Malaya Journal of Matematik, Vol. 8, No. 4, 1728-1734, 2020
https://doi.org/10.26637/MJM0804/0068

Eccentric domination number of some path related graphs

S. K. Vaidya[1]* and D. M. Vyas[2]

Abstract
In a graph G, a vertex u is said to be an eccentric vertex of a vertex v if $d(u,v)$ = eccentricity of vertex v. A dominating set D of a graph $G=(V,E)$ is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of the minimal eccentric dominating sets of graph G is said to be eccentric domination number of graph G which is denoted by $\gamma_{ed}(G)$. Here, exact value of $\gamma_{ed}(G)$ for some path related graphs, have been investigated.

Keywords
Dominating set, eccentric dominating set, eccentric domination number.

AMS Subject Classification
05C38, 05C69, 05C76.

[1] *Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India.*
[2] *Department of Mathematics, V.V.P. Engineering College, Rajkot-360005, Gujarat, India.*
***Corresponding author**: [1] samirkvaidya@yahoo.co.in; [2] dhavalvyas77@gmail.com
Article History: Received **23** July **2020**; Accepted **27** September **2020**

Contents

1. Introduction

The domination in graphs is one of the most emerging concepts within and out side of graph theory. It has attracted many researchers to work on it due to its diversified applications in various fields.

There are various domination models available in the literature. Total domination [3], equitable domination [7], global domination [10], steiner domination [9], independent domination [4], restrained domination [8], eccentric domination [6] are among worth to mention.

The present work is focused on eccentric domination of some path related graphs.

A graph $G=(V,E)$, we mean a finite, simple and connected graph with vertex set $V=\{v_1,v_2,v_3,\cdots,v_n\}$ and edge set E. For any graph theoretic terminology and notation we refer to West [13] while the terms related to domination are used in the sense of Haynes *et al.* [5].

Definition 1.1. A set $D \subseteq V$ of vertices in a graph $G=(V,E)$ is said to be a **dominating set** if every vertex in $V-D$ is adjacent to at least one vertex in D.

Definition 1.2. A dominating set D is said to be a **minimal dominating set** if no proper subset $D' \subset D$ is a dominating set. The set of all minimal dominating sets of a graph G is denoted by $MDS(G)$. The minimum cardinality of a set in $MDS(G)$ is called **domination number** of graph G and is denoted by $\gamma(G)$.

Definition 1.3. Let G be a connected graph and v be a vertex of G. The **eccentricity of** v is denoted by $e(v)$ is defined by $e(v)=max\{d(u,v): u \in V\}$.

The **radius of graph** G is defined as $rad(G)=min\{e(v): v \in V\}$ while the **diameter of graph** G is defined as $diam(G)=max\{e(v): v \in V\}$

In a graph G, a vertex u is said to be an eccentric vertex of a vertex v if $d(u,v)=e(v)$ = eccentricity of vertex v. The eccentric set of a vertex v is denoted by $E(v)$ and is defined as $E(v)=\{u \in V(G): d(u,v)=e(v)\}$.

Definition 1.4. A set $D \subseteq V(G)$ is an **eccentric dominating set of** G if D is a dominating set of G and for every vertex $v \in V-D$, there exists at least one eccentric vertex of v in D.

An eccentric dominating set D of graph G is a minimal eccentric dominating set if no proper subset $D' \subset D$ is an eccentric dominating set of graph G.

The cardinality of a minimal eccentric dominating set of a graph G is called eccentric domination number of G which is denoted as $\gamma_{ed}(G)$.

The concept of eccentric domination was introduced by Janakiraman *et al* [6].

Illustration 1.5. The set $D = \{v_1, v_2, v_7\}$ is an eccentric dominating set of the graph G given in Figure 1, which is also a minimal eccentric dominating set.

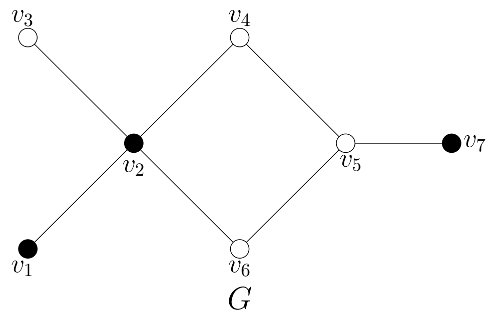

Figure 1. Minimal eccentric dominating set of graph G.

The sets $\{v_3, v_2, v_7\}$ and $\{v_1, v_3, v_4, v_6, v_7\}$ are also minimal eccentric dominating sets of graph G.

But the minimum cardinality of minimal eccentric dominating set is 3.

So, $\gamma_{ed}(G) = 3$

2. Some Definitions and Existing Results

Definition 2.1. [11] The m**-shadow graph** $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \cdots, G_m$, then join each vertex u in G_i to the neighbours of the corresponding vertex v in G_j, $1 \leq i, j \leq m$.

Definition 2.2. [12] The **extended** m**-shadow graph** $D^*_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \cdots, G_m$, then join each vertex u in G_i to the neighbours of the corresponding vertex v and with v in G_j, $1 \leq i, j \leq m$.

