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Generalized class of k-starlike functions of order α related to a quantum calculus operator

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Abstract

Post quantum calculus or (p,q)-calculus is the generalisation of the quantum calculus(q-calculus). In this paper we define Rucheweyh post-quantum differential operator $R^{\delta}_{p,q}$ and the subclass $k-S^{*}(\alpha,\delta,p,q)$ using the operator $R^{\delta}_{p,q}$. We prove necessary and sufficient condition for a function to be in the subclass, convolution condition, discuss interesting properties such as sharp coefficient bounds and solve Fekete Szego problem.

Keywords

Starlike, quantum calculus, differential operator, Rucheweyh operator.

AMS Subject Classification

30C45, 30C50.

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1. Introduction

Quantum calculus is the approach similar to the idea of deriving the q-analog in the usual calculus, but without the use of limit. At recent times, there has been a spurt of activities in Geometric Function Theory using q-calculus techniques. Kanas and Dorina in [10] introduced and studied a class of k-starlike functions using the q-calculus operator. Around 1991 Chakrabarti and Jaganathan [4], Brodimas *et al.* [3], Wachs And White [15] and Arik *et al.* [1] separately studied the(p,q)-calculus using the (p,q)-numbers, with two independent numbers p and q. For the basic ideas and results on q-differential calculus we refer Jackson F.H [5] and [6]. We in this paper, motivated by works of earlier authors, introduce and study a generalised class applying post quantum differential operator and prove many interesting results. We consider p and q to be in $(0,1)$ such that both are not simultaneously equal. Denote by A the family of regular functions defined in the unit disk $\Delta := \{z\in\mathbb{C}/|z|<1\}$ with the series expansion

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

normalised by the conditions $f(0)=f'(0)-1=0$ and let S denote the class of univalent functions in A. Let γ be a positively oriented circular arc contained in Δ with center $\xi\in\Delta$. Then $f\in A$ is said to be unifomly convex(UCV) if f maps γ univalently onto a convex arc and f is said to be uniformly starlike(UST) if $f(\gamma)$ is starlike with respect to $f(\xi)$. In 1992, Ma and Minda [11] gave the following one variable characterization for the class UCV, whereas the one variable analytic characterization for UST is still open. If

$$\Re\{1+\frac{zf''(z)}{f'(z)}\}>|\frac{zf''(z)}{f'(z)}|$$

for $f(z)\in A$ and $z\in\Delta$, then $f\in UCV$. Ronning in [12] independent of Ma and Minda gave the single variable analytic characterization of UCV and using the well known Alexander relation Ronning characterised the parabolic starlike function S_p which satisfies the inequality

$$\Re\{\frac{zf'(z)}{f(z)}\}>|\frac{zf'(z)}{f(z)}-1|$$

for $f(z)\in A$ and $z\in\Delta$. Kanas and Wisniowska in [7] extended the class UCV to the k-Uniformly convex functions denoted by k-UCV and proved analytic charecterisation for the class k-UCV. A function $f \in A$ is said to be k-UCV in Δ if the image of every positively oriented circular arc of the form $\{z \in \Delta / |z-\xi| = r\}$ with $\xi \in \Delta$ and $0 \leq |\xi| \leq k$, is mapped univalently onto a convex arc by f. An analytic characterization for the members of k-UCV is given in [7] as,

$$\Re\{1+\frac{zf''(z)}{f'(z)}\} > k\left|\frac{zf''(z)}{f'(z)}\right|$$

for $f(z) \in A$, $z \in \Delta$ and $0 \leq k < \infty$. In [8], Kanas and Wisniowska introduced a class of k-starlike functions denoted by k-ST using the Alexander relation. Such a class consists of functions $f(z) \in A$ satisfing inequality

$$\Re\{\frac{zf'(z)}{f(z)}\} > k\left|\frac{zf''(z)}{f'(z)}-1\right|$$

for $z \in \Delta$. Note that when $k=1$, k-ST=ST. k-ST can be further generalized as follows [2]. A function $f(z) \in A$ is said to be in the class $ST(k,\alpha)$ of k-starlike functions of order α, $0 \leq \alpha < 1$, if

$$\Re\{\frac{zf'(z)}{f(z)}\} > k\left|\frac{zf''(z)}{f'(z)}-1\right|+\alpha \quad (k \geq 0 \; and \; z \in \Delta).$$

