



Common fixed point theorem for four self maps of S -metric spaces by employing compatibility of type(R)

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Abstract

In this paper, we define an almost generalized weakly contraction condition and prove a common fixed point theorem for two pairs of compatible mappings of type(R) in an S -metric space. As an application, we deduce a common fixed point result for four finite families of self mappings.

Keywords

S -metric space, common fixed point, associated sequence, compatibility mappings of type(R). **AMS Subject**

Classification

54H25, 47H10.

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1. Introduction

In an attempt to generalize the notion of metric space, several mathematicians proposed different generalizations and studied fixed and common fixed point results under different contractive conditions. In this direction, Mustafa and Sims [1] introduced G-metric spaces as a generalization of metric spaces in 2006 and proved existence of some fixed points. In 2012, Sedghi, Shobe and Aliouche [2] introduced S -metric spaces by generalizing G-metric spaces and investigated some of its properties. But, in 2014 Dung, Hieu and Radojevic [3] showed by an example that the class of S -metric spaces and the class of G-metric spaces are not the same. Thereafter, in 2014, S.Sedghi, N.V.Dung [4] generalized some results in [2] and in [5], J.K. Kim, S. Sedghi, N. Shobkolaei established a common fixed point theorem for R-weakly commuting maps in S -metric spaces. For more fixed point results on S -metric

spaces, we refer to ([6]-[10]). On the other hand, in 2004, Rohan et al. [11] combined the definitions of compatible, compatible of type(P) and introduced compatible of type(R) in Banach spaces and studied some fixed point results for such mappings.

The aim of this paper is to define an almost weakly generalized contraction and compatibility of type(R) in S -metric spaces and to prove a common fixed point theorem using associated sequence[12] with respect to four self maps.

2. Preliminaries

In this section, we recall some definitions and results which will be used in the results.

Definition 2.1 ([2]). *Let X be a non-empty set. Then we say that a function $S : X^3 \rightarrow [0, \infty)$ is an S -metric on X iff it satisfies the following for all α, β, γ and θ in X ,*

$$(P1) \quad S(\alpha, \beta, \gamma) = 0 \text{ iff } \alpha = \beta = \gamma$$

$$(P2) \quad S(\alpha, \beta, \gamma) \leq S(\alpha, \theta, \theta) + S(\beta, \theta, \theta) + S(\gamma, \theta, \theta).$$

Here (X, S) is called a S -metric space.

Example 2.2 ([2]). *Let (X, d) be a metric space. Define $S : X^3 \rightarrow [0, \infty)$ by*

$$S(\alpha, \beta, \gamma) = d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha)$$

for $\alpha, \beta, \gamma \in X$. Then S is an S -metric on X and S is called the S -metric induced by the metric d .

Example 2.3 ([8]). (X, S) is a S -metric space where $X = [0, 1]$ and

$$S(\alpha, \beta, \gamma) := \begin{cases} 0, & \text{if } \alpha = \beta = \gamma \\ \max\{\alpha, \beta, \gamma\}, & \text{otherwise} \end{cases}$$

for $\alpha, \beta, \gamma \in X$.

Definition 2.4 ([4]). We say that a sequence (α_n) in S -metric space converges to a point $\alpha \in X$ iff $S(\alpha_n, \alpha_n, \alpha) \rightarrow 0$, as $n \rightarrow \infty$.

Definition 2.5 ([4]). We say that a sequence (α_n) in S -metric space (X, S) is a Cauchy sequence in X iff $S(\alpha_n, \alpha_n, \alpha_\xi) \rightarrow 0$, as $n, \xi \rightarrow \infty$.

Definition 2.6 ([4]). We say that a S -metric space (X, S) is said to be complete iff every Cauchy sequence in X converges in X .

Lemma 2.7 ([6]). In S -metric space (X, S) , we have

$$S(\alpha, \alpha, \gamma) = S(\gamma, \gamma, \alpha),$$

for $\alpha, \gamma \in X$.

Lemma 2.8 ([4]). Let (X, S) be a S -metric space. If there exist sequences (α_n) and (β_n) in X such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ and $\lim_{n \rightarrow \infty} \beta_n = \beta$, then $\lim_{n \rightarrow \infty} S(\alpha_n, \alpha_n, \beta_n) = S(\alpha, \alpha, \beta)$.

Definition 2.9. We say that two self maps f and R of an S -metric space (X, S) are compatible of type R if

$$\lim_{n \rightarrow \infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n) = 0$$

and

$$\lim_{n \rightarrow \infty} S(ff\alpha_n, ff\alpha_n, RR\alpha_n) = 0,$$

whenever a sequence $\{\alpha_n\}$ in X such that

$$\lim_{n \rightarrow \infty} f\alpha_n = \lim_{n \rightarrow \infty} R\alpha_n = \gamma,$$

for some $\gamma \in X$.

Definition 2.10 ([10]). Let f, g, R and T be four self maps of a S -metric space (X, S) such that $fX \subset TX$ and $gX \subset RX$ and let $\alpha_0 \in X$. since $fX \subset TX$ and $\alpha_0 \in X$, there exists a point $\alpha_1 \in X$ such that $f\alpha_0 = T\alpha_1$. Now $g\alpha_1 \in gX$ and $gX \subset RX$ implies that there exists a point $\alpha_2 \in X$ such that $g\alpha_1 = R\alpha_2$. This will imply that there exists a point $\alpha_3 \in X$ such that $f\alpha_2 = T\alpha_3$, since $fX \subset TX$. similarly, $g\alpha_3 \in gX$ and $gX \subset RX$ implies that there is a point $\alpha_4 \in X$ such that $g\alpha_3 = R\alpha_4$. By continuing in this process, we get a sequence (α_n) in X such that $f\alpha_{2n} = T\alpha_{2n+1}$ and $g\alpha_{2n+1} = R\alpha_{2n+2}$ for $n \geq 0$. We call this sequence (α_n) as an associated sequence of α_0 with respect to the four self-maps f, g, R and T .