Definition 2.3. [11] The $m-$**splitting graph** $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G.

Motivating through above three concepts, we have introduced the concept of the extended $m-$splitting graph which is defined as follows.

Definition 2.4. The **extended** $m-$**splitting graph** $Spl^*_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G and also adjacent to v.

Definition 2.5. The **square of graph** G**, denoted by** G^2, is defined to be the graph with the same vertex set as G and in which two vertices u and v are joined by an edge if and only if in G we have $1 \leq d(u, v) \leq 2$.

Janakiraman *et al* [6] have proved the following result.

Theorem 2.6. [6] For any path P_n,

$$\gamma_{ed}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & ; n = 3k+1 \\ \left\lceil \frac{n}{3} \right\rceil + 1 & ; n = 3k \text{ or } n = 3k+2 \end{cases}$$

3. Main Results

We have improved the result of Theorem 2.6 as follows.

Theorem 3.1. For any path P_n, $\gamma_{ed}(P_n) = \begin{cases} 1 & ; n = 2 \\ \left\lceil \frac{n+2}{3} \right\rceil & ; n \geq 3 \end{cases}$

Proof: Let $V = V(P_n) = \{v_1, v_2, \cdots, v_n\}$ is the set of all vertices of path $P_n, \forall n \geq 2$ and let D is a minimal eccentric dominating set of P_n.

Case-1: $n = 2$
For path P_2, the set $D = \{v_1\}$ is a minimal eccentric dominating set of path P_2, as v_1 dominates the remaining vertex v_2 and v_1 is also the eccentric vertex of v_2.
Thus, $\gamma_{ed}(P_2) = 1$

Case-2: $n = 3$
For path P_3, the set $D = \{v_1, v_2\}$ is a minimal eccentric dominating set of path P_3, as v_2 dominates the remaining vertex v_3 and v_1 is the eccentric vertex of v_3.
Thus, $\gamma_{ed}(P_3) = 2 = \left\lceil \frac{5}{3} \right\rceil = \left\lceil \frac{3+2}{3} \right\rceil = \left\lceil \frac{n+2}{3} \right\rceil$

Case-3: $n \geq 4$
If n is an even number then the vertex v_1 is the only eccentric vertex of all vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \cdots, v_n$ and the vertex v_n is the only eccentric vertex of all vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}}$.
Similarly if n is an odd number then the vertex v_1 is the only eccentric vertex of all vertices $v_{\frac{n+1}{2}+1}, v_{\frac{n+1}{2}+2}, v_{\frac{n+1}{2}+3}, \cdots, v_n$ and the vertex v_n is the only eccentric vertex of all vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}-1}$. And v_1 and v_n both are eccentric vertices of the vertex $v_{\frac{n+1}{2}}$.
So, Both end vertices v_1 and v_n of path P_n must be included in D.
Thus, The vertices v_2 and v_{n-1} will be dominated by the vertices v_1 and v_n respectively.
Now, we know that for any path P_n, the domination number $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$.
So, we need to take the $\left\lceil \frac{n-4}{3} \right\rceil$ vertices from $v_3, v_4, v_5, \cdots,$ v_{n-2} to dominate the remaining $n-4$ vertices $v_3, v_4, v_5, \cdots,$

v_{n-2}, as it constructs the path P_{n-4} of $n-4$ vertices.
Then, D contains $\left\lceil \frac{n-4}{3} \right\rceil$ from $v_3, v_4, v_5, \cdots, v_{n-2}$ vertices and v_1 and v_n.
So, D contains $\left\lceil \frac{n-4}{3} \right\rceil + 2$ vertices.
Hence, $\gamma_{ed}(P_n) = |D| = \left\lceil \frac{n-4}{3} \right\rceil + 2 = \left\lceil \frac{n-4}{3} + 2 \right\rceil$
$= \left\lceil \frac{n-4+6}{3} \right\rceil = \left\lceil \frac{n+2}{3} \right\rceil$

Thus, $\gamma_{ed}(P_n) = \begin{cases} 1 & ; n = 2 \\ \left\lceil \frac{n+2}{3} \right\rceil & ; n \geq 3 \end{cases}$

Illustration 3.2. Minimal eccentric dominating set of path $P_{11} = \{v_1, v_4, v_7, v_{10}, v_{11}\}$, given in Figure 2.

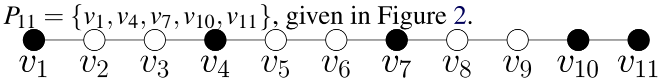

Figure 2. Minimal eccentric dominating set of path P_{11}

Where, $\gamma_{ed}(P_{11}) = 5 = \left\lceil \frac{13}{3} \right\rceil = \left\lceil \frac{11+2}{3} \right\rceil$

Theorem 3.3. Eccentric domination number of extended $m-$shadow graph of path P_n is same as eccentric domination number of path P_n.

i.e. $\gamma_{ed}[D^*_m(P_n)] = \gamma_{ed}(P_n) = \begin{cases} 1 & ; n = 2 \\ \left\lceil \frac{n+2}{3} \right\rceil & ; n \geq 3 \end{cases}$