Denote by $\mathscr{P}$ the class of normalized Caratheodory functions and denote by $\Omega_{k,\alpha}$ the following conic domain,

$$\Omega_{k,\alpha} = \{w = u+iv : u > k\sqrt{(u-1)^2+v^2}+\alpha\} \quad (1.2)$$

where $0 \leq k < \infty$ and $0 \leq \alpha < 1$. The domain $\Omega_{k,\alpha}$ is convex and symmetric about the real axis and $1 \in \Omega_{k,\alpha}$ for all k. For $k=0$, $\Omega_{k,\alpha}$ is the right half plane $\Re(w) > \alpha$, for $k=1$, the domain is an unbounded parabola, for $0<k<1$, the domain is a hyperbola and for $k>1$, the domain is a bounded portion, the interior of the ellipse. We denote by $\mathscr{P}(p_{k,\alpha})$ the following class:

$$\mathscr{P}(p_{k,\alpha}) = \{p \in \mathscr{P} : p(\Delta) \subset \Omega_{k,\alpha}\}.$$

The extremal function of the above class is given with slight modification in [9] as follows

$$p_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z} & , if \; k=0, \\ 1+\frac{2(1-\alpha)}{\pi^2}(\Theta)^2 & , if \; k=1, \\ \frac{1-\alpha}{1-k^2}cos(A(k)i\Theta)-\frac{k^2-\alpha}{1-k^2} & , if \; k \in (0,1), \\ \frac{1-\alpha}{k^2-1}sin(\frac{\pi}{2K(t)}\int_0^{\frac{u(z)}{\sqrt{t}}} Y dx)+\frac{k^2-\alpha}{k^2-1} & , if, k>1. \end{cases}$$

with $A(k) = \frac{2}{\pi}arccosk$, $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $\Theta = log\frac{1+\sqrt{z}}{1-\sqrt{(z)}}$ and $Y = \frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}$ $(0<t<1, z \in \Delta)$, where t is chosen such that $k = cosh\frac{\pi K'(t)}{4K(t)}$, and $K(t)$ is Legendre's complete elliptic integral of first kind and $K'(t)$ is complementary integral of $K(t)$. The series expansion of $p_{k,\alpha}$ is given by

$$p_{k,\alpha}(z) = 1+P_1z+P_2z^2+...$$

where $P_i = P_i(k,\alpha)$.

The (p,q) analog of the number k is defined as

$$[k]_{p,q} = \frac{p^k-q^k}{p-q} \; for \; p \neq q.$$

Then $[k]_{1,q} = \frac{1-q^k}{1-q}$, which is the q integer number k and $\lim_{q\to 1}[k]_{1,q} = k$, the ordinary integer k. For our alleviation we use the notation Υ_k instead of $[k]_{p,q}$ throughout this paper.

Definition 1.1. *[5] The (p,q) derivative of a function $f(z)$ with respect to z denoted by $D_{p,q}f(z)$ is defined as*

$$D_{p,q}f(z) = \frac{f(pz)-f(qz)}{(p-q)z} \quad (z \neq 0, p \neq q)$$

and $(D_{p,q}f)(0) = f'(0)$, provided that $f(z)$ is differentiable at 0.

In particular if $f(z) \in A$, then $(D_{p,q}f)(0) = f'(0) = 1$. Note that $D_{1,q}$ is the q-derivative operator defined in [10]. Also it can be easily seen that the operator $D_{p,q}$ operator is a linear operator.

Example 1.2. *Let $f(z) = \frac{1+z}{1-z}$. Then*

$$D_{p,q}f(z) = \frac{2}{(1-pz)(1-qz)}$$

$$D_{1,q}f(z) = \frac{2}{(1-qz)^2} = D_qf(z)$$

and

$$\lim_{q\to 1} D_{1,q}f(z) = \frac{2}{(1-z)^2} = f'(z).$$

For $f(z)$ of the form (1.1)

$$D_{p,q}f(z) = 1+\sum_{n=1}^{\infty} a_n[n]_{p,q}z^{n-1}.$$

The (p,q)-gamma function is defined as $\Gamma_{p,q}(n+1) = [n]_{p,q}!$ and the generalised (p,q)-Pochhamer symbol is defined as

$$[t]_n = \begin{cases} 1 & , if \; n=0 \\ [t]_{p,q}[t+1]_{p,q}...[t+n-1]_{p,q} & , if \; n \neq 0 \end{cases}$$

2. Preliminaries

We need the following results to prove our main result.

