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Perfect dominating sets and perfect domination polynomial of a star graph

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Abstract

The paper probes the perfect dominating sets for a star Graph and how the construction of the family of set forms by using this perfect dominating set. This collection of families of sets becomes the on coefficient of novel perfect dominating polynomial. The relations which gets identified with this on coefficients helps to develop the perfect dominating polynomial. The properties of the polynomial is also mentioned.

Keywords

Perfect dominating set, star graph, polynomial, perfect domination number, coefficient of perfect dominating set.

AMS Subject Classification

05C05, 05C.

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1. Introduction

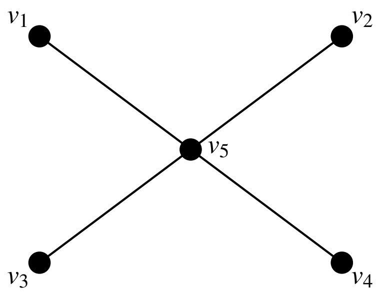
Let $G=(V,E)$ be a simple graph of order $|V|=n$. For any vertex $u \in V$, the open neighborhood of u is the set $N(u)=\{v \in V \; uv \in E\}$. A set $S \subseteq V$ is a dominating set of G, if every vertex $u \in V$ is a element of S or is adjacent to an element of S [4]. A graph $G=(V,E)$ is said to be a star graph if there exists a fixed vertex v (called the center of the star graph) such that $E=\{vu/u \in V \text{ and } u \neq v\}$ and a star graph is said to be an n-star graph if the number of vertices of the graph is n [8]. We denote the Star Graph by S_n. Let S_n be a Star Graph with n vertices. $\mathscr{S}_{pf}(n,i)$ be the family of perfect dominating sets of a Star Graph S_n with cardinality i and let $d_{pf}(S_n,i)=|\mathscr{S}_{pf}(n,i)|$. We call the polynomial $D_{pf}(S_n,x)=\sum_{i=1}^{n} d_{pf}(S_n,i)x^i$, the perfect domination polynomial of a Star Graph S_n. In this article the set $\{1,2,\ldots,n\}$ is represented by $[n]$.

2. Preliminaries

Let $\mathscr{S}_{pf}(n,i)$ be the family of perfect dominating sets of the Star Graph S_n with cardinality i. Then the perfect dominating sets of the Star Graph S_n is investigated by as follows;

Definition 2.1. *[7] The dominating set S is a perfect dominating set if $|N(u) \cap S|=1$ for each $u \in V-S$, or equivalently, if every vertex u in $V-S$ is adjacent to exactly one vertex in S. The perfect domination number γ_{pf} is the minimum cardinality of a perfect dominating set in G.*

Example 2.2. *Consider S_5*



Here, $\{v_5\}$ is the only perfect dominating set of cardinality 1.
The perfect dominating sets of cardinality 2 are $\{v_1,v_5\}$, $\{v_2,v_5\}$, $\{v_3,v_5\}$ and $\{v_4,v_5\}$.
The perfect dominating sets of cardinality 3 are $\{v_1,v_5,v_2\}$, $\{v_1,v_5,v_3\}$, $\{v_1,v_5,v_4\}$, $\{v_2,v_5,v_3\}$, $\{v_2,v_5,v_4\}$ and $\{v_3,v_5,v_4\}$.
The perfect dominating sets of cardinality 4 are $\{v_1,v_2,v_3,v_5\}$, $\{v_1,v_2,v_4,v_5\}$, $\{v_1,v_3,v_4,v_5\}$ and $\{v_2,v_3,v_4,v_5\}$.
The perfect dominating sets of cardinality 5 is $\{v_1,v_2,v_3,v_4,v_5\}$.

The following lemmas are required to prove our main results in this article.

Lemma 2.3. $\gamma_{pf}(S_n) = 1$ *for* $n \in \mathbb{N}$.

Proof. As centre is always in a perfect dominating set of a graph and it dominate all other vertices of a graph S_n. Therefore $\gamma_{pf}(S_n) = 1$ for $n \in \mathbb{N}$. □

By the defnition of perfect domination number, we have the following lemma.