Proof: Let $P_n^1, P_n^2, P_n^3, \cdots, P_n^m$ are the m copies of path P_n in the extended $m-$shadow graph of P_n.
Let $\{v_1^i, v_2^i, v_3^i, \cdots, v_n^i,\}$ is the vertex set of path $P_n^i, 1 \leq i \leq m$.
Let the set D is a minimal eccentric dominating set of the extended $m-$shadow graph of P_n, i.e. $D^*_m(P_n)$.
Now, in the graph $D^*_m(P_n)$, for $1 \leq i \leq m$ the vertices v_j^i are adjacent to the vertices v_{j-1}^k and v_{j+1}^k, for all $1 \leq k \leq m$ and $2 \leq j \leq n-1$ and v_j^i are also adjacent to $v_j^l, 1 \leq i \neq l \leq m$.
For $j = 1$, the vertices v_1^i are adjacent to the vertices $v_2^k, 1 \leq k \leq m$ and v_1^i are also adjacent to the vertices $v_1^l, 1 \leq i \neq l \leq m$.
And similarly for $j = n$, the vertices v_n^i are adjacent to the vertices $v_{n-1}^k, 1 \leq k \leq m$ and v_n^i are also adjacent to the vertices $v_n^l, 1 \leq i \neq l \leq m$.

Thus, if we find the minimal eccentric dominating set for any one copy of path P_n^i then that same set will be the minimal eccentric dominating set for the extended $m-$shadow graph of P_n.

Hence, $\gamma_{ed}[D^*_m(P_n)] = \gamma_{ed}(P_n) = \begin{cases} 1 & ; n = 2 \\ \left\lceil \frac{n+2}{3} \right\rceil & ; n \geq 3 \end{cases}$

Illustration 3.4. Minimal eccentric dominating set of extended $3-$shadow graph of $P_9 = \{v_1, v_4, v_7, v_9\}$, given in Figure 3.

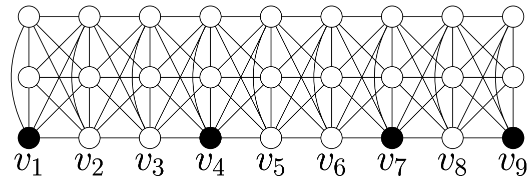

Figure 3. Minimal eccentric dominating set of $D^*_3(P_9)$

Where, $\gamma_{ed}[D^*_3(P_9)] = 4 = \left\lceil \frac{11}{3} \right\rceil = \left\lceil \frac{9+2}{3} \right\rceil$

Theorem 3.5. Eccentric domination number of extended $m-$splitting graph of path P_n is same as eccentric domination number of path P_n.

i.e. $\gamma_{ed}[Spl^*_m(P_n)] = \gamma_{ed}(P_n) = \begin{cases} 1 & ; n = 2 \\ \left\lceil \frac{n+2}{3} \right\rceil & ; n \geq 3 \end{cases}$

Proof: Let $v_1, v_2, v_3, \cdots, v_n$ is the vertex set of path P_n.
Let $v_1^i, v_2^i, v_3^i, \cdots, v_n^i$ are the m copies of the vertex set of path $P_n, 1 \leq i \leq m$ in the extended $m-$splitting graph of P_n.

Let the set D is a minimal eccentric dominating set of the extended $m-$splitting graph of P_n, i.e. $Spl^*_m(P_n)$.

Now, in the graph $Spl^*_m(P_n)$, for $1 \leq i \leq m$ the vertices v_j and v_j^i are adjacent to the vertices v_{j-1} and v_{j+1} of path P_n, for all $2 \leq j \leq n-1$ and v_j^i are also adjacent to v_j.

For $j = 1$, the vertices v_1 and v_1^i are adjacent to the vertex v_2 and v_1^i are also adjacent to the vertex $v_1, 1 \leq i \leq m$.

Similarly for $j = n$, the vertices v_n and v_n^i are adjacent to the vertex v_{n-1} and v_n^i are also adjacent to the vertex $v_n, 1 \leq i \leq m$.

Thus, if we find the minimal eccentric dominating set for the path P_n then that same set will be the minimal eccentric dominating set for the extended $m-$splitting graph of P_n.

Hence, $\gamma_{ed}[Spl^*_m(P_n)] = \gamma_{ed}(P_n) = \begin{cases} 1 & ; n = 2 \\ \left\lceil \frac{n+2}{3} \right\rceil & ; n \geq 3 \end{cases}$

Illustration 3.6. Minimal eccentric dominating set of extended $3-$splitting graph of $P_{10} = \{v_1, v_4, v_7, v_{10}\}$, given in Figure 4.