Lemma 2.1. *[10] If $q(z) = 1+q_1z+q_2z^2+...$ is an analytic function with positive real part in Δ, then*

$$\left|q_2-\mu q_1^2\right| \leq 2max\{1; \left|2\mu-1\right|\}.$$

Lemma 2.2. *[9] If $q(z) = 1+q_1z+q_2z^2+... \in \mathscr{P}(p_{k,\alpha})$ is an analytic function in Δ, then*

$$\left|q_2-\mu q_1^2\right| \leq \begin{cases} P_1-\mu P_1^2 & if \; \mu \leq 0, \\ P_1 & if \; 0<\mu<1, \\ P_1+(\mu-1)P_1^2 & if \; \mu \geq 1. \end{cases}$$

Lemma 2.3. *[9] Let $0 \leq k < \infty$ be fixed and $0 \leq \alpha < 1$. If a function $q(z) = 1 + q_1 z + q_2 z^2 + ... \in \mathscr{P}(p_{k,\alpha})$, then*

$$\left|q_1^2 - q_2\right| \leq P_1.$$

Definition 2.4. *For $f(z) \in A$, the generalised Ruscheweyh-(p,q) differential operator is defined as,*

$$R^{\delta}_{p,q} f(z) = f(z) * F_{p,q,\delta+1}(z) \quad (z \in \Delta, \delta > -1) \tag{2.1}$$

where

$$\begin{aligned} F_{p,q,\delta+1}(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_{p,q}(n+\delta)}{[n-1]_{p,q}!\Gamma_{p,q}(1+\delta)} z^n \\ &= z + \sum_{n=2}^{\infty} \frac{[\delta+1]_{n-1}}{[n-1]_{p,q}!} z^n \end{aligned} \tag{2.2}$$

The symbol $*$ stands for convolution. As $p \to q$ and $q \to 1$, the $R^{\delta}_{p,q} f(z)$ is the Ruscheweyh derivative operator defined by Ruscheweyh in [13]. As $p \to 1$, $R^{\delta}_{p,q} f(z)$ reduces to the $R^{\lambda}_{q} f(z)$ as in [7]. From (2.1) we can see that, $D^0_{p,q} f(z) = f(z)$, $D^1_{p,q} f(z) = zD_{p,q} f(z)$, . . . and $D^m_{p,q} f(z) = \frac{zD^m_{p,q}(z^{m-1} f(z))}{[m]_{p,q}!}$ for $m \in \mathbb{N}$. The power series for $R^{\delta}_{p,q} f(z)$ is given by

$$R^{\delta}_{p,q} f(z) = z + \sum_{n=2}^{\infty} \frac{[\delta+1]_{n-1}}{[n-1]_{p,q}!} a_n z^n$$

using (2.1) and (2.2). We can easily check that

$$zD_{p,q}(F_{p,q,\delta+1}(z)) = (1+A)F_{p,q,\delta+2}(z) - AF_{p,q,\delta+1}(z) \quad z \in \Delta \tag{2.3}$$

where $A = pq[\delta]_{p,q} \frac{[n-1]_{p,q}}{[\delta+n]_{p,q} - [\delta+1]_{p,q}}$.
Also making use of Hadamard product we obtain

$$zD_{p,q}(R^{\delta}_{p,q} f(z)) = (1+A)R^{\delta+1}_{p,q} f(z) - AR^{\delta}_{p,q} f(z) \quad z \in \Delta$$

where $A = pq[\delta]_{p,q} \frac{[n-1]_{p,q}}{[\delta+n]_{p,q} - [\delta+1]_{p,q}}$. As $p \to q$ and $q \to 1$ the above equality reduces to the well known recurrent formula for the Ruscheweyh differential operator. Now we define the following subclass as it specify the regions for various values of k as in [9].