Example 2.11. Consider S -metric space (X, S) , where $X = \mathbb{R}$ and

$$S(\alpha, \beta, \gamma) = |\alpha - \gamma| + |\beta - \gamma|$$

for $\alpha, \beta, \gamma \in X$. Now we define four self-maps f, g, R and T on X by $f\alpha = \frac{\alpha^4}{5}, g\alpha = \frac{\alpha^4}{5}, R\alpha = \alpha^4$ and $T\alpha = \alpha^4$ for $\alpha \in X$. Let $\alpha_0 \in X$ and choose α_n as $\alpha_n = \frac{\alpha_0}{(\sqrt[4]{5})^n}$ for $n \in \mathbb{N}$. Then

$$\text{we have } f\alpha_{2n} = \frac{\alpha_{2n}^4}{5} = \frac{\alpha_0^4}{5^{2n+1}} \text{ and } T\alpha_{2n+1} = \alpha_{2n+1}^4 = \frac{\alpha_0^4}{5^{2n+1}}$$

$$\text{for } n \in \mathbb{N}. \text{ similarly, } g\alpha_{2n+1} = \frac{\alpha_{2n+1}^4}{5} = \frac{\alpha_0^4}{5^{2n+2}} \text{ and } R\alpha_{2n+2} =$$

$$\alpha_{2n+2}^4 = \frac{\alpha_0^4}{5^{2n+2}} \text{ for } n \in \mathbb{N}. \text{ Therefore } f\alpha_{2n} = T\alpha_{2n+1} \text{ and } g\alpha_{2n+1} = R\alpha_{2n+2} \text{ for all } n \in \mathbb{N} \cup \{0\} \text{ and hence } (\alpha_n) \text{ is an}$$

associated sequence of the point $\alpha_0 \in X$ with respect to f, g, R and T . Similarly, one can easily get that the sequence (α_n) defined by $\alpha_n = -\frac{\alpha_0}{(\sqrt[4]{5})^n}$ for $n \in \mathbb{N}$, is also an associated sequence of the same point α_0 of f, g, R and T .

Proposition 2.12. Suppose f and R are two self maps of S -metric space. If f and R are compatible of type (R) and

$$\lim_{n \rightarrow \infty} f\alpha_n = \lim_{n \rightarrow \infty} R\alpha_n = \gamma,$$

for some $\gamma \in X$, then $\lim_{n \rightarrow \infty} Rf\alpha_n = f\gamma$ if f is continuous.

Proof. Since f and R are compatible of type (R) , we have

$$\lim_{n \rightarrow \infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n) = 0$$

and $\lim_{n \rightarrow \infty} S(ff\alpha_n, ff\alpha_n, RR\alpha_n) = 0$ whenever a sequence $\{\alpha_n\}$ in X such that $\lim_{n \rightarrow \infty} f\alpha_n = \lim_{n \rightarrow \infty} R\alpha_n = \gamma$ for some $\gamma \in X$. Now, since f is continuous,

$$\lim_{n \rightarrow \infty} ff\alpha_n = \lim_{n \rightarrow \infty} fR\alpha_n = f\gamma.$$

So by triangle inequality, we have

$$S(Rf\alpha_n, Rf\alpha_n, f\gamma) \leq 2S(Rf\alpha_n, Rf\alpha_n, fR\alpha_n) + S(f\gamma, f\gamma, fR\alpha_n).$$

Letting $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} S(Rf\alpha_n, Rf\alpha_n, f\gamma) = 0$, which implies that $\lim_{n \rightarrow \infty} Rf\alpha_n = f\gamma$. \square

Proposition 2.13. Suppose f and R are two self maps of S -metric space. If f and R are compatible of type (R) and $f\gamma = R\gamma$ for some γ in X , then $fR\gamma = ff\gamma = Rf\gamma = RR\gamma$.

Proof. Since f and R are compatible of type (R) , we have $\lim_{n \rightarrow \infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n) = 0$ and

$$\lim_{n \rightarrow \infty} S(ff\alpha_n, ff\alpha_n, RR\alpha_n) = 0,$$

whenever a sequence $\{\alpha_n\}$ in X such that $\lim_{n \rightarrow \infty} f\alpha_n = \lim_{n \rightarrow \infty} R\alpha_n = \gamma$ for some $\gamma \in X$. Let $\alpha_n = \gamma$ for $n \geq 1$. Then $\lim_{n \rightarrow \infty} f\alpha_n = \lim_{n \rightarrow \infty} R\alpha_n = f\gamma$, since $f\gamma = R\gamma$. As f and R are compatible of type (R) , we have

$$S(fR\gamma, fR\gamma, Rf\gamma) = \lim_{n \rightarrow \infty} S(fR\alpha_n, fR\alpha_n, Rf\alpha_n) = 0,$$

which implies that $fR\gamma = Rf\gamma$ and hence $fR\gamma = ff\gamma = Rf\gamma = RR\gamma$, since $f\gamma = R\gamma$. \square



3. Main Results

Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a function such that

- (i) it is continuous
- (ii) non-decreasing and
- (iii) $\psi(k) = 0$ iff $k = 0$.

Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be also a continuous function satisfying $\phi(k) = 0$ iff $k = 0$. We begin with the following lemma which plays important role in the main result.

Lemma 3.1. *Suppose in a S -metric space (X, S) , there are four self maps f, g, R and T on X and $D \geq 0$ satisfying,*

$$\psi(S(f\alpha, f\alpha, g\beta)) \leq \psi(P(\alpha, \beta)) - \phi(P(\alpha, \beta)) + DK(\alpha, \beta) \tag{3.1}$$

for $\alpha, \beta \in X$, where

$$P(\alpha, \beta) = \max \left\{ S(R\alpha, R\alpha, T\beta), \frac{1}{2} [S(R\alpha, R\alpha, f\alpha) + S(T\beta, T\beta, g\beta)] \right. \\ \left. \frac{1}{3} [S(R\alpha, R\alpha, g\beta) + S(T\beta, T\beta, f\alpha)] \right\}$$

and

$$K(\alpha, \beta) = \min\{S(R\alpha, R\alpha, g\beta), S(T\beta, T\beta, f\alpha)\},$$

and (X, S) is complete. Then for any associated sequence (α_η) of a point $\alpha_0 \in X$ with respect to f, g, R and T , the sequence $(f\alpha_0, g\alpha_1, f\alpha_2, g\alpha_3, \dots)$ converges in (X, S) .

