Malaya Journal of Matematik, Vol. 8, No. 4, 1756-1765, 2020
https://doi.org/10.26637/MJM0804/0072

Symmetric and generating functions for some generalized polynomials

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Abstract
In this paper, we define generalized Gaussian Padovan numbers, generalized Gaussian Padovan polynomials and generalized trivariate Fibonacci polynomials, we present the new generating functions of these generalizations, and as special cases, we investigate Gaussian Padovan numbers and polynomials, Gaussian Pell Padovan numbers and polynomials and trivariate Fibonacci and Lucas polynomials with their generating functions. Moreover, we give the new generating functions of some generalized Vieta polynomials.

Keywords
Symmetric functions, generating functions, generalized Gaussian Padovan numbers, generalized trivariate Fibonacci polynomials, generalized Vieta-Pell polynomials, generalized Vieta-Jacobsthal polynomials.

AMS Subject Classification
05E05, 11B39.

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Article History: Received **24** June **2020**; Accepted **13** September **2020**

Contents

1. Introduction

Many numbers and polynomials sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. The third order recurrence sequence have been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation. There is a many generalizations of numbers and polynomials of third order in literature which are defined recursively. For exemple, Soykan and Tasdemir in [13], given the Gaussian generalized Tribonacci numbers $\{GV_n\}_{n\geq 0}=\{GV_n(GV_0,GV_1,GV_2)\}_{n\geq 0}$ by

$$\begin{cases} GV_n = GV_{n-1}+GV_{n-2}+GV_{n-3},\ n\geq 3 \\ GV_0 = c_0+i(c_2-c_1-c_0),\ GV_1=c_1+ic_0, \\ GV_2 = c_2+ic_1. \end{cases}$$

Let now we define some generalizations for some numbers and polynomials of third order linear recurrence sequences, we begin by the recurrence relation of generalized Gaussian Padovan numbers $\{GN_n\}_{n\geq 0}=\{GN_n(a,b)\}_{n\geq 0}$ as:

$$\begin{cases} GN_n = aGN_{n-2}+GN_{n-3},\ n\geq 3 \\ GN_0 = 1+bi,\ GN_1=GN_2=1+i \end{cases}.$$

Special cases of $\{GN_n\}_{n\geq 0}$ are Gaussian Padovan numbers $GN_n(1,0)=GP_n$ and Gaussian Pell Padovan numbers $GN_n(2,-1)=GR_n$. We formally define them as follows:

Gaussian Padovan numbers is defined by

$$GP_n = GP_{n-2}+GP_{n-3},\ \text{for all } n\geq 3,$$

with initial conditions $GP_0=1$, $GP_1=GP_2=1+i$ and Gaussian Pell Padovan numbers is defined by

$$GR_n = 2GR_{n-2}+GR_{n-3},\ \text{for all } n\geq 3,$$

with initial conditions $GR_0=1-i$, $GR_1=GR_2=1+i$, see the paper [5].

Also, we define the generalized trivariate Fibonacci polynomials $\{W_n(x,y,t)\}_{n\geq 0}$ as follows

$$\begin{cases} W_n(x,y,t) = xW_{n-1}(x,y,t) + yW_{n-2}(x,y,t) + \\ tW_{n-3}(x,y,t),\ n \geq 3 \\ W_0(x,y,t) = a,\ W_1(x,y,t) = b + cx, \\ W_2(x,y,t) = dx + ey + fx^2. \end{cases} \tag{1.1}$$

If we take $\{a = c = e = f = 0,\ b = d = 1\}$ and $\{a = 3,\ b = d = 0,\ c = f = 1,\ e = 2\}$ in (1.1) we give the following definitions.

Definition 1.1. *[6] For any integer $n \geq 0$, the trivariate Fibonacci polynomials, denoted by $\{H_n(x,y,t)\}_{n\geq 0}$ is defined recursively by*

$$\begin{aligned} H_n(x,y,t) = &xH_{n-1}(x,y,t) + yH_{n-2}(x,y,t) \\ &+ tH_{n-3}(x,y,t), \end{aligned}$$

with the initials

$$H_0(x,y,t) = 0,\ H_1(x,y,t) = 1 \text{ and } H_2(x,y,t) = x.$$

Definition 1.2. *[6] For any integer $n \geq 0$, the trivariate Lucas polynomials, denoted by $\{K_n(x,y,t)\}_{n\geq 0}$ is defined recursively by*

$$\begin{aligned} K_n(x,y,t) = &xK_{n-1}(x,y,t) + yK_{n-2}(x,y,t) \\ &+ tK_{n-3}(x,y,t), \end{aligned}$$

with the initials

$$K_0(x,y,t) = 3,\ K_1(x,y,t) = x \text{ and } K_2(x,y,t) = x^2 + 2y.$$

Next, we define the generalized Gaussian Padovan polynomials $\{GM_n(x)\}_{n\geq 0}$ by the following third order recurrence relation:

$$\begin{cases} GM_n(x) = axGM_{n-2}(x) + GM_{n-3}(x),\ n \geq 3 \\ GM_0(x) = 1 + bi,\ GM_1(x) = GM_2(x) = 1 + i \end{cases}. \tag{1.2}$$

If we take $\{a = 1,\ b = 0\}$ and $\{a = 2,\ b = -1\}$ in the relationship (1.2), then we get the recurrence relations of Gaussian Padovan and Gaussian Pell Padovan polynomials $\{GP_n(x)\}_{n\geq 0}$ and $\{GR_n(x)\}_{n\geq 0}$ given in the Table 1.