Definition 2.5. *Let $0 \leq \alpha < 1$, $k \geq 0$ and $\delta > -1$. Then $f(z) \in A$ is said to be in the class $k - S^*(\alpha, \delta, p, q)$ if*

$$\Re\left\{\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)}\right\} > k\left|\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} - 1\right| + \alpha \tag{2.4}$$

As $p \to q$ $k - S^*(\alpha, \delta, p, q)$ reduces to $ST(k, \alpha, \delta, q)$ and as $p \to q$ and $q \to 1$, $k - S^*(\alpha, \delta, p, q)$ reduces to the class $ST(k, \alpha)$.

3. Properties of the class $k - S^*(\alpha, \delta, p, q)$

The following theorem provides the necessary and sufficient condition for $f(z)$ to be in $k - S^*(\alpha, \delta, p, q)$.

Theorem 3.1. *Let $f(z) \in A$ be given by (1.1).Then $f \in k - S^*(\alpha, \delta, p, q)$ if and only if the inequality*

$$\sum_{n=2}^{\infty} ([n]_{p,q}(k+1) - k - \alpha) \frac{\Gamma_{p,q}(n+\delta)}{[n-1]!\Gamma_{p,q}(1+\delta)} \left|a_n\right| \leq 1 - \alpha \tag{3.1}$$

holds true for some k $(0 \leq k < \infty)$, $\delta > -1$ and α $(0 \leq \alpha < 1)$. The inequality is sharp for the function

$$f_n(z) = z - \frac{(1-\alpha)[n-1]!\Gamma p,q(1+\delta)}{[n](k+1) - k - \alpha\Gamma_{p,q}(n+\delta)} z^n. \tag{3.2}$$

Proof. From (2.4), it is enough to prove that

$$k\left|\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} - 1\right| - \Re\left\{\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} - 1\right\} < 1 - \alpha$$

Now consider,
$k\left|\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} - 1\right| - \Re\left\{\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} - 1\right\}$

$$\begin{aligned} &\leq (k+1)\left|\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} - 1\right| \\ &= (k+1)\left|\frac{\sum_{n=2}^{\infty} ([n]_{p,q} - 1) \frac{\Gamma_{p,q}(n+\delta)}{[n-1]!\Gamma_{p,q}(n+1)} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \frac{\Gamma_{p,q}(n+\delta)}{[n-1]!\Gamma_{p,q}(n+1)} a_n z^{n-1}}\right| \\ &< (k+1) \frac{\sum_{n=2}^{\infty} ([n]_{p,q} - 1) \frac{\Gamma_{p,q}(n+\delta)}{[n-1]!\Gamma_{p,q}(n+1)} \left|a_n\right|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma_{p,q}(n+\delta)}{[n-1]!\Gamma_{p,q}(n+1)} \left|a_n\right|} \end{aligned}$$

The last equation is bounded by $1 - \alpha$ only if the inequality (3.1) holds. We can easily verify that the result is sharp for the functions given in (3.2). Now we have to prove that the function $f_n(z) \in k - S^*(\alpha, \delta, p, q)$. Consider,

$$\begin{aligned} k\left|\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} - 1\right| &= k\left|\frac{(1-\alpha)(1-[n]_{p,q})z^{n-1}}{(\Lambda) - (1-\alpha)z^{n-1}}\right| \\ &< \frac{k(1-\alpha)}{k+1} \end{aligned}$$

and

$$\begin{aligned} \Re\left\{\frac{zD_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)}\right\} &= \Re\left\{\frac{\Lambda - [n]_{p,q}(1-\alpha)z^{n-1}}{\Lambda - (1-\alpha)z^{n-1}}\right\} \\ &> \frac{k+\alpha}{k+1} \end{aligned}$$

where $\Lambda = [n]_{p,q}(k+1) - k - \alpha$, the condition (2.4) holds true for $f_n(z)$. Thus $f_n \in k - S^*(\alpha, \delta, p, q)$. ☐

Corollary 3.2. *Let* $f(z) = z + a_n z^n$. *If*

$$a_n \leq \frac{(1-\alpha)[n-1]!\Gamma p,q(1+\delta)}{[n](k+1)-k-\alpha\Gamma_{p,q}(n+\delta)} \quad (n \geq 2),$$

then $f \in k - S^*(\alpha,\delta,p,q)$.