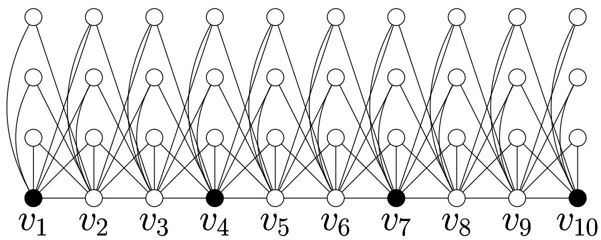

Figure 4. Minimal eccentric dominating set of $Spl_3^*(P_{10})$

Where, $\gamma_{ed}[Spl_3^*(P_{10})] = 4 = \left\lceil \frac{12}{3} \right\rceil = \left\lceil \frac{10+2}{3} \right\rceil$

Theorem 3.7. For any square of a path P_n,

$$\gamma_{ed}(P_n^2) = \begin{cases} 1 & ;2 \le n \le 3 \\ \left\lceil \frac{n+4}{5} \right\rceil & ;n \ge 4 \end{cases}$$

Proof: Let $V = V(P_n^2) = \{v_1, v_2, \cdots, v_n\}$ is the set of all vertices of graph $P_n^2, \forall n \ge 2$ and let D is a minimal eccentric dominating set of P_n^2.

Now, $2 \le n \le 3$ that means $n = 2$ or $n = 3$.

Case-1: $n = 2$
The graph P_2^2 is same as path P_2 and so using Theorem 3.1, $\gamma_{ed}(P_2^2) = \gamma_{ed}(P_2) = 1$.

Case-2: $n = 3$
For the graph P_3^2, the set $D = \{v_1\}$ is a minimal eccentric dominating set, as v_1 dominates the remaining vertices v_2 and v_3 and v_1 is also the eccentric vertex of v_2 and v_3.
Thus, $\gamma_{ed}(P_3^2) = 1$

Case-3: $n = 4$
For the graph P_4^2, the set $D = \{v_1, v_4\}$ is a minimal eccentric dominating set, as v_1 and v_4 dominates the remaining vertices v_2 and v_3 and v_1 is an eccentric vertex of v_2 and v_3.
Thus, $\gamma_{ed}(P_4^2) = 2 = \left\lceil \frac{8}{5} \right\rceil = \left\lceil \frac{4+4}{5} \right\rceil = \left\lceil \frac{n+4}{5} \right\rceil$

Case-4: $n = 5$
For the graph P_5^2, the set $D = \{v_1, v_5\}$ is a minimal eccentric dominating set, as v_1 and v_5 dominates the remaining vertices v_2, v_3 and v_4 and v_1 is the eccentric vertex of v_3, v_4 and v_5 is the eccentric vertex of v_2, v_3.
Thus, $\gamma_{ed}(P_5^2) = 2 = \left\lceil \frac{9}{5} \right\rceil = \left\lceil \frac{5+4}{5} \right\rceil = \left\lceil \frac{n+4}{5} \right\rceil$

Case-5: $n = 6$
For the graph P_6^2, the set $D = \{v_1, v_6\}$ is a minimal eccentric dominating set, as v_1 and v_6 dominates the remaining vertices v_2, v_3, v_4 and v_5 and v_1 is the eccentric vertex of v_4, v_5 and v_6 is the eccentric vertex of v_2, v_3.

Thus, $\gamma_{ed}(P_6^2) = 2 = \left\lceil \frac{10}{5} \right\rceil = \left\lceil \frac{6+4}{5} \right\rceil = \left\lceil \frac{n+4}{5} \right\rceil$

Case-6: $n \ge 7$
If n is an even number then the vertex v_1 is the only eccentric vertex of all vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \cdots, v_n$ and the vertex v_n is the only eccentric vertex of all vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}}$. Similarly if n is an odd number then the vertex v_1 is the only eccentric vertex of all vertices $v_{\frac{n+1}{2}+1}, v_{\frac{n+1}{2}+2}, v_{\frac{n+1}{2}+3}, \cdots, v_n$ and the vertex v_n is the only eccentric vertex of all vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}-1}$. And v_1 and v_n both are eccentric vertices of the vertex $v_{\frac{n+1}{2}}$.
So, Both end vertices v_1 and v_n of graph P_n^2 must be taken in D.
Then, The vertices v_2, v_3 and v_{n-1}, v_{n-2} will be dominated by the vertices v_1 and v_n respectively.

Now we can observe that for each $k, (3 \le k \le n-2)$ the vertex v_k dominates the **five** vertices $v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}$.

So, to dominate remaining $n-6$ vertices $v_4, v_5, v_6, \cdots, v_{n-3}$, we need to take one vertex in D for each five vertices. That means we have to take $\left\lceil \frac{n-6}{5} \right\rceil$ vertices in D.

Thus, D contains $\left\lceil \frac{n-6}{5} \right\rceil$ vertices from $v_4, v_5, \cdots, v_{n-3}$ vertices with v_1 and v_n.
So, D contains $\left\lceil \frac{n-6}{5} \right\rceil + 2$ vertices.
Thus, $\gamma_{ed}(P_n^2) = |D| = \left\lceil \frac{n-6}{5} \right\rceil + 2 = \left\lceil \frac{n-6}{5} + 2 \right\rceil$
$= \left\lceil \frac{n-6+10}{5} \right\rceil = \left\lceil \frac{n+4}{5} \right\rceil$

Hence, $\gamma_{ed}(P_n^2) = \begin{cases} 1 & ;2 \le n \le 3 \\ \left\lceil \frac{n+4}{5} \right\rceil & ;n \ge 4 \end{cases}$

Illustration 3.8. Minimal eccentric dominating set of $P_{11}^2 = \{v_1, v_6, v_{11}\}$, given in Figure 5.