Proof. Let (α_η) be an associated sequence of a point $\alpha_0 \in X$ with respect to the four self maps f, g, R and T . Then we have $f\alpha_{2\eta} = T\alpha_{2\eta+1}$ and $g\alpha_{2\eta+1} = R\alpha_{2\eta+2}$ for $\eta \geq 0$. Now we construct a sequence (β_η) as follows:

$\beta_{2\eta+1} := f\alpha_{2\eta} = T\alpha_{2\eta+1}$ and $\beta_{2\eta+2} := g\alpha_{2\eta+1} = R\alpha_{2\eta+2}$ for $\eta \geq 0$. Clearly (β_η) is same as $(f\alpha_0, g\alpha_1, f\alpha_2, g\alpha_3, \dots)$. Now let us show that (β_η) converges in (X, S) . Since (X, S) is complete, it is enough to show that (β_η) is Cauchy sequence in (X, S) .

Case(i). For this, we assume that $\beta_{2\eta} \neq \beta_{2\eta+1}$ for all $\eta \in \mathbb{N} \cup \{0\}$. Now for each $\eta \in \mathbb{N} \cup \{0\}$, we have

$$\psi(S(f\alpha_{2\eta}, f\alpha_{2\eta}, g\alpha_{2\eta+1})) \leq \psi(P(\alpha_{2\eta}, \alpha_{2\eta+1})) - \phi(P(\alpha_{2\eta}, \alpha_{2\eta+1})) + DK(\alpha_{2\eta}, \alpha_{2\eta+1})$$

where

$$P(\alpha_{2\eta}, \alpha_{2\eta+1}) = \max \left\{ S(R\alpha_{2\eta}, R\alpha_{2\eta}, T\alpha_{2\eta+1}), \right. \\ \left. \frac{1}{2} [S(R\alpha_{2\eta}, R\alpha_{2\eta}, f\alpha_{2\eta}) + S(T\alpha_{2\eta+1}, T\alpha_{2\eta+1}, f\alpha_{2\eta})], \right. \\ \left. \frac{1}{3} [S(R\alpha_{2\eta}, R\alpha_{2\eta}, g\alpha_{2\eta+1}) + S(T\alpha_{2\eta+1}, T\alpha_{2\eta+1}, f\alpha_{2\eta})] \right\}$$

and

$$K(\alpha_{2\eta}, \alpha_{2\eta+1}) = \min \left\{ S(R\alpha_{2\eta}, R\alpha_{2\eta}, g\alpha_{2\eta+1}), S(T\alpha_{2\eta+1}, T\alpha_{2\eta+1}, f\alpha_{2\eta}) \right\}.$$

Then we have

$$P(\alpha_{2\eta}, \alpha_{2\eta+1}) = \max \left\{ S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}), \right. \\ \left. \frac{1}{2} [S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}) + S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})], \right. \\ \left. \frac{1}{3} [S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+2}) + S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+1})] \right\}$$

and

$$K(\alpha_{2\eta}, \alpha_{2\eta+1}) = \min \left\{ S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}), S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+1}) \right\} = 0.$$

This will imply that

$$P(\alpha_{2\eta}, \alpha_{2\eta+1}) \leq \max \left\{ S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}) \right. \\ \left. \frac{1}{2} [S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}) + S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})] \right. \\ \left. \frac{1}{3} [2S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}) + S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})] \right\}.$$

If

$$S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}) < S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}),$$

then we must have

$$P(\alpha_{2\eta}, \alpha_{2\eta+1}) \leq S(\beta_{2\eta+1}, \beta_{2\eta+2}, \beta_{2\eta+2}).$$

Note that

$$\psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \leq \psi(P(\alpha_{2\eta}, \alpha_{2\eta+1})) - \phi(P(\alpha_{2\eta}, \alpha_{2\eta+1})) \leq \psi(P(\alpha_{2\eta}, \alpha_{2\eta+1})).$$

By definition of non-decreasing of ψ , we have

$$S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) \leq P(\alpha_{2\eta}, \alpha_{2\eta+1}).$$

Therefore we must have

$$P(\alpha_{2\eta}, \alpha_{2\eta+1}) = S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}).$$

By (3.1) we have

$$\psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \leq \psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) - \phi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})).$$

This implies that, $-\phi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \geq 0$. It follows that $\phi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) = 0$. By definition of ϕ , we have $S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) = 0$ and hence $\beta_{2\eta+1} = \beta_{2\eta+2}$ -contradiction. Therefore we have

$$S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) \leq S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1})$$



and hence

$$P(\alpha_{2\eta}, \alpha_{2\eta+1}) = S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}).$$

By using (3.1) we get that

$$\begin{aligned} \psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \\ \leq \psi(S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1})) - \phi(S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1})) \\ \leq \psi(S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1})). \end{aligned}$$

Therefore

$$\psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \leq \psi(S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}))$$

for each $\eta \in \mathbb{N} \cup \{0\}$. By definition of ψ we have

$$S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) \leq S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1})$$

for all $\eta \in \mathbb{N} \cup \{0\}$. This shows that the sequence

$$(S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}))$$

is a decreasing sequence of real numbers. Then there exists $\chi \geq 0$ such that $\lim_{\eta \rightarrow \infty} S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}) = \chi$.

Now letting $\eta \rightarrow \infty$ in (3.1) we have $\psi(\chi) \leq \psi(\chi) - \phi(\chi)$. This will imply that $-\phi(\chi) \geq 0$. It follows that $\phi(\chi) = 0$ and hence $\chi = 0$. Therefore we must have

$$\lim_{\eta \rightarrow \infty} S(\beta_{2\eta}, \beta_{2\eta}, \beta_{2\eta+1}) = 0.$$

Similarly, by taking $\alpha = \alpha_{2\eta+1}, \beta = \alpha_{2\eta+2}$ in (3.1) we can get that

$$\lim_{\eta \rightarrow \infty} S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) = 0.$$

Therefore $\lim_{\eta \rightarrow \infty} S(\beta_\eta, \beta_\eta, \beta_{\eta+1}) = 0$. This shows that (β_η) is a Cauchy sequence in (X, S) . This implies that (β_η) converges in (X, S) , since (X, S) is complete.