Now consider $p(z) = zD_{p,q}(R^{\delta}_{p,q}f(z))/R^{\delta}_{p,q}f(z)$. We can rewrite (2.4) as

$$Rep(z) > k\left|p(z)-1\right| + \alpha. \tag{3.3}$$

Then the range of $p(z)$ for $z \in \Delta$ is the conic domain(1.2) and $\partial\Omega_{k,\alpha}$ is a curve defined by

$$\partial\Omega_{k,\alpha} = \{w = u + iv/(u-\alpha)^2 = k^2(u-1)^2 + k^2v^2\}.$$

From (2.4) and (3.3) we obtain that

$$\frac{zD_{p,q}(R^{\delta}_{p,q}f(z))}{R^{\delta}_{p,q}} \in \Omega_{k,\alpha}. \tag{3.4}$$

Using the properties of the domain $\Omega_{k,\alpha}$ and (3.4) it follows that if $f \in k - S^*(\alpha,\delta,p,q)$, then

$$\frac{zD_{p,q}(R^{\delta}_{p,q}f(z))}{R^{\delta}_{p,q}} < \frac{k+\alpha}{k+1} \quad (z \in \Delta)$$

and

$$\left|Arg\frac{zD_{p,q}(R^{\delta}_{p,q}f(z))}{R^{\delta}_{p,q}}\right| \leq \begin{cases} arctan\frac{1-\alpha}{\sqrt{|k^2-\alpha^2|}} & ,\alpha \in I, k > 0 \\ \frac{\pi}{2} & ,k = 0 \end{cases}$$

where $I = [0,1)$. Let $f_{k,\alpha} = z + A_2z^2 + A_3z^3 + ...$ be the extremal function in the class $k - S^*(\alpha,\delta,p,q)$. Then the relation between the extremal functions in the classes $\mathscr{P}(p_{k,\alpha})$ and $k - S^*(\alpha,\delta,p,q)$ is given by

$$p_{k,\alpha}(z) = \frac{zD_{p,q}(R^{\delta}_{p,q}f(z))}{R^{\delta}_{p,q}f(z)} \quad (z \in \Delta). \tag{3.5}$$

Making use of (2.4), (3.3) and (3.5) we obtain for $p_{k,\alpha}(z)$ the following coefficient relation

$$\frac{[\delta+1]_{n-1}([n]_{p,q}-1)A_n}{[n-1]_{p,q}!} = \sum_{m=1}^{n-1}\frac{[\delta+1]_{m-1}A_nP_{n-m}}{[m-1]_{p,q}!}, \ A_1 = 1.$$

In particular, we get

$$\begin{aligned} A_2 &= \frac{P_1}{[1+\delta]_{p,q}([2]_{p,q}-1)} \ \textit{and} \\ A_3 &= \frac{[2]_{p,q}!\{([2]_{p,q}-1)P_2+P_1^2\}}{[1+\delta]_{p,q}[2+\delta]_{p,q}([3]_{p,q}-1)([2]_{p,q}-1)} \end{aligned} \tag{3.6}$$

A_n are nonnegative, since $\delta > -1$, p *and* $q \in (0,1)$ and P_n's are nonnegative.

Theorem 3.3. *Let* $k \in [0,\infty)$ *and* $\alpha \in [0,1)$. *If* $f(z)$ *is of the form (1.1) belongs to the class* $k - S^*(\alpha,\delta,p,q)$, *then*

$$\left|a_2\right| \leq A_2 \textit{ and } \left|a_3\right| \leq A_3.$$

Proof. Let $p(z) = zD_{p,q}(R^{\delta}_{p,q}f(z))/R^{\delta}_{p,q}f(z)$. Using the relation (2.3) and for $p(z) = 1 + p_1z + p_2z^2 + ...$, we have

$$\frac{[\delta+1]_{p,q}([n]_{p,q}-1)}{[n-1]_{p,q}!}a_n = \sum_{m=1}^{n-1}\frac{[\delta+1]_{m-1}}{[m-1]_{p,q}!}a_np_{n-m}. \tag{3.7}$$