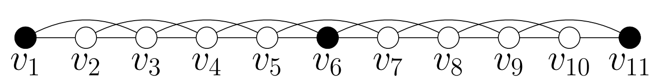

Figure 5. Minimal eccentric dominating set of path P_{11}^2

Where, $\gamma_{ed}(P_{11}^2) = 3 = \left\lceil \frac{15}{5} \right\rceil = \left\lceil \frac{11+4}{5} \right\rceil$

Theorem 3.9. Eccentric domination number of $m-$shadow graph of path P_n is given by

$$\gamma_{ed}[D_m(P_n)] = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1 & ;n = 3 \text{ } or \text{ } n \equiv 2(mod 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 2 & ;n \equiv 0 \text{ or } 1 \text{ or } 3(mod 4), n \ne 3 \end{cases}$$

Proof: Let $P_n^1, P_n^2, P_n^3, \cdots, P_n^m$ are the m copies of path P_n in the $m-$shadow graph of P_n.

Let $\{v_1^i, v_2^i, v_3^i, \cdots, v_n^i\}$ is the vertex set of path $P_n^i, 1 \leq i \leq m$ and V is the vertex set of the $m-$shadow graph of P_n, i.e. $D_m(P_n)$.
Let D is a minimal eccentric dominating set of the graph $D_m(P_n)$.

Now, in the graph $D_m(P_n)$, for $1 \leq i \leq m$, the vertices v_j^i are adjacent to the vertices v_{j-1}^k and v_{j+1}^k, for all $1 \leq k \leq m$ and $2 \leq j \leq n-1$.
For $j = 1$, the vertices v_1^i are adjacent to the vertices $v_2^k, 1 \leq k \leq m$.
And similarly for $j = n$, the vertices v_n^i are adjacent to the vertices $v_{n-1}^k, 1 \leq k \leq m$.
Here v_j^i is not adjacent to v_j^k for $1 \leq i \neq k \leq m$ and $1 \leq j \leq n$.
But, we can dominate all vertices of all copies of path P_n using the vertices of any one copy of path P_n.
Without lose of generality, let we take the vertices from P_n^1 to construct D.

Case-1: $n = 2$
The minimal eccentric dominating set of graph $D_m(P_2)$ will be $D = \{v_1^1, v_2^1\}$, as v_1^1 is dominating all vertices v_2^k and v_2^1 is dominating all vertices v_1^k of the set $V - D$ with v_1^1 is the eccentric vertex of all vertices v_1^k and v_2^1 is the eccentric vertex of all vertices v_2^k for $2 \leq k \leq m$ of the set $V - D$.
Thus, $\gamma_{ed}[D_m(P_2)] = 2 = 1 + 1 = \left\lfloor \frac{2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1.$

Case-2: $n = 3$
The minimal eccentric dominating set of graph $D_m(P_3)$ will be $D = \{v_1^1, v_2^1\}$, as v_1^1 is dominating all vertices v_2^k and v_2^1 is dominating all vertices v_1^k and v_3^l of the set $V - D$ with v_1^1 is the eccentric vertex of all vertices v_1^k and v_3^l and v_2^1 is the eccentric vertex of all vertices v_2^k for $2 \leq k \leq m, 1 \leq l \leq m$ of the set $V - D$.
Thus, $\gamma_{ed}[D_m(P_3)] = 2 = 1 + 1 = \left\lfloor \frac{3}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1.$

Case-3: $n \geq 4$
If n is an even number then the vertices v_1^i are the only eccentric vertices of all vertices $v_{\frac{n}{2}+1}^k, v_{\frac{n}{2}+2}^k, v_{\frac{n}{2}+3}^k, \cdots, v_n^k$ and the vertices v_n^i are the only eccentric vertices of all vertices $v_1^k, v_2^k, v_3^k, \cdots, v_{\frac{n}{2}}^k$ for $1 \leq i, k \leq m$.
Similarly if n is an odd number then the vertices v_1^i are the only eccentric vertices of all vertices $v_{\frac{n+1}{2}+1}^k, v_{\frac{n+1}{2}+2}^k$, $v_{\frac{n+1}{2}+3}^k, \cdots, v_n^k$ and the vertices v_n^i are the only eccentric vertices of all vertices $v_1^k, v_2^k, v_3^k, \cdots, v_{\frac{n+1}{2}-1}^k$. And v_1^i and v_n^i both are eccentric vertices of the vertices $v_{\frac{n+1}{2}}^k$ for $1 \leq i, k \leq m$.
So, One end vertex from v_1^i and one end vertex form v_n^i of graph $D_m(P_n)$ must be taken in D for $1 \leq i \leq m$.
Thus, The vertices v_2^k and v_{n-1}^k will be dominated by the vertices v_1^i and v_n^i respectively for $1 \leq i, k \leq m$.
But to dominate the all vertices v_1^k, we have to take the vertex v_2^1 and to dominate the all vertices v_n^k we have to take the vertex v_{n-1}^1 in D, for $2 \leq k \leq m$.
So, the all vertices v_3^k and v_{n-2}^k for $1 \leq k \leq m$ will be dominated.
To dominate remaining $m(n-6)$ vertices $v_4^k, v_5^k, v_6^k, \cdots, v_{n-3}^k$ for $1 \leq k \leq m$, we need to take two middle vertices in D for each four vertices.