Case(ii). Now let $\beta_{2\eta} = \beta_{2\eta+1}$ for some $\eta \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned} P(\alpha_{2\eta}, \alpha_{2\eta+1}) &= \frac{1}{2} S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) \\ &\leq S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}). \end{aligned}$$

Therefore, we must have

$$P(\alpha_{2\eta}, \alpha_{2\eta+1}) \leq S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}).$$

From (3.1) we can write that

$$\psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \leq \psi(P(\alpha_{2\eta}, \alpha_{2\eta+1})).$$

By definition of non-decreasing of ψ , we have

$$S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) \leq P(\alpha_{2\eta}, \alpha_{2\eta+1}).$$

Hence $P(\alpha_{2\eta}, \alpha_{2\eta+1}) = S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})$ and

$$K(\alpha_{2\eta}, \alpha_{2\eta+1}) = 0.$$

By (3.1), we have

$$\begin{aligned} \psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \\ \leq \psi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) - \phi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})). \end{aligned}$$

It follows that $-\phi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) \geq 0$. This will imply that $\phi(S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2})) = 0$ and hence $\beta_{2\eta+1} = \beta_{2\eta+2}$. Now for each $\eta \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \psi(S(f\alpha_{2\eta+1}, f\alpha_{2\eta+1}, g\alpha_{2\eta+2})) \\ \leq \psi(P(\alpha_{2\eta+1}, \alpha_{2\eta+2})) - \phi(P(\alpha_{2\eta+1}, \alpha_{2\eta+2})) \\ + DK(\alpha_{2\eta+1}, \alpha_{2\eta+2}), \end{aligned}$$

where,

$$\begin{aligned} P(\alpha_{2\eta+1}, \alpha_{2\eta+2}) &= \max \{S(R\alpha_{2\eta+1}, R\alpha_{2\eta+1}, T\alpha_{2\eta+2}) \\ &\frac{1}{2} [S(R\alpha_{2\eta+1}, R\alpha_{2\eta+1}, f\alpha_{2\eta+1}) + S(T\alpha_{2\eta+2}, T\alpha_{2\eta+2}, f\alpha_{2\eta+2})] \\ &\frac{1}{3} [S(R\alpha_{2\eta+1}, R\alpha_{2\eta+1}, g\alpha_{2\eta+2}) + S(T\alpha_{2\eta+2}, T\alpha_{2\eta+2}, f\alpha_{2\eta+1})] \} \end{aligned}$$

and

$$\begin{aligned} K(\alpha_{2\eta+1}, \alpha_{2\eta+2}) &= \min \{S(R\alpha_{2\eta+1}, R\alpha_{2\eta+1}, g\alpha_{2\eta+2}), \\ &S(T\alpha_{2\eta+2}, T\alpha_{2\eta+2}, f\alpha_{2\eta+1}) \}. \end{aligned}$$

Then we have

$$\begin{aligned} P(\alpha_{2\eta+1}, \alpha_{2\eta+2}) &= \max \{S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) \\ &\frac{1}{2} [S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) + S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})] \\ &\frac{1}{3} [S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+3}) + S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+2})] \} \end{aligned}$$

and

$$\begin{aligned} K(\alpha_{2\eta+1}, \alpha_{2\eta+2}) &= \min \{S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}), \\ &S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+2}) \} = 0. \end{aligned}$$

Then we have

$$\begin{aligned} P(\alpha_{2\eta+1}, \alpha_{2\eta+2}) &= \max \{S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) \\ &\frac{1}{2} [S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+2}) + S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})] \\ &\frac{1}{3} [S(\beta_{2\eta+1}, \beta_{2\eta+1}, \beta_{2\eta+3}) + S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+2})] \} \\ &= \frac{1}{2} S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3}), \quad (\text{by } \beta_{2\eta+1} = \beta_{2\eta+2}) \\ &\leq S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3}). \end{aligned}$$

Therefore

$$P(\alpha_{2\eta+1}, \alpha_{2\eta+2}) \leq S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3}).$$

Notice that

$$\begin{aligned} \psi(S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})) \\ \leq \psi(P(\alpha_{2\eta+1}, \alpha_{2\eta+2})) - \phi(P(\alpha_{2\eta+1}, \alpha_{2\eta+2})) \\ \leq \psi(P(\alpha_{2\eta+1}, \alpha_{2\eta+2})). \end{aligned}$$



By definition of non-decreasing of ψ , we have

$$S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3}) \leq P(\alpha_{2\eta+1}, \alpha_{2\eta+2}).$$

Therefore we must have

$$P(\alpha_{2\eta+1}, \alpha_{2\eta+2}) = S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3}).$$

By using (3.1) we get that

$$\begin{aligned} &\psi(S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})) \\ &\leq \psi(S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})) - \phi(S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})). \end{aligned}$$

This will imply that $-\phi(S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})) \geq 0$. It follows that $\phi(S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})) = 0$. Thus we have $\phi(S(\beta_{2\eta+2}, \beta_{2\eta+2}, \beta_{2\eta+3})) = 0$ and hence $\beta_{2\eta+2} = \beta_{2\eta+3}$. By continuing in this manner, we get that $\beta_{2\eta} = \beta_{2\eta+1} = \beta_{2\eta+2} = \beta_{2\eta+3} = \beta_{2\eta+4}$ and going on. Thus we proved that (β_η) is eventually constant sequence in (X, S) and hence (β_η) converges in (X, S) . From both cases, we conclude that (β_η) converges in (X, S) and hence the result is proved. \square

Remark 3.2. The converse of the Lemma 3.1 may not hold and the following example illustrates this.

Example 3.3. Consider the S -metric space (X, S) , where $X = [2, 13]$ with S -metric

$$S(\alpha, \beta, \gamma) := |\alpha - \beta| + |\beta - \gamma| + |\gamma - \alpha|,$$

for $\alpha, \beta, \gamma \in X$. Clearly (X, S) is not complete. Now we define four self-maps f, g, R and T on X as follows:

$$f(\alpha) := \begin{cases} 2, & \text{if } \alpha = 2 \\ 3, & \text{if } \alpha \neq 2 \end{cases} \quad g(\alpha) := \begin{cases} 2, & \text{if } \alpha = 2 \\ 3, & \text{if } \alpha \neq 2 \end{cases}$$

$$R(\alpha) := \frac{\alpha + 2}{2} \text{ and } T(\alpha) := \frac{\alpha + 2}{2}, \text{ for } \alpha \in X.$$

Also we define $\phi, \psi : [0, \infty) \rightarrow [0, \infty)$ by $\phi(k) := \frac{k}{8}$ and $\psi(k) := 3k$ for $\alpha \in [0, \infty)$ respectively.

