcap_{j=1,j\neq i}^{n} A_j)} = \sqrt{(A_i : \chi_M)}.$$

Therefore, $\sqrt{(A : B)} = \sqrt{(A_i : \chi_M)}$. Hence the set of intuitionistic fuzzy prime ideals $\{\sqrt{(A_i : \chi_M)} | i = 1, 2, ..., n\}$ is independent of the particular irredundant intuitionistic fuzzy primary decomposition of A. □

From Theorem (3.3) and Theorem (3.4) we see that an intuitionistic fuzzy submodule of M that is an intersection of intuitionistic fuzzy primary submodules of M if and only if it has only one associated intuitionistic fuzzy prime ideal.

Theorem 3.5. *Let $A \in IFM(M)$. If A has an intuitionistic fuzzy primary decomposition, then A has a normal intuitionistic fuzzy primary decomposition.*

Proof. We assume that A has an intuitionistic fuzzy primary decomposition $A = \cap_{i=1}^{n} A_i$. If $A_{i_1}, A_{i_2}, ..., A_{i_k} \in \{A_1, A_2, ..., A_n\}$ are such that $\sqrt{(A_{i_1} : \chi_M)} = \sqrt{(A_{i_2} : \chi_M)} = ... = \sqrt{(A_{i_k} : \chi_M)}$, let $\acute{A_i} = \bigcap_{j=1}^{k} A_{i_j}$. Then $\acute{A_i}$ is an intuitionistic fuzzy primary submodule of M and $\sqrt{(\acute{A_i} : \chi_M)} = \sqrt{(A_i : \chi_M)}$, by Theorem (3.3). Thus $A = \acute{A_1} \cap \acute{A_2} \cap ...... \cap \acute{A_m}$, where the $\acute{A_i}$ have distinct associated intuitionistic fuzzy prime ideals. If $\acute{A_i} \supseteq \bigcap_{j=1,j\neq i}^{m} \acute{A_j}$, for some i, then $\acute{A_i}$ is deleted. Therefore A has a normal intuitionistic fuzzy primary decomposition. □

Theorem 3.6. *Let $A \in IFM(M)$. Suppose that A has an intuitionistic fuzzy primary decomposition, $A = \cap_{i=1}^{n} A_i$. Then $A_* = \cap_{i=1}^{n} (A_i)_*$ is a primary decomposition of A_*, and if it is normal , then the decomposition of $A = \cap_{i=1}^{n} A_i$ is normal.*

Proof. It follows from Theorem (3.5) and Theorem (2.22). □

Theorem 3.7. *Let $A \in IFM(M)$. Suppose that $A = \cap_{i=1}^{n} A_i$ is a normal intuitionistic fuzzy primary decomposition of A. Then there exists a finite set $\{\sqrt{(A_i : \chi_M)} | i = 1, 2, ..., m\}$, $m \leq n$, where the $\sqrt{(A_i : \chi_M)}$ are minimal in the set of associated intuitionistic fuzzy prime ideals of $A = \cap_{i=1}^{n} A_i$, such that $\sqrt{(A : \chi_M)} = \bigcap_{i=1}^{m} \sqrt{(A_i : \chi_M)}$ and $(A : (\bigcup_{i=1}^{m} \sqrt{(A_i : \chi_M)})) = A$ when $m \geq 2$.*

Proof. Suppose that $A = \cap_{i=1}^{n} A_i$ is a normal intuitionistic fuzzy primary decomposition of A. Then, by Theorem (2.7)

$$\sqrt{(A : \chi_M)} = (\sqrt{\bigcap_{i=1}^{n} A_i : \chi_M}) = \bigcap_{i=1}^{n} \sqrt{(A_i : \chi_M)}$$

Let C be an intuitionistic fuzzy prime ideal of R such that $C \supseteq \sqrt{(A : \chi_M)}$. Then
$C \supseteq \bigcap_{i=1}^{n} \sqrt{(A_i : \chi_M)} \supseteq \sqrt{(A_1 : \chi_M)}\sqrt{(A_2 : \chi_M)}..(\sqrt{A_n : \chi_M})$, by Theorem (3.18) of [5]. So $C \supseteq \sqrt{(A_i : \chi_M)}$ for some i. Thus C contains some $\sqrt{(A_i : \chi_M)}$ that is minimal among $\sqrt{(A_1 : \chi_M)}, \sqrt{(A_2 : \chi_M)}, ..., \sqrt{(A_n : \chi_M)}$.
Hence if we select those $\sqrt{(A_i : \chi_M)}$ in $\{\sqrt{(A_1 : \chi_M)}, \sqrt{(A_2 : \chi_M)}, ..., \sqrt{(A_n : \chi_M)}\}$ that are minimal and reindex, then we have

$$\sqrt{(A : \chi_M)} = \bigcap_{i=1}^{m} \sqrt{(A_i : \chi_M)}.$$

If $m \geq 2$, then $(A : \bigcup_{i=1}^{m} \sqrt{(A_i : \chi_M)}) = \bigcap_{i=1}^{m} \sqrt{(A_i : (A_i : \chi_M))}$, by Theorem (2.10)