Since $p_{k,\alpha}$ is univalent in Δ, the function

$$s(z) = \frac{1+p^{-1}_{k,\alpha}(p(z))}{1-p^{-1}_{k,\alpha}(p(z))} = 1 + c_1z + c_2z^2 + ...$$

is analytic in Δ and $Re\ q(z) > 0$. From

$$\begin{aligned} p(z) &= p_{k,\alpha}(\frac{s(z)-1}{s(z)+1}) \\ &= 1 + \frac{1}{2}c_1P_1z + (\frac{1}{2}c_2p_1 + \frac{1}{4}c_1^2(P_2-P_1))z^2 + \\ &\quad (\frac{1}{2}P_1c_3 + c_1c_2(\frac{P_2}{2}-P_1) + \frac{1}{4}c_1^3(\frac{P_3}{2}-P_2+\frac{P_1}{2}))z^3 \\ &\quad + ... \end{aligned}$$

we have

$$a_2 = \frac{c_1p_1}{2[\delta+1]_{p,q}([2]_{p,q}-1)} \tag{3.8}$$

and

$$\left|a_2\right| = \frac{\left|c_1p_1\right|}{2[\delta+1]_{p,q}([2]_{p,q}-1)} \leq \frac{P_1}{[\delta+1]_{p,q}([2]_{p,q}-1)} = A_2$$

using upon (3.6) and $\left|c_n\right| \leq 2$. We consider a_3 and use lemma.2.3 to get

$$a_3 = \frac{[2]_{p,q}![p_2 + \frac{p_1^2}{([2]_{p,q}-1)}]}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)} \tag{3.9}$$

and

$$\begin{aligned} \left|a_3\right| &\leq \frac{[2]_{p,q}!\{[([2]_{p,q}-1)\left|p_2-p_1^2\right|]-([2]-2)\left|p_1^2\right|\}}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)([2]_{p,q}-1)} \\ &\leq \frac{[2]_{p,q}!\{[([2]_{p,q}-1)(P_2-P_1^2)]-([2]-2)P_1^2\}}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)([2]_{p,q}-1)} \\ &= \frac{[2]_{p,q}!\{[([2]_{p,q}-1)P_2+P_1^2]\}}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)([2]_{p,q}-1)} = A_3 \end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let* $0 \leq k < \infty$, $\delta > -1$, *and* $\alpha \in [0,1)$. *If* $f(z)$ *of the form (1.1) belongs to the class* $k - S^*(\alpha,\delta,p,q)$, *then*

$$\left|a_n\right| \leq \frac{[n-1]_{p,q}!P_1(([2]_{p,q}-1)+P_1)...(([n-1]_{p,q}-1)+P_1)}{[\delta+1]_{n-2}([n]_{p,q}-1)...([2]_{p,q}-1)} \tag{3.10}$$

Proof. We prove this result using induction on n. The result is clearly true for n=2. Let n be any integer number with $n \geq 2$, and assume that the inequality is true for all $k \leq n-1$. Making use of (3.7), we have

$$\begin{aligned} a_n &= \frac{[n-1]_{p,q}!}{[\delta+1]_{n-1}([n]_{p,q}-1)}\{p_{n-1} \\ &\quad + \sum_{m=2}^{n-1}\frac{[\delta+1]_{m-1}}{[n-1]_{p,q}!}a_m p_{n-m}\} \\ |a_n| &\leq \frac{[n-1]_{p,q}!}{[\delta+1]_{n-1}([n]_{p,q}-1)}\{P_1 \\ &\quad + \sum_{m=2}^{n-1}\frac{[\delta+1]_{m-1}}{[n-1]_{p,q}!}|a_m|P_1\} \\ &\leq \frac{[n-1]_{p,q}!}{[\delta+1]_{n-1}([n]_{p,q}-1)}P_1\{1+\sum_{m=2}^{n-1}\frac{[\delta+1]_{m-1}}{[n-1]_{p,q}!}\times \\ &\quad \frac{[m-2]_{p,q}!P_1(([2]_{p,q}-1)+P_1)...((\Omega)+P_1)}{[\delta+1]_{m-2}([m]_{p,q}-1)...([2]_{p,q}-1)}\} \end{aligned}$$

where $\Omega = [m-1]_{p,q}-1$, using the induction hypothesis and $|p_n| \leq P_1$. Again applying mathematical induction, we find

$$\begin{aligned} 1 &+ \sum_{m=2}^{n-1}\frac{[m-2]_{p,q}![\delta+m-1]_{m-1}}{[n-1]_{p,q}!} \\ &\times\frac{P_1(([2]_{p,q}-1)+P_1)...(([m-1]_{p,q}-1)+P_1)}{[\delta+1]_{m-2}([m]_{p,q}-1)...([2]_{p,q}-1)} \\ &= \frac{(([2]_{p,q}-1)+P_1)...(([n-2]_{p,q}-1)+P_1)}{([n-1]_{p,q}-1)...([2]_{p,q}-1)} \end{aligned}$$