(i) Clearly $fX = gX = \{2, 3\}$ and hence $fX \subset TX$ and $gX \subset TX$.

(i) Let $\alpha, \beta \in X$ be such that $\alpha = \beta = 2$.

Then we have

$$\begin{aligned} \psi(S(f\alpha, f\alpha, g\beta)) &= 3S(2, 2, 2) = 0 \\ &\leq 3P(\alpha, \beta) - \frac{P(\alpha, \beta)}{8} + DK(\alpha, \beta), \end{aligned}$$

for any $D \geq 0$. Now consider $\alpha = 2$ and $\beta \neq 2$. Then we have

$$\psi(S(f\alpha, f\alpha, g\beta)) = 3S(2, 2, 3) = 3 \cdot 2 \cdot |3 - 2| = 6.$$

Now we compute the following.

$$\begin{aligned} P(\alpha, \beta) &= \max \left\{ S\left(2, 2, \frac{\beta + 2}{2}\right), \right. \\ &\quad \left. \frac{1}{2} \left[S(2, 2, 2) + S\left(\frac{\beta + 2}{2}, \frac{\beta + 2}{2}, 3\right) \right] \right. \\ &\quad \left. \frac{1}{3} \left[S(2, 2, 3) + S\left(\frac{\beta + 2}{2}, \frac{\beta + 2}{2}, 2\right) \right] \right\} \\ &= \max \left\{ 2 \left| \frac{\beta + 2}{2} - 2 \right|, \frac{1}{2} \left[2 \left| \frac{\beta + 2}{2} - 3 \right| \right], \right. \\ &\quad \left. \frac{1}{3} \left[2|2 - 3| + 2 \left| \frac{\beta + 2}{2} - 2 \right| \right] \right\} \\ &= \max \left\{ |\beta - 2|, \frac{|\beta - 4|}{2}, \frac{2}{3} \left[1 + \frac{|\beta - 2|}{2} \right] \right\} = 11. \end{aligned}$$

Therefore $P(\alpha, \beta) = 11$. This will imply that $\psi(P(\alpha, \beta)) = 3(11) = 33$ and $\phi(P(\alpha, \beta)) = \frac{11}{8}$. Thus we have

$$\begin{aligned} \psi(S(f\alpha, f\alpha, g\beta)) &= 6 < \left(33 - \frac{11}{8} + DK(\alpha, \beta) \right) \\ &= \psi(P(\alpha, \beta)) - \phi(P(\alpha, \beta)) + DK(\alpha, \beta), \end{aligned}$$

for any $D \geq 0$. Let us consider $\alpha \neq 2$ and $\beta = 2$. Then we have

$$\psi(S(f\alpha, f\alpha, g\beta)) = 3S(3, 3, 2) = 6.$$

Now look at the following.

$$\begin{aligned} P(\alpha, \beta) &= \max \left\{ S\left(\frac{\alpha + 2}{2}, \frac{\alpha + 2}{2}, 2\right), \right. \\ &\quad \left. \frac{1}{2} \left[S\left(\frac{\alpha + 2}{2}, \frac{\alpha + 2}{2}, 3\right) + S(2, 2, 2) \right], \right. \\ &\quad \left. \frac{1}{3} \left[S(2, 2, 3) + S\left(\frac{\alpha + 2}{2}, \frac{\alpha + 2}{2}, 2\right) \right] \right\} \\ &= \max \left\{ 2 \left| \frac{\alpha + 2}{2} - 2 \right|, \frac{1}{2} \left[2 \left| \frac{\alpha + 2}{2} - 3 \right| \right], \right. \\ &\quad \left. \frac{1}{3} \left[2 \left| \frac{\alpha + 2}{2} - 2 \right| + 2|2 - 3| \right] \right\} \\ &= \max \left\{ |\alpha - 2|, \frac{|\alpha - 4|}{2}, \frac{2}{3} \left[1 + \frac{|\alpha - 2|}{2} \right] \right\} \\ &= \max \left\{ |\alpha - 2|, \frac{|\alpha - 4|}{2}, \frac{2}{3} \left[1 + \frac{|\alpha - 2|}{2} \right] \right\}. \end{aligned}$$

Thus $P(\alpha, \beta) = 11$. It follows that that $\psi(P(\alpha, \beta)) = 3(11) = 33$ and $\phi(P(\alpha, \beta)) = \frac{11}{8}$. Thus

$$\begin{aligned} \psi(S(f\alpha, f\alpha, g\beta)) &= 6 < \left(33 - \frac{11}{8} + DK(\alpha, \beta) \right) \\ &= \psi(P(\alpha, \beta)) - \phi(P(\alpha, \beta)) + DK(\alpha, \beta), \end{aligned}$$

for any $D \geq 0$. Let $\alpha \neq 2$ and $\beta \neq 2$. Then we have

$$\begin{aligned} \psi(S(f\alpha, f\alpha, g\beta)) &= 3S(3, 3, 3) = 0 \\ &\leq 3P(\alpha, \beta) - \frac{P(\alpha, \beta)}{8} + DK(\alpha, \beta), \end{aligned}$$



for any $D \geq 0$. In all the cases, $D = 0$ satisfying the inequality

$$\psi(S(f\alpha, f\alpha, g\beta)) \leq 3P(\alpha, \beta) - \frac{P(\alpha, \beta)}{8} + DK(\alpha, \beta)$$

for all $\alpha, \beta \in X$.

(iii) Let (α_η) be an associated sequence of $\alpha_0 \in X$ with respect to the four self-maps f, g, R and T . Now consider $\alpha_0 = 2$. Then we have $f\alpha_0 = 2 = T\alpha_1$ for some $\alpha_1 \in X$. This will imply that $\alpha_1 = 2$ and $g\alpha_1 = 2$. Also we have $g\alpha_1 = R\alpha_2$ for some $\alpha_2 \in X$. It follows that $\alpha_2 = 2$ and $f\alpha_2 = 2$. Then we have $f\alpha_2 = 2 = T\alpha_3$ for some $\alpha_3 \in X$. This implies that $\alpha_3 = 2$ and $g\alpha_3 = 2$. By continuing this process, we get the sequence $(f\alpha_0, g\alpha_1, f\alpha_2, g\alpha_3, \dots)$ as $(2, 2, 2, 2, \dots)$ converging to 2.