Gaussian polynomials	Linear recurrence sequences	Initial conditions
Gaussian Padovan polynomials	$GP_n(x) = xGP_{n-2}(x) + GP_{n-3}(x), n \geq 3$	$GP_0(x) = 1$, $GP_1(x) = 1+i$, $GP_2(x) = 1+i$
Gaussian Pell Padovan polynomials	$GR_n(x) = 2xGR_{n-2}(x) + GR_{n-3}(x), n \geq 3$	$GR_0(x) = 1-i$, $GR_1(x) = 1+i$, $GR_2(x) = 1+i$

Table 1. Gaussian Padovan and Gaussian Pell Padovan polynomials.

In this part we define some generalized Vieta polynomials.

Definition 1.3. *[11] For $n \in \mathbb{N}$, the generalized Vieta-Jacobsthal polynomials, denoted by $\left\{GJ_{k,n}(x)\right\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\begin{cases} G_{k,n}(x) = G_{k,n-1}(x) - 2^k xG_{k,n-2}(x),\ n \geq 2 \\ G_{k,0}(x) = 0,\ G_{k,1}(x) = 1 \end{cases}.$$

Definition 1.4. *[11] For $n \in \mathbb{N}$, the generalized Vieta-Jacobsthal-Lucas polynomials, denoted by $\left\{g_{k,n}(x)\right\}_{n\in\mathbb{N}}$ is defined recursively by*

$$\begin{cases} g_{k,n}(x) = g_{k,n-1}(x) - 2^k xg_{k,n-2}(x),\ n \geq 2 \\ g_{k,0}(x) = 2,\ g_{k,1}(x) = 1 \end{cases}.$$

Definition 1.5. *[12] For $n \in \mathbb{N}$, the generalized Vieta-Pell polynomials, denoted by $\left\{P_{k,n}(x)\right\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\begin{cases} P_{k,n}(x) = 2^k xP_{k,n-1}(x) - P_{k,n-2}(x),\ n \geq 2 \\ P_{k,0}(x) = 0,\ P_{k,1}(x) = 1 \end{cases}.$$

Definition 1.6. *[12] For $n \in \mathbb{N}$, the generalized Vieta-Pell-Lucas polynomials, denoted by $\left\{Q_{k,n}(x)\right\}_{n\in\mathbb{N}}$ is defined recursively by*

$$\begin{cases} Q_{k,n}(x) = 2^k xQ_{k,n-1}(x) - Q_{k,n-2}(x),\ n \geq 2 \\ Q_{k,0}(x) = 2,\ Q_{k,1}(x) = 2^k x \end{cases}.$$

Definition 1.7. *[12] For $n \in \mathbb{N}$, the generalized Vieta-modified Pell polynomials, denoted by $\left\{q_{k,n}(x)\right\}_{n\in\mathbb{N}}$ is defined recursively by*

$$\begin{cases} q_{k,n}(x) = 2^k xq_{k,n-1}(x) - q_{k,n-2}(x),\ n \geq 2 \\ q_{k,0}(x) = 1,\ q_{k,1}(x) = 2^{k-1} x \end{cases}.$$

The remainder of this paper is organized as follows:

- **In section 2**, we first give the notion of the symmetric function and then we present and prove our main result which relates the symmetric function defined in this section with the symmetrizing operator $\delta_{p_1 p_2}$.
- **In section 3**, we derive the new generating functions of generalized Gaussian Padovan numbers, generalized Gaussian Padovan polynomials and generalized trivariate Fibonacci polynomias. In particular, the generating functions of Gaussian Padovan numbers and polynomials, Gaussian Pell Padovan numbers and polynomials, trivariate Fibonacci polynomials and trivariate Lucas polynomials are obtained.
- **In section 4**, by making use of the symmetric function we obtain the new generating functions of generalized Vieta-Jacobsthal polynomials, generalized Vieta-Jacobsthal-Lucas-polynomials, generalized Vieta-Pell polynomials, generalized Vieta-Pell-Lucas polynomials and generalized Vieta-modified Pell polynomials.

2. Preliminaries and main results

In this section, we give definitions and properties of the symmetric functions (for more details, we can see [7]). Let us now start at the following definitions.

Definition 2.1. *Let k and n be two positive integers and $\{p_1, p_2, ..., p_n\}$ are set of given variables the k-th complete homogeneous symmetric function $h_k(p_1, p_2, ..., p_n)$ is defined by*

$$h_k(p_1, p_2, ..., p_n) = \sum_{i_1+i_2+...+i_n=k} p_1^{i_1} p_2^{i_2} ... p_n^{i_n} \quad (k \geq 0),$$

with $i_1, i_2, ..., i_n \geq 0$.

Remark 2.2. *Set $h_0(p_1, p_2, ..., p_n) = 1$, by usual convention. For $k < 0$, we set $h_k(p_1, p_2, ..., p_n) = 0$.*

Definition 2.3. *[1] Let A and P be any two alphabets. We define $S_n(A-P)$ by the following form:*

$$\frac{\prod_{p \in P}(1-pz)}{\prod_{a \in A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-P) z^n, \tag{2.1}$$

with the condition $S_n(A-P) = 0$ for $n < 0$.

Equation (2.1) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A-P) z^n = \left(\sum_{n=0}^{\infty} S_n(A) z^n\right)\left(\sum_{n=0}^{\infty} S_n(-P) z^n\right), \tag{2.2}$$

where

$$S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P) S_j(A).$$

Remark 2.4. *Taking $A = \{0\}$ in (2.1) gives*

$$\sum_{n=0}^{\infty} S_n(-P) z^n = \