implies the inequality (3.10). □

4. Fekete-Szego probelm

Theorem 4.1. *Let $k \in [0,\infty)$, $\delta \geq -1$ and $\alpha \in [0,1)$. For $f(z) \in k-S^*(\alpha,\delta,p,q)$ of the form (1.1)*

$$\begin{aligned} |a_3-\mu a_2^2| &\leq \frac{2[2]_{p,q}!}{[\delta+1]_{p,q}([3]_{p,q}-1)}\times max\{1; \\ &\quad |\frac{2\mu([3]_{p,q}-1)[\delta+2]_{p,q}}{[2]_{p,q}![\delta+1_{p,q}]([2]_{p,q}-1)^2}-\kappa|\}. \end{aligned} \tag{4.1}$$

For a real parameter μ, we get

$$|a_3-\mu a_2^2| \leq \begin{cases} P_1-\Delta\{\frac{C}{[2]_{p,q}!D}-1\}P_1^2 & ,\mu \leq \frac{[2]_{p,q}!}{\rho} \\ P_1 & ,\mu \in X \\ P_1-\Delta\{\frac{C}{[2]_{p,q}!D}-[2]_{p,q}\}P_1^2 & ,\mu \geq \frac{[2]_{p,q}^2!}{\rho}. \end{cases} \tag{4.2}$$

where $\Delta = \frac{1}{([2]_{p,q}-1)}$, $\rho = \frac{D}{[\delta+2]_{p,q}([3]_{p,q}-1)}$, $\kappa = \frac{([2]_{p,q}+1)}{([2]_{p,q}-1)}$, $B = \frac{1}{([2]_{p,q}-1)![\delta+1_{p,q}]([2]_{p,q}-1)^2}$, $C = \mu([3]_{p,q}-1)[\delta+2]_{p,q}$, $D = [\delta+1_{p,q}]([2]_{p,q}-1)$ *and* $X = B(C-([2]_{p,q}-1)!D, C-([2]_{p,q}-1)!^2D)$

Proof. Let μ be complex. From (3.8) and (3.9) we have

$$\begin{aligned} |a_3-\mu a_2^2| &= \frac{[2]_{p,q}}{[\delta+1]_2([3]_{p,q}-1)}|p_2-\frac{p_1^2}{([2]_{p,q}-1)} \\ &\quad (\frac{\mu([3]_{p,q}-1)[\delta+2]_{p,q}}{[2]_{p,q}![\delta+1]_{p,q}([2]_{p,q}-1)}-1) \end{aligned}$$

using lemma.2.1 we arrive at the inequality (4.1). Now making μ real and using lemma.2.2 we have the inequality (4.2). □

A necessary and sufficient condition for a function $f(z) \in A$ to be in the class $S^*(k,\alpha,\delta,p,q)$ using Hadamard product is given by

Theorem 4.2. *Let $0 \leq k < \infty$, $\delta > -1$ and $0 \leq \alpha < 1$. Then the function $f(z)$ belongs to the class $k-S^*(\alpha,\delta,p,q)$ if and only if $(f*H_{p,q,\delta}(z)/z) \neq 0$ in Δ where*

$$H_{p,q,\delta} = F_{p,q,\delta+2}(z)\{1-(1-\frac{F_{p,q,\delta+1}(z)}{F_{p,q,\delta+2}(z)})(\frac{w(t)+A}{w(t)-1})\} \tag{4.3}$$

where $A = \frac{[\delta+n]_{p,q}-[\delta+1]_{p,q}}{pq[n-1]_{p,q}}$, $t^2-(kt+\alpha-1)^2 > 0$ *and* $w(t) = (kt+\alpha) \pm i\sqrt{t^2-(kt+\alpha-1)^2}$