Let us consider the case $\alpha_0 \neq 2$. Then we have $f\alpha_0 = 3 = T\alpha_1$ for some $\alpha_1 \in X$. This will imply that $\alpha_1 = 4$ and $g\alpha_1 = 3$. Also we have $g\alpha_1 = R\alpha_2$ for some $\alpha_2 \in X$. It follows that $\alpha_2 = 4$ and $f\alpha_2 = 3$. Then we have $f\alpha_2 = 3 = T\alpha_3$ for some $\alpha_3 \in X$. This implies that $\alpha_3 = 4$ and $g\alpha_3 = 3$. By continuing this process, we get the sequence $(f\alpha_0, g\alpha_1, f\alpha_2, g\alpha_3, \dots)$ as $(3, 3, 3, 3, \dots)$ converging to 3. From above cases, we conclude that any associated sequence (α_η) of any $\alpha_0 \in X$ with respect to the four self-maps such that $(f\alpha_0, g\alpha_1, f\alpha_2, g\alpha_3, \dots)$ converges in (X, S) .

Theorem 3.4. In a S -metric space (X, S) , if there are four self-maps f, g, R and T on X and $D \geq 0$ satisfying

(A1)

$$\psi(S(f\alpha, f\alpha, g\beta)) \leq \psi(P(\alpha, \beta)) - \phi(P(\alpha, \beta)) + DK(\alpha, \beta)$$

for $\alpha, \beta \in X$ where

$$P(\alpha, \beta) = \max\{S(R\alpha, R\alpha, T\beta), S(R\alpha, R\alpha, f\alpha) + S(T\beta, T\beta, g\beta), S(R\alpha, R\alpha, g\beta)\}$$

and

$$K(\alpha, \beta) = \min\{S(R\alpha, R\alpha, T\beta), S(T\beta, T\beta, g\beta)\}.$$

(A2) $fX \subset TX$ and $gX \subset RX$

(A3) the sequence $(f\alpha_0, g\alpha_1, f\alpha_2, g\alpha_3, \dots)$ converges to a point $\gamma \in X$ in (X, S) for any associated sequence (α_η) of a point $\alpha_0 \in X$ with respect to the four self-maps and

(A4) the pairs (f, R) and (g, T) are compatible of type(R), then γ is the unique common fixed point for f, g, R and T provided one of the four mappings is continuous.

Proof. Suppose that the condition (A3) holds. Then $f\alpha_{2\eta} = T\alpha_{2\eta+1}$ and $g\alpha_{2\eta+1} = R\alpha_{2\eta+2}$ for $\eta \geq 0$ and hence $f\alpha_{2\eta} \rightarrow$

$\gamma, T\alpha_{2\eta+1} \rightarrow \gamma, g\alpha_{2\eta+1} \rightarrow \gamma$ and $R\alpha_{2\eta+2} \rightarrow \gamma$.

Assume that R is continuous. Since (f, R) is compatible of type (R), it follows that $RR\alpha_{2\eta}, fR\alpha_{2\eta} \rightarrow R\gamma$. We first claim that $R\gamma = \gamma$. For each $\eta \in \mathbb{N} \cup \{0\}$, we have

$$\psi(S(fR\alpha_{2\eta}, fR\alpha_{2\eta}, g\alpha_{2\eta+1})) \leq \psi(P(R\alpha_{2\eta}, \alpha_{2\eta+1})) - \phi(P(R\alpha_{2\eta}, \alpha_{2\eta+1})) + DK(R\alpha_{2\eta}, \alpha_{2\eta+1}),$$

where

$$P(R\alpha_{2\eta}, \alpha_{2\eta+1}) = \max\{S(RR\alpha_{2\eta}, RR\alpha_{2\eta}, T\alpha_{2\eta+1}), S(RR\alpha_{2\eta}, RR\alpha_{2\eta}, fR\alpha_{2\eta}) + S(T\alpha_{2\eta+1}, T\alpha_{2\eta+1}, g\alpha_{2\eta+1}), S(RR\alpha_{2\eta}, RR\alpha_{2\eta}, g\alpha_{2\eta+1})\}$$

and

$$K(R\alpha_{2\eta}, \alpha_{2\eta+1}) = \min\{S(RR\alpha_{2\eta}, RR\alpha_{2\eta}, T\alpha_{2\eta+1}), S(T\alpha_{2\eta+1}, T\alpha_{2\eta+1}, g\alpha_{2\eta+1})\}.$$

On letting $\eta \rightarrow \infty$, we have

$$\begin{aligned} \lim_{\eta \rightarrow \infty} P(R\alpha_{2\eta}, \alpha_{2\eta+1}) &= \max\{S(R\gamma, R\gamma, \gamma), S(\gamma, \gamma, \gamma) + S(\gamma, \gamma, \gamma), S(R\gamma, R\gamma, \gamma)\} \\ &= \max\{S(R\gamma, R\gamma, \gamma), 0 + 0, S(R\gamma, R\gamma, \gamma)\} \\ &= S(R\gamma, R\gamma, \gamma) \end{aligned}$$

and

$$\lim_{\eta \rightarrow \infty} K(R\alpha_{2\eta}, \alpha_{2\eta+1}) = \min\{S(R\gamma, R\gamma, \gamma), S(\gamma, \gamma, \gamma)\} = 0.$$

This will imply that

$$\psi(S(R\gamma, R\alpha\gamma, \gamma)) \leq \psi(S(R\gamma, R\alpha\gamma, \gamma)) - \phi(S(R\gamma, R\alpha\gamma, \gamma)) + D = 0.$$

It follows that $\phi(S(R\gamma, R\gamma, \gamma)) = 0$ and hence $S(R\gamma, R\gamma, \gamma) = 0$. Therefore $R\gamma = \gamma$. Let us now show that $f\gamma = \gamma$. For each $\eta \in \mathbb{N} \cup \{0\}$, we have

$$\psi(S(f\gamma, f\gamma, g\alpha_{2\eta+1})) \leq \psi(P(\gamma, \alpha_{2\eta+1})) - \phi(P(\gamma, \alpha_{2\eta+1})) + DK(\gamma, \alpha_{2\eta+1}),$$

where

$$P(\gamma, \alpha_{2\eta+1}) = \max\{S(R\gamma, R\gamma, T\alpha_{2\eta+1}), S(R\gamma, R\gamma, f\gamma) + S(T\alpha_{2\eta+1}, T\alpha_{2\eta+1}, g\alpha_{2\eta+1}), S(R\gamma, R\gamma, g\alpha_{2\eta+1})\}$$

and

$$K(\gamma, \alpha_{2\eta+1}) = \min\{S(R\gamma, R\gamma, T\alpha_{2\eta+1}), S(T\alpha_{2\eta+1}, T\alpha_{2\eta+1}, g\alpha_{2\eta+1})\}.$$