Case-(i): $n \equiv 2(mod 4)$
Let $n = 4t + 2,\ t \in \mathbb{N}$.
Then $D = \{v_1^1, v_2^1, v_5^1, v_6^1, v_9^1, v_{10}^1, \cdots, v_{4t+1}^1, v_{4t+2}^1\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1^1, v_5^1, v_9^1, \cdots, v_{4t+1}^1\}$ then $|D'| = t + 1$.
Moreover, $|D| = 2|D'| = 2(t+1) = 2\left(\frac{n-2}{4} + 1\right) = \frac{n}{2} + 1$
$= \left\lfloor \frac{n}{2} \right\rfloor + 1$, as $\frac{n}{2}$ is an integer.

Case-(ii): $n \equiv 0(mod 4)$
Let $n = 4t,\ t \in \mathbb{N}$.
Then $D = \{v_1^1, v_2^1, v_5^1, v_6^1, v_9^1, v_{10}^1, \cdots, v_{4t-3}^1, v_{4t-2}^1, v_{4t-1}^1, v_{4t}^1\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1^1, v_5^1, v_9^1, \cdots, v_{4t-3}^1\}$ then $|D'| = t$.
Moreover, $|D| = 2|D'| + 2 = 2(t) + 2 = 2\left(\frac{n}{4}\right) + 2 = \frac{n}{2} + 2$
$= \left\lfloor \frac{n}{2} \right\rfloor + 2$, as $\frac{n}{2}$ is an integer.

Case-(iii): $n \equiv 1(mod 4)$
Let $n = 4t + 1,\ t \in \mathbb{N}$.
Then $D = \{v_1^1, v_2^1, v_5^1, v_6^1, v_9^1, v_{10}^1, \cdots, v_{4t-3}^1, v_{4t-2}^1, v_{4t}^1, v_{4t+1}^1\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1^1, v_5^1, v_9^1, \cdots, v_{4t-3}^1\}$ then $|D'| = t$.
Moreover, $|D| = 2|D'| + 2 = 2(t) + 2 = 2\left(\frac{n-1}{4}\right) + 2$
$= \left(\frac{n}{2} - \frac{1}{2}\right) + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2$, as $\left(\frac{n}{2} - \frac{1}{2}\right)$ is an integer.

Case-(iv): $n \equiv 3(mod 4)$
Let $n = 4t + 3,\ t \in \mathbb{N}$.
Then $D = \{v_1^1, v_2^1, v_5^1, v_6^1, v_9^1, v_{10}^1, \cdots, v_{4t+1}^1, v_{4t+2}^1, v_{4t+3}^1\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1^1, v_5^1, v_9^1, \cdots, v_{4t+1}^1\}$ then $|D'| = t + 1$.
Moreover,
$|D| = 2|D'| + 1 = 2(t+1) + 1 = 2\left(\frac{n-3}{4} + 1\right) + 1$
$= \left(\frac{n}{2} - \frac{1}{2}\right) + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2$, as $\left(\frac{n}{2} - \frac{1}{2}\right)$ is an integer.
Hence,
$\gamma_{ed}[D_m(P_n)] = |D|$
And so,

$$\gamma_{ed}[D_m(P_n)] = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1 & ; n = 3 \text{ or } n \equiv 2(mod 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 2 & ; n \equiv 0 \text{ or } 1 \text{ or } 3(mod 4), n \neq 3 \end{cases}$$

Illustration 3.10. Minimal eccentric dominating set of $3-$shadow graph of path P_{12} $[i.e. D_3(P_{12})] = D = \{v_1, v_2, v_5, v_6, v_9, v_{10}, v_{11}, v_{12}\}$ given in Figure 6.

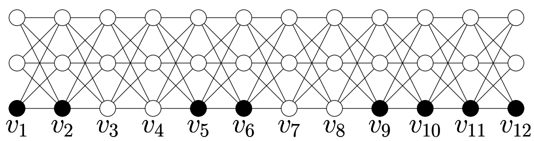

Figure 6. Minimal eccentric dominating set of $D_3(P_{12})$

Where, $\gamma_{ed}[D_3(P_{12})] = 8 = 6+2 = \left\lfloor \frac{12}{2} \right\rfloor + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2$

Theorem 3.11. Eccentric domination number of $m-$splitting graph of path P_n is given by

$$\gamma_{ed}[Spl_m(P_n)] = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1 & ; n = 3 \text{ or } n \equiv 2(mod 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 2 & ; n \equiv 0 \text{ or } 1 \text{ or } 3(mod 4), n \neq 3 \end{cases}$$