As $\sqrt{(A_i : \chi_M)} \nsubseteq \sqrt{(A : \chi_M)} = \bigcap_{i=1}^{m} (A_i : \sqrt{(A_i : \chi_M)})$, $(A : \sqrt{(A_i : \chi_M)}) = A, \forall i \in \{1, 2, ..., n\}$, by Theorem (2.17)

Hence $A = (A : \bigcup_{i=1}^{m} \sqrt{(A_i : \chi_M)})$. □

Theorem 3.8. *Let $A = \cap_{i=1}^{n} A_i$ be a normal intuitionistic fuzzy prime decomposition of A and A_i be isolated intuitionistic fuzzy $\sqrt{(A_i : \chi_M)}$-primary submodules of M. Then*

$$A = (A : \bigcap_{j=1,j\neq i}^{n} \sqrt{(A_j : \chi_M)}), \forall i = 1, 2, ..., n.$$

Proof. Since
$\sqrt{(A_1 : \chi_M)} ... \sqrt{(A_{i-1} : \chi_M)} \sqrt{(A_{i+1} : \chi_M)} ... \sqrt{(A_n : \chi_M)} \subseteq \bigcap_{j=1,j\neq i}^{n} \sqrt{(A_j : \chi_M)}$, it follows from the minimality of $\sqrt{(A_i : \chi_M)}$ that $\bigcap_{j=1,j\neq i}^{n} \sqrt{(A_j : \chi_M)} \nsubseteq \sqrt{(A_i : \chi_M)}$ and hence

$$\bigcap_{j=1,j\neq i}^{n} \sqrt{(A_j : \chi_M)} \nsubseteq \bigcap_{j=1}^{n} \sqrt{(A_j : \chi_M)} = \sqrt{(A : \chi_M)}$$

Thus by Theorem (2.16), we have
$(A : \bigcap_{j=1,j\neq i}^{n} \sqrt{(A_j : \chi_M)}) = A, \forall i = 1, 2, ..., n.$ □

Example 3.9. *Let G be a finite group of order $n = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$, where p_i are distinct primes. Then by the structure theorem of finitely generated group we have $G \cong Z_{p_1^{n_1}} \oplus Z_{p_2^{n_2}} \oplus ..... \oplus Z_{p_k^{n_k}}$. Take $M = G$, then M is a $\mathbb{Z}$-module. Let $M = < x_1, x_2, ..., x_k >$ such that $o(x_i) = p_i^{n_i}$, for $1 \leq i \leq k$. Let $M_0 = < 0 >$, $M_1 = < x_1 >$, $M_2 = < x_1, x_2 >$,........,$M_k = < x_1, x_2, ..., x_k > = M$ be the chain of maximal submodules of M such that $M_0 \subset M_1 \subset .... \subset M_{k-1} \subset M_k$.*
Let A be any intuitionistic fuzzy submodule of M defined by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in M_0 \\ \alpha_1 & \text{if } x \in M_1 \backslash M_0 \\ \alpha_2 & \text{if } x \in M_2 \backslash M_1 \\ .... & \\ \alpha_k & \text{if } x \in M_k \backslash M_{k-1} \end{cases}$$

$$g_A(x) = \begin{cases} 0 & \text{if } x \in M_0 \\ \beta_1 & \text{if } x \in M_1 \backslash M_0 \\ \beta_2 & \text{if } x \in M_2 \backslash M_1 \\ \dots\dots\dots & \\ \beta_k & \text{if } x \in M_k \backslash M_{k-1}. \end{cases}$$

where $1 = \alpha_0 \geq \alpha_1 \geq \ldots.. \geq \alpha_k$ *and* $0 = \beta_0 \leq \beta_1 \leq \ldots. \leq \beta_k$ *and the pair* (α_i, β_i) *are called double pins and the set* $\wedge(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots\ldots, (\alpha_k, \beta_k)\}$ *is called the set of double pinned flags for the IFSM A of M.*

Define IFSs A_i *on M as follows:*

$$f_{A_i}(x) = \begin{cases} 1, & \text{if } x \in < p_i^{n_i} > \\ \alpha_{i+1}, & \text{if otherwise} \end{cases};$$

$$g_{A_i}(x) = \begin{cases} 0, & \text{if } x \in < p_i^{n_i} > \\ \beta_{i+1}, & \text{otherwise}. \end{cases}$$

where $\alpha_i, \beta_i \in (0,1)$ *such that* $\alpha_i + \beta_i \leq 1$, *for* $1 \leq i \leq k$ *and* $\alpha_{k+1} = \alpha_1$, $\beta_{k+1} = \beta_1$. *Clearly,* A_i *are intuitionistic fuzzy primary submodules of M. It can be easily checked that* $A = \cap_{i=1}^{n} A_i$ *is an intuitionistic fuzzy primary decomposition of A.*

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