Lemma 2.4. *For $n \geq 3$, $\mathscr{S}_{pf}(n,i) = \emptyset$ if and only if $i > n$.*

Proof. Let S_n be a star graph with n vertices.
Any member of $\mathscr{S}_{pf}(n,i)$ contains atmost n vertices.
Therefore, $\mathscr{S}_{pf}(n,i) = \emptyset$ for $i > n$.
Conversely, when $i > n$ then by definition of perfect dominating set $\mathscr{S}_{pf}(n,i) = \emptyset$. □

3. Main Results

3.1 Perfect dominating sets

Lemma 3.1. *Let $\mathscr{S}_{pf}(n,i)$ the family of perfect dominating sets of a star graph S_n with cardinality i then,*

i) $\mathscr{S}_{pf}(n,n) = \{1,2,3,\ldots,n\}$ *for all* $n \in \mathbb{N}$

ii) $\mathscr{S}_{pf}(2,1) = \{\{1\},\{2\}\}$

iii) $\mathscr{S}_{pf}(n,0) = \emptyset$ *for all* $n \in \mathbb{N}$

iv) $\mathscr{S}_{pf}(n,1) = \{n\}$ *for $n \geq 3$, where n is a centre of S_n*

v) $\mathscr{S}_{pf}(n,2) = \{\{1,n\},\{2,n\},\ldots,\{n-1,n\}\}$ *for $n \geq 3$, where n is a centre of S_n*

Proof. i) For, $V(G)$ is always a perfect dominating set of graph G then $\mathscr{S}_{pf}(n,n) = \{1,2,3,\ldots,n\}$.

ii) Here,S_2 is a path with two vertices then by the definition of a perfect dominating set we have $\{\{1\},\{2\}\}$ are the perfect dominating set of S_2.

iii) Since, there does not exist a perfect dominating set with cardinality 0 for a star graph S_n. Hence, $\mathscr{S}_{pf}(n,0) = \emptyset$ for all $n \in N$.

iv) Since, n is a centre of S_n the only vertex adjacent to $\{V(S_n) - \{n\}\}$ is n.
Hence, $\mathscr{S}_{pf}(n,1) = \{n\}$.

v) Every vertices from $\{V(G) - \{n\}\}$ is adjacent to $\{n\}$ also $\{n\}$ is a center of S_n therefore, $\mathscr{S}_{pf}(n,2) = \{\{1,n\}, \{2,n\},\ldots,\{n-1,n\}\}$.

□

Lemma 3.2. *Let S_n be a star graph with n vertices , then for $n \geq 3$, $d_{pf}(S_n) = \binom{n-1}{i-1}$.*

Proof. Let S_n be a star graph and $v \in V(S_n)$ be the centre of S_n . The number of subsets of $V(S_n)$ with cardinality i is $\binom{n}{i}$.
Also, every perfect dominating sets of S_n has a vertex v.
Now, number of subsets of $V(S_n) - v$ with cardinality i is $\binom{n-1}{i}$. Therefore, we get

$$\begin{aligned}
& d_{pf}(S_n,i) \\
= & \binom{n}{i} - \binom{n-1}{i} \\
= & \frac{n!}{i!(n-i)!} - \frac{(n-1)!}{i![(n-1)-i]!} \\
= & \frac{(n-i+1)(n-i+2)\ldots n}{i!} \\
& -\frac{(n-i)(n-i+1)\ldots(n-1)}{i!} \\
= & \left[\frac{(n-i+1)(n-i+2)\ldots(n-1)}{(i-1)!}\right] \\
& \left[\frac{n-(n-i)}{i}\right] \\
= & \binom{n-1}{i-1}
\end{aligned}$$

□

Lemma 3.3. *Let S_n be a star graph with n vertices , then for $n \geq 3$, $d_{pf}(S_n,i) = d_{pf}(S_{n-1},i) + d_{pf}(S_{n-1},i-1)$*

Proof. By lemma 3.2 we have, $d_{pf}(S_{n-1},i) = \binom{n-2}{i-1}$

and $d_{pf}(S_{n-1}, i-1) = \binom{n-2}{i-2}$ Therefore,

$$
\begin{aligned}
& d_{pf}(S_{n-1}, i) + d_{pf}(S_{n-1}, i-1) \\
= & \binom{n-2}{i-1} + \binom{n-2}{i-2} \\
= & \frac{(n-2)!}{(i-1)![n-2-(i-1)]!} \\
& - \frac{(n-2)!}{(i-2)![n-2-(i-2)]!} \\
= & \frac{(n-2)!}{(i-1)!(n-i-1)!} \\
& + \frac{(n-2)!}{(i-2)!(n-i)!} \\
= & \frac{(n-i)(n-i+1)\dots(n-2)}{(i-1)!} \\
& + \frac{(n-i+1)(n-i+2)\dots(n-2)}{(i-2)!} \\
= & \left[\frac{(n-i+1)(n-i+2)\dots(n-2)}{(i-2)!}\right] \\
& \left[\frac{n-i}{i-1} + 1\right] \\
= & \binom{n-1}{i-1}
\end{aligned}
$$

□

Lemma 3.4. *Let S_n be a star graph with n vertices , then*

i) $d_{pf}(S_2, 1) = 2$

ii) $d_{pf}(S_n, 0) = 0$ *for every* $n \in N$

iii) $d_{pf}(S_n, 1) = 1$ *for every* $n \geq 3$

Proof. The proof is follows from lemma 3.1 (ii),(iii) & (iv). Using the above lemma we obtain $d_{pf}(S_n, i)$ for $1 \leq n \leq 15$ as shown in the following **Table 1**.