Proof: Let $v_1, v_2, v_3, \cdots, v_n$ is the vertex set of path P_n. Let $v_1^i, v_2^i, v_3^i, \cdots, v_n^i$ are the m copies of the vertex set of path $P_n, 1 \leq i \leq m$ in the $m-$splitting graph of P_n.
Let the set D is a minimal eccentric dominating set of the $m-$splitting graph of P_n, i.e. $Spl_m(P_n)$.
Now, in the graph $Spl_m(P_n)$, for $1 \leq i \leq m$ the vertices v_j, v_j^i are adjacent to the vertices v_{j-1} and v_{j+1} of path P_n, for all $2 \leq j \leq n-1$.
For $j = 1$, the vertices v_1, v_1^i are adjacent to the vertex $v_2, 1 \leq i \leq m$.
And similarly for $j = n$, the vertices v_n, v_n^i are adjacent to the vertex $v_{n-1}, 1 \leq i \leq m$.
Here v_j is not adjacent to v_j^i for $1 \leq i \leq m$ and $1 \leq j \leq n$.
But, we can dominate all vertices of $m-$splitting graph of path P_n using the vertices of path P_n.

Case-1: $n = 2$
The minimal eccentric dominating set of graph $Spl_m(P_2)$ will be $D = \{v_1, v_2\}$, as v_1 is dominating all vertices v_2^k and v_2 is dominating all vertices v_1^k of the set $V - D$ with v_1 is the eccentric vertex of all vertices v_1^k and v_2 is the eccentric vertex of all vertices v_2^k for $1 \leq k \leq m$ of the set $V - D$.
Thus, $\gamma_{ed}[Spl_m(P_2)] = 2 = 1+1 = \left\lfloor \frac{2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Case-2: $n = 3$
The minimal eccentric dominating set of graph $Spl_m(P_3)$ will be $D = \{v_1, v_2\}$, as v_1 is dominating all vertices v_2^k and v_2 is dominating all vertices v_1^k, v_3^k and v_3 of the set $V - D$ with v_1 is the eccentric vertex of all vertices v_1^k, v_3^k and v_3 and v_2 is the eccentric vertex of all vertices v_2^k for $1 \leq k \leq m$ of the set $V - D$.
Thus, $\gamma_{ed}[Spl_m(P_3)] = 2 = 1+1 = \left\lfloor \frac{3}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Case-3: $n \geq 4$
If n is an even number then the vertices v_1 and v_1^i are the only eccentric vertices of all vertices $v_{\frac{n}{2}+1}^k, v_{\frac{n}{2}+2}^k, v_{\frac{n}{2}+3}^k, \cdots, v_n^k$ and $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \cdots, v_n$ and the vertices v_n and v_n^i are the only eccentric vertices of all vertices $v_1^k, v_2^k, v_3^k, \cdots, v_{\frac{n}{2}}^k$ and $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}}$ for $1 \leq i, k \leq m$.
Similarly if n is an odd number then the vertices v_1 and v_1^i are the only eccentric vertices of all vertices $v_{\frac{n+1}{2}+1}^k, v_{\frac{n+1}{2}+2}^k, v_{\frac{n+1}{2}+3}^k, \cdots, v_n^k$ and $v_{\frac{n+1}{2}+1}, v_{\frac{n+1}{2}+2}, v_{\frac{n+1}{2}+3}, \cdots, v_n$ and the vertices v_n and v_n^i are the only eccentric vertices of all vertices $v_1^k, v_2^k, v_3^k, \cdots, v_{\frac{n+1}{2}-1}^k$ and $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}-1}$. And v_1, v_1^i, v_n, v_n^i are eccentric vertices of the vertices $v_{\frac{n+1}{2}}^k$ for $1 \leq i, k \leq m$.
So, one end vertex from v_1, v_1^i and one end vertex form v_n, v_n^i of graph $Spl_m(P_n)$ must be taken in D for $1 \leq i \leq m$.
We choose the vertices v_1 and v_n as eccentric vertices in D.
Then, The vertices v_2, v_2^k and v_{n-1}, v_{n-1}^k will be dominated by the vertices v_1 and v_n respectively for $1 \leq k \leq m$.
But to dominate the all vertices v_1^k, we have to take the vertex v_2 and to dominate the all vertices v_n^k we have to take the vertex v_{n-1} in D, for $1 \leq k \leq m$.
So, the all vertices v_3, v_3^k and v_{n-2}, v_{n-2}^k for $1 \leq k \leq m$ will be dominated.
To dominate remaining $(m+1)(n-6)$ vertices $v_4, v_5, v_6, \cdots, v_{n-3}$ and $v_4^k, v_5^k, v_6^k, \cdots, v_{n-3}^k$ for $1 \leq k \leq m$, we need to take two middle vertices in D for each four vertices of path P_n.

Case-(i): $n \equiv 2(mod 4)$
Let $n = 4t+2$, $t \in \mathbb{N}$.
Then $D = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \cdots, v_{4t+1}, v_{4t+2}\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1, v_5, v_9, \cdots, v_{4t+1}\}$ then $|D'| = t+1$.
Moreover, $|D| = 2|D'| = 2(t+1) = 2\left(\frac{n-2}{4} + 1\right) = \frac{n}{2} + 1$
$= \left\lfloor \frac{n}{2} \right\rfloor + 1$, as $\frac{n}{2}$ is an integer.