Proof. Let $f \in k-S^*(\alpha,\delta,p,q)$. Then by (3.4) we have

$$\frac{zD_{p,q}(R_{p,q}^{\delta}f(z))}{R_{p,q}^{\delta}} \in \Omega_{k,\alpha}$$

so that

$$\frac{zD_{p,q}(R_{p,q}^{\delta}f(z))}{R_{p,q}^{\delta}} \neq (kt+\alpha) \pm i\sqrt{t^2-(kt+\alpha-1)^2} = w(t) \tag{4.4}$$

where $z \in \Delta$ and $t^2-(kt+\alpha-1)^2 > 0$. Using (2.1) and the properties of Hadamard product, the condition (4.4) is satisfied only if

$$f(z)*[zD_{p,q}(F_{p,q,\delta+1}(z))-w(t)F+p,q,\delta+1(z)]/z \neq 0. \tag{4.5}$$

Making use of the recurrent formula for $zD_{p,q}(F_{p,q,\delta+1}(z))$ from (2.3), it follows from (4.5) that $(f*H_{p,q,\delta}(z)/z) \neq 0$ where $H_{p,q,\delta}(z)$ is given by (4.3). Conversely, assume that $(f*H_{p,q,\delta}(z)/z) \neq 0$ for $z \in \Delta$. Then the value of $\frac{zD_{p,q}(R_{p,q}^{\delta}f(z))}{R_{p,q}^{\delta}}$ lies completely inside $\Omega_{k,\alpha}$ or on its complement. But at $z = 0$ the value of $\frac{zD_{p,q}(R_{p,q}^{\delta}f(z))}{R_{p,q}^{\delta}}$ is $1 \in \Omega_{k,\alpha}$ and therefore $\frac{zD_{p,q}(R_{p,q}^{\delta}f(z))}{R_{p,q}^{\delta}} \in \Omega_{k,\alpha}$ impling that $f \in k-S^*(\alpha,\delta,p,q)$. □

Theorem 4.3. *Let $0 \leq k < \infty$, $\delta > -1$ and $0 \leq \alpha < 1$. The coefficients h_n of the function $H_{p,q,\delta}$ given by (4.3) satisfies the inequality*

$$|h_n| = \frac{[\delta+1]_{n-1}(1-\alpha+[n]_{p,q}(k+1))}{(1-\alpha)[n-1]_{p,q}!}, \quad n \geq 2. \quad (4.6)$$

Proof. From the series expansion of (4.3) we have

$$h_n = \frac{[\delta+1]_{n-1}}{[n-1]_{p,q}!}\frac{[n]_{p,q}-w(t)}{1-w(t)}$$

and hence

$$\begin{aligned} |h_n|^2 &= (\frac{[\delta+1]_{n-1}}{[n-1]_{p,q}!})^2(1-\frac{2k([n]_{p,q}-1)}{t}+([n]_{p,q}) \\ &\quad -1\frac{[n]_{p,q}+1-2\alpha}{t^2}) \\ &:= (\frac{[\delta+1]_{n-1}}{[n-1]_{p,q}!})^2\Phi(t). \end{aligned}$$

The function $\Phi(t)$ attain its minimum at $t=t_0$ where $t_0 = \frac{[n]_{p,q}+1-2\alpha}{k}$ and $\Phi(t)$ is decreasing in the interval $(\frac{1-\alpha}{1+k}, t_0)$ and increasing in the interval (t_0, ∞). As t becomes large, $\Phi(t)$ approaches 1 and

$$\begin{aligned} \Phi(\frac{1-\alpha}{k+1}) &= 1-2k([n]_{p,q}-1)\frac{1+k}{1-\alpha} \\ &\quad +([n]_{p,q}+1-2\alpha)\frac{(1+k)^2}{(1-\alpha)^2} \\ &\geq 1 \end{aligned}$$

So the maximum value of $\Phi(t)$ is attained at the point $\frac{1-\alpha}{k+1}$. But $\Phi(\frac{1-\alpha}{k+1}) \leq [\frac{1-\alpha+[n]_{p,q}(k+1)}{1-\alpha}]^2$ so the coefficients of $H_{p,q,\delta}$ satisfies the inequality (4.6). □

Corollary 4.4. *Let $h(z) = z + a_n z^n$. If*

$$|a_n| \leq \frac{(1-\alpha)[n-1]_{p,q}!}{[\delta+1]_{n-1}(1-\alpha+[n]_{p,q}(k+1))}$$

then $h \in k - S^*(\alpha, \delta, p, q)$.

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