Table 1

□

3.2 Perfect Domination Polynomial

Definition 3.5. *Let S_n be a Star Gaph with n vertices. Let $\mathscr{S}_{pf}(n,i)$ be the family of perfect dominating sets of the Star Graph with cardinality i and $d_{pf}(S_n, i) = |\mathscr{S}_{pf}(n,i)|$. Then the Perfect dominating polynomial of the Star Graph S_n is given by $D_{pf}(S_n, x) = \sum_{i=1}^{n} d_{pf}(S_n, i)x^i$.*

Theorem 3.6. *If $\mathscr{S}_{pf}(n,i)$ be a family of perfect dominating sets with cardinality i then, for every $n \geq 4$, $D_{pf}(S_n, x) = D_{pf}(S_{n-1}, x) + xD_{pf}(S_{n-1}, x)$ with initial values $D_{pf}(S_3, x) = x + 2x^2 + x^3$, $D_{pf}(S_2, x) = 2x + x^2$, $D_{pf}(S_1, x) = x$.*

Proof.

$$
\begin{aligned}
& D_{pf}(S_n, x) \\
= & \sum_{i=1}^{n} d_{pf}(S_n, i)x^i \\
= & \sum_{i=1}^{n} [d_{pf}(S_{n-1}, i) \\
& \quad + d_{pf}(S_{n-1}, i-1)]x^i \\
= & \sum_{i=1}^{n} d_{pf}(S_{n-1}, i)x^i \\
& \quad + \sum_{i=1}^{n} d_{pf}(S_{n-1}, i-1)x^i \\
= & \sum_{i=1}^{n-1} d_{pf}(S_{n-1}, i)x^i \\
& \quad + x\sum_{i=1}^{n} d_{pf}(S_{n-1}, i)x^{i-1} \\
= & D_{pf}(S_{n-1}, x) + xD_{pf}(S_{n-1}, x)
\end{aligned}
$$

□

i \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7	8	9	10	11	12	13	14
3			1	3	6	10	15	21	28	36	45	55	66	78	91
4				1	4	10	20	35	56	84	120	165	220	286	364
5					1	5	15	35	70	126	210	330	495	715	1001
6						1	6	21	56	126	252	462	792	1287	2002
7							1	7	28	84	210	462	924	1716	3003
8								1	8	36	120	330	792	1716	3432
9									1	9	45	165	495	1287	3003
10										1	10	55	220	715	2002
11											1	11	66	286	1001
12												1	12	78	364
13													1	13	91
14														1	14
15															1

Theorem 3.7. *Sum of the Coefficient of a perfect dominating polynomial of a graph S_n is 2^{n-1}. ie.,* $\sum_{i=1}^{n} d_{pf}(S_n,i) = 2^{n-1}$

Proof.

$$\begin{aligned}
& \sum_{i=1}^{n} d_{pf}(S_n,i) \\
= \;& d_{pf}(S_n,1) + d_{pf}(S_n,2) \\
& \quad + \cdots + d_{pf}(S_n,n) \\
= \;& \binom{n-1}{0} + \binom{n-1}{1} \\
& \quad + \cdots + \binom{n-1}{i-1} \\
= \;& 2^{n-1}
\end{aligned}$$

□

Example 3.8. *Consider the Star Graph S_5 with 5 vertices. We construct a perfect dominating polynomial $D_{pf}(S_5,x)$ using **Theorem 3.6** & **Table 1***

Then the Perfect dominating polynomial of a Star Graph S_5 is given by $D_{pf}(S_5,x) = \sum_{i=1}^{5} d_{pf}(S_5,i)x^i$. From the Table 1 we have $d_{pf}(S_5,1) = 1$, $d_{pf}(S_5,2) = 4$, $d_{pf}(S_5,3) = 6$, $d_{pf}(S_5,4) = 4$, $d_{pf}(S_5,5) = 1$.
Hence, $D_{pf}(S_5,x) = x + 4x^2 + 6x^3 + 4x^4 + x^5$.