Now letting $\eta \rightarrow \infty$, we have

$$\begin{aligned} & \lim_{\eta \rightarrow \infty} P(R\gamma, \alpha_{2\eta+1}) \\ &= \max\{S(R\gamma, R\gamma, \gamma), S(R\gamma, R\gamma, f\gamma) + S(\gamma, \gamma, \gamma), S(R\gamma, R\gamma, \gamma)\} \\ &= \max\{S(\gamma, \gamma, \gamma), S(\gamma, \gamma, f\gamma) + S(\gamma, \gamma, \gamma), S(\gamma, \gamma, \gamma)\} \\ &= \max\{0, S(\gamma, \gamma, f\gamma) + 0, 0\} \\ &= S(\gamma, \gamma, f\gamma) \\ &= S(f\gamma, f\gamma, \gamma) \end{aligned}$$

and

$$\lim_{\eta \rightarrow \infty} K(\gamma, \alpha_{2\eta+1}) = \min\{S(R\gamma, R\gamma, \gamma), S(\gamma, \gamma, \gamma)\} = 0.$$

Therefore, we must have

$$\psi(S(f\gamma, f\gamma, \gamma)) \leq \psi(S(f\gamma, f\gamma, \gamma)) - \phi(S(f\gamma, f\gamma, \gamma)).$$

This will imply that $-\phi(S(f\gamma, f\gamma, \gamma)) \geq 0$. It follows that $\phi(S(f\gamma, f\gamma, \gamma)) = 0$ and hence $S(f\gamma, f\gamma, \gamma) = 0$. This will imply that $f\gamma = \gamma$. Since $f\gamma = \gamma$ and $fX \subset TX$, there exists $\theta_1 \in X$ such that $\gamma = T\theta_1 = f\gamma$.

For each $\eta \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} & \psi(S(f\gamma, f\gamma, g\theta_1)) \\ & \leq \psi(P(\gamma, \theta_1)) - \phi(P(\gamma, \theta_1)) + DK(\gamma, \theta_1), \end{aligned}$$

where

$$\begin{aligned} P(\gamma, \theta_1) &= \max\{S(R\gamma, R\gamma, T\theta_1), S(R\gamma, R\gamma, f\gamma) \\ &+ S(T\theta_1, T\theta_1, g\theta_1), S(R\gamma, R\gamma, g\theta_1)\} \\ &= \max\{S(\gamma, \gamma, \gamma), S(\gamma, \gamma, \gamma) + S(\gamma, \gamma, g\theta_1), S(\gamma, \gamma, g\theta_1)\} \\ &= S(\gamma, \gamma, g\theta_1). \end{aligned}$$

and

$$\begin{aligned} K(\gamma, \theta_1) &= \min\{S(\gamma, \gamma, T\theta_1), S(T\theta_1, T\theta_1, g\theta_1)\} \\ &= \min\{S(\gamma, \gamma, \gamma), S(\gamma, \gamma, g\theta_1)\} \\ &= 0. \end{aligned}$$

Therefore we must have

$$\psi(S(\gamma, \gamma, g\theta_1)) \leq \psi(S(\gamma, \gamma, g\theta_1)) - \phi(S(\gamma, \gamma, g\theta_1)).$$

This will imply that $-\phi(S(\gamma, \gamma, g\theta_1)) \geq 0$. It follows that $\phi(S(\gamma, \gamma, g\theta_1)) = 0$ and hence $S(\gamma, \gamma, g\theta_1) = 0$. This will imply that $g\theta_1 = \gamma$. Since the pair (g, T) is compatible of type (R) , we have $gT\theta_1 = Tg\theta_1$ and hence $g\gamma = T\gamma$. Finally we show that $T\gamma = \gamma$. For each $\eta \in \mathbb{N} \cup \{0\}$, we have

$$\psi(S(f\gamma, f\gamma, g\gamma)) \leq \psi(P(\gamma, \gamma)) - \phi(P(\gamma, \gamma)) + DK(\gamma, \gamma),$$

where

$$\begin{aligned} P(\gamma, \gamma) &= \max\{S(R\gamma, R\gamma, T\gamma), S(R\gamma, R\gamma, f\gamma) \\ &+ S(T\gamma, T\gamma, g\gamma), S(R\gamma, R\gamma, g\gamma)\} \\ &= \max\{S(\gamma, \gamma, T\gamma), S(\gamma, \gamma, \gamma) \\ &+ S(T\gamma, T\gamma, T\gamma), S(\gamma, \gamma, T\gamma)\} \\ &= S(\gamma, \gamma, T\gamma) \\ K(\gamma, \gamma) &= \min\{S(\gamma, \gamma, T\gamma), S(T\gamma, T\gamma, g\gamma)\} \\ &= \min\{S(\gamma, \gamma, T\gamma), S(T\gamma, T\gamma, T\gamma)\} \\ &= 0. \end{aligned}$$

Therefore we have,

$$\psi(S(\gamma, \gamma, T\gamma)) \leq \psi(S(\gamma, \gamma, T\gamma)) - \phi(S(\gamma, \gamma, T\gamma)).$$