Case-(ii): $n \equiv 0(mod 4)$
Let $n = 4t$, $t \in \mathbb{N}$.
Then $D = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \cdots, v_{4t-3}, v_{4t-2}, v_{4t-1}, v_{4t}\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1, v_5, v_9, \cdots, v_{4t-3}\}$ then $|D'| = t$.
Moreover, $|D| = 2|D'| + 2 = 2(t) + 2 = 2\left(\frac{n}{4}\right) + 2 = \frac{n}{2} + 2$
$= \left\lfloor \frac{n}{2} \right\rfloor + 2$, as $\frac{n}{2}$ is an integer.

Case-(iii): $n \equiv 1(mod 4)$
Let $n = 4t+1$, $t \in \mathbb{N}$.
Then $D = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \cdots, v_{4t-3}, v_{4t-2}, v_{4t}, v_{4t+1}\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1, v_5, v_9, \cdots, v_{4t-3}\}$ then $|D'| = t$.
Moreover, $|D| = 2|D'| + 2 = 2(t) + 2 = 2\left(\frac{n-1}{4}\right) + 2$
$= \left(\frac{n}{2} - \frac{1}{2}\right) + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2$, as $\left(\frac{n}{2} - \frac{1}{2}\right)$ is an integer.

Case-(iv): $n \equiv 3(mod 4)$
Let $n = 4t+3$, $t \in \mathbb{N}$.

Then $D = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \cdots, v_{4t+1}, v_{4t+2}, v_{4t+3}\}$ is a minimal eccentric dominating set.
Let $D' = \{v_1, v_5, v_9, \cdots, v_{4t+1}\}$ then $|D'| = t+1$.
Moreover,
$|D| = 2|D'| + 1 = 2(t+1) + 1 = 2\left(\frac{n-3}{4} + 1\right) + 1$
$= \left(\frac{n}{2} - \frac{1}{2}\right) + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2$, as $\left(\frac{n}{2} - \frac{1}{2}\right)$ is an integer.
Hence, $\gamma_{ed}[Spl_m(P_n)] = |D|$
And so,

$$\gamma_{ed}[Spl_m(P_n)] = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1 & ; n = 3 \text{ } or \text{ } n \equiv 2(mod 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 2 & ; n \equiv 0 \text{ or } 1 \text{ or } 3(mod 4), n \neq 3 \end{cases}$$

Illustration 3.12. Minimal eccentric dominating set of $2-$ splitting graph of path P_{14} $[i.e. Spl_2(P_{14})] = D = \{v_1, v_2, v_5, v_6, v_9, v_{10}, v_{13}, v_{14}\}$ given in Figure 7.

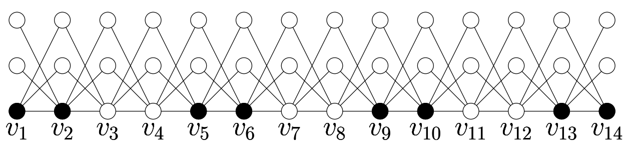

Figure 7. Minimal eccentric dominating set of $Spl_2(P_{14})$

$$\text{Where, } \gamma_{ed}[Spl_2(P_{14})] = 8 = 7 + 1 = \left\lfloor \frac{14}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$$

4. Concluding Remarks

The concept of eccentric domination relates a dominating set with eccentricity of a vertex. We have investigated eccentric domination number of some path related graphs.

References

[1] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw-Hill, Boston, (2005).

[2] J. Clark and D. A. Holton, *A First Look at Graph Theory*, Allied Publishers Ltd., (1995).

[3] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, *Total Domination in Graphs, Networks*, (10) (1980), 211-219.

[4] S. Ao, E. J. Cockayne, G. MacGillivray and C. M. Mynhardt, *Domination Critical Graphs with Higher Independent Domination Numbers*, J. Graph Theory, (22) (1996), 9-14.

[5] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., (1998).

[6] T. N. Janakiraman, M. Bhanumathi and S. Muthammai, *Eccentric Domination in Graphs*, International Journal of Engineering Science, Advanced Computing and Bio Technology, 1(2) (2010), 1-16.

[7] V. Swaminathan and K. M. Dharmalingam, *Degree Equitable Dominations on Graphs*, Kragujevac Journal of Mathematics, 35(1) (2011), 191-197.

[8] J. A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees, SIAM J. Discrete Mathematics, (10) (1997), 529-550.

[9] S. K. Vaidya and R. N. Mehta, *Steiner Domination Number of Some Wheel Related Graphs*, International Journal of Mathematics and Soft Computing, 5(2) (2015), 15-19.

[10] S. K. Vaidya and R. M. Pandit, *Some New Results on Global Dominating Sets*, ISRN Discrete Mathematics, (2012), 1-6.

[11] S. K. Vaidya and K. M. Popat, *On Equienergetic, Hyperenergetic and Hypoenergetic Graphs*, Kragujevac Journal of Mathematics, Volume 44(4) (2020), 523-532.

[12] S. K. Vaidya and K. M. Popat, *Some Borderenergetic and Equienergetic Graphs of Arbitrarily Large Order*, (Communicated).

[13] D. B. West, *Introduction to Graph Theory*, Prentice - Hall of India, New Delhi, (2003).