Theorem 3.9. *The coefficients of $D_{pf}(S_n,x)$ have the following properties*

i) $d_{pf}(S_n,2) = n-1$ *for every* $n \geq 3$

ii) $d_{pf}(S_n,n-2) = \frac{(n-1)(n-2)}{2}$ *for every* $n \geq 3$

iii) $d_{pf}(S_n,n-3) = \frac{(n-1)(n-2)(n-3)}{6}$ *for every* $n \geq 4$

iv) $d_{pf}(S_n,n-1) = n-1$ *for every* $n \geq 3$

v) $d_{pf}(S_n,n) = 1$ *for every* $n \geq 3$

vi) $d_{pf}(S_n,n-4) = \frac{(n-1)(n-2)(n-3)(n-4)}{24}$ *for every* $n \geq 5$

Theorem 3.10. *For every $n \in N$ and $1 \leq i \leq n$. $|\mathscr{S}_{pf}(n,i)|$ is the coefficient of $u^n v^i$ in the expansion of the function* $f(u,v) = \frac{uv(1+uv+u+2u^2+u^3+3u^2v+u^2v^2+3u^3v+3u^3v^2+u^3v^3)}{(1-u-uv)}$

Proof. First we set $f(u,v) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} |\mathscr{S}_{pf}(n,i)|u^n v^i$.

By using the recursive formula for $|\mathscr{S}_{pf}(n,i)|$ we can write $f(u,v)$ as follows:

$$\begin{aligned}
& f(u,v) \\
= \;& \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}(|\mathscr{S}_{pf}(n-1,i)| + \\
& \quad |\mathscr{S}_{pf}(n-1,i-1)|)u^n v^i \\
= \;& u\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}|\mathscr{S}_{pf}(n-1,i)|u^{n-1}v^i + \\
& \quad uv\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}|\mathscr{S}_{pf}(n-1,i-1)| \\
& \quad\quad u^{n-1}v^{i-1} \\
= \;& u(|\mathscr{S}_{pf}(1,1)|uv + |\mathscr{S}_{pf}(1,2)|uv^2 \\
& +|\mathscr{S}_{pf}(2,1)|u^2v + |\mathscr{S}_{pf}(2,2)|u^2v^2 \\
& +|\mathscr{S}_{pf}(2,3)|u^2v^3 + |\mathscr{S}_{pf}(3,1)|u^3v \\
& +|\mathscr{S}_{pf}(3,2)|u^3v^2 + |\mathscr{S}_{pf}(3,3)|u^3v^3 \\
& +|\mathscr{S}_{pf}(3,4)|u^3v^4) + uf(u,v) \\
& +uv(|\mathscr{S}_{pf}(0,0)| + |\mathscr{S}_{pf}(1,0)|u \\
& +|\mathscr{S}_{pf}(1,1)|uv + |\mathscr{S}_{pf}(1,2)|uv^2 \\
& +|\mathscr{S}_{pf}(2,0)|u^2 + |\mathscr{S}_{pf}(2,1)|u^2v \\
& +|\mathscr{S}_{pf}(2,2)|u^2v^2 + |\mathscr{S}_{pf}(2,3)|u^2v^3 \\
& +|\mathscr{S}_{pf}(3,0)|u^3 + |\mathscr{S}_{pf}(3,1)|u^3v \\
& +|\mathscr{S}_{pf}(3,2)|u^3v^2 + |\mathscr{S}_{pf}(3,3)|u^3v^3 \\
& +|\mathscr{S}_{pf}(3,4)|u^3v^4) + uvf(u,v).
\end{aligned}$$

Substituting the values from **Table 1** also for $|\mathscr{S}_{pf}(n,0)| = 0$ for all $n \in N$ and $|\mathscr{S}_{pf}(0,0)| = 1$ we have,

$$\begin{aligned}
& f(u,v) \\
= \;& u(uv + 2u^2v + u^2v^2 + u^3v + 2u^3v^2 \\
& +u^3v^3) + uf(u,v) + uv(1 + uv \\
& +2u^2v + u^2v^2 + u^3v + 2u^3v^2 \\
& +u^3v^3) + uvf(u,v) \\
= \;& uv(1 + uv + 2u^2v + u^2v^2 + u^3v \\
& +2u^3v^2 + u^3v^3 + u + 2u^2 + u^2v \\
& +u^3 + 2u^3v + u^3v^2) + uf(u,v) \\
& +uvf(u,v) \\
& f(u,v)(1-u-uv) \\
= \;& uv(1 + uv + u + 2u^2 + u^3 + 3u^2v \\
& +u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3) \\
& \text{Hence,} f(u,v) \\
= \;& \left[\frac{1}{(1-u-uv)}\right][uv(1 + uv + u \\
& +2u^2 + u^3 + 3u^2v + u^2v^2 + 3u^3v \\
& +3u^3v^2 + u^3v^3)]
\end{aligned}$$

□

4. Conclusion

The paper sums up findings of how perfect dominating polynomial is structured up by perfect dominating set.

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