This will imply $-\phi(S(\gamma, \gamma, T\gamma)) \geq 0$. It follows that

$$\phi(S(\gamma, \gamma, T\gamma)) = 0.$$

This implies that $S(\gamma, \gamma, T\gamma) = 0$ and hence $T\gamma = \gamma$. Therefore $f\gamma = g\gamma = R\gamma = T\gamma = \gamma$, showing that γ is a common fixed point for f, g, R and T . Now let us show the uniqueness of common fixed point for f, g, R and T . For this, let $\rho \in X$ be another common fixed point for f, g, R and T . Then we have $f\rho = g\rho = R\rho = T\rho = \rho$. Now we consider,

$$\begin{aligned} \psi(S(\gamma, \gamma, \rho)) &= \psi(S(f\gamma, f\gamma, g\rho)) \\ & \leq \psi(P(\gamma, \rho)) - \phi(P(\gamma, \rho)) + DK(\gamma, \rho), \end{aligned}$$

where

$$\begin{aligned} P(\gamma, \rho) &= \max\{S(R\gamma, R\gamma, T\rho), S(R\gamma, R\gamma, f\gamma) \\ &+ S(T\rho, T\rho, g\rho), S(R\gamma, R\gamma, g\rho)\} \\ &= \max\{S(\gamma, \gamma, \rho), S(\gamma, \gamma, \gamma) + S(\rho, \rho, \rho), S(\gamma, \gamma, \rho)\} \\ &= S(\gamma, \gamma, \rho) \end{aligned}$$

and

$$K(\gamma, \rho) = \min\{S(R\gamma, R\gamma, T\rho), S(T\rho, T\rho, g\rho)\} = 0.$$

It follows that

$$\psi(S(\gamma, \gamma, \rho)) \leq \psi(S(\gamma, \gamma, \rho)) - \phi(S(\gamma, \gamma, \rho)).$$

This will imply that $-\phi(S(\gamma, \gamma, \rho)) \geq 0$. This implies that $\phi(S(\gamma, \gamma, \rho)) = 0$ and hence $S(\gamma, \gamma, \rho) = 0$. Therefore $\gamma = \rho$ and hence the result proved. As an application of the above Theorem, we deduce the following result. \square

Corollary 3.5. Suppose that $\{f_k\}_{k=1}^r, \{g_l\}_{l=1}^s, \{R_m\}_{m=1}^t$ and $\{T_n\}_{n=1}^v$ are finite families of self maps of an S -metric space (X, S) with $f = \prod_{k=1}^r f_k, g = \prod_{l=1}^s g_l, R = \prod_{m=1}^t R_m$ and $T = \prod_{n=1}^v T_n$ such that all the conditions of the above Theorem are satisfied. Then f, g, R and T have a unique common fixed point.

Corollary 3.6. Suppose in a S -metric space (X, S) , there are four self-maps f, g, R and T on X and $D \geq 0$ satisfying



(H1)

$$S(f\alpha, f\alpha, g\beta) \leq P(\alpha, \beta) - \phi(P(\alpha, \beta)) + DK(\alpha, \beta)$$

for $\alpha, \beta \in X$ where

$$P(\alpha, \beta) = \max\{S(R\alpha, R\alpha, T\beta), S(R\alpha, R\alpha, f\alpha) + S(T\beta, T\beta, g\beta), S(R\alpha, R\alpha, g\beta)\}$$

and

$$K(\alpha, \beta) = \min\{S(R\alpha, R\alpha, T\beta), S(T\beta, T\beta, g\beta)\}.$$

(H2) $fX \subset TX$ and $gX \subset RX$

(H3) The sequence $(f\alpha_0, g\alpha_1, f\alpha_2, g\alpha_3, \dots)$ converges to a point $\gamma \in X$ in (X, S) for any associated sequence (α_n) of a point $\alpha_0 \in X$ with respect to the four self-maps

(H4) The pairs (f, R) and (g, T) are compatible of type(R), then γ is the unique common fixed point for f, g, R and T provided one of the four mappings is continuous. Then γ is the unique common fixed point for f, g, R and T .

Proof. We define $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(k) = k$ for $k \in [0, \infty)$. Therefore, the hypothesis of Theorem 3.4 is satisfied and hence the result proved. \square

4. Conclusion

We obtained a common fixed point result for four self maps by defining an almost generalized weakly contractive condition and compatible mappings of type(R) in S -metric spaces. Further, this result was utilized to prove a fixed point theorem for four families of self maps.

References

- [1] Z. Mustafa, B. Sims, A New Approach To Generalized Metric Spaces, *J. Nonlinear and Convex Anal*, 7(2006), 289–297.
- [2] S. Sedghi, N. Shobe and A. Aliouche, A Generalization of Fixed Point Theorem in S -Metric Spaces, *Math. Vesnik*, 64(2012), 258–266.
- [3] N. V. Dung, N.T. Hieu, and S.Radojevic, Fixed Point Theorems for g -Monotone Maps on Partially Ordered S -metric spaces, *Filomat*, 28(9), (2014), 1885–1898.
- [4] S.Sedghi, N.V.Dung, Fixed point theorems on S -metric spaces, *Mat. Vesnik*, 66(2014), 113–124.
- [5] J.K. Kim, S. Sedghi, N. Shobkolaei, Common Fixed Point Theorems for the R-weakly Commuting Mappings in S -metric Spaces, *J.Comput. Anal. Appl*, 19(2015), 751–759.
- [6] S. Sedghi, I. Altun, N. Shobe, M.A. Salahshour, Some Properties of S -metric Spaces and Fixed Point Results, *Kyungpook Math. J*, 54(2014), 113–122.
- [7] S. Chaipornjareansri, Fixed Point Theorems for Fw-Contractions in Complete S -Metric Spaces, *Thai J. Math*, 14(2016), 98–109.

- [8] G. V. R. Babu, Leta Bekere Kumssa, Fixed Points of α, ψ, ϕ -Generalized Weakly Contractive Maps and Property (P) in S -metric spaces, *Filomat*, 1(14), (2017).
- [9] S. Sedghi, N. Shobkolaei, M. Shahraki I, T. Dosenovic, Common fixed point of four maps in S -metric spaces, *Mathematical Sciences*, 12(2018), 137–143.
- [10] Gutti Venkata Ravindranadh Babu, Pericherla Durga Sailaja and Gadhavajjala Srichandana, Common Fixed Points of α, ψ, ϕ -Almost Generalized Weakly Contractive Maps in S -metric spaces, *Commun. Nonlinear Anal*, 7(1), (2019), 17–35.
- [11] Y Rohan, M. R. Singh and Shambhu, Common Fixed Point Of Compatible Mapping of Type (R) in Banach spaces, *Proc.of ath. Soci. Bhu*, 20(2004), 77–87.
- [12] V. Srinivas and V. Nagaraju, Common fixed point theorem on compatible mappings of type (P), *Gen. Math. Notes*, 21(2), (2014), 87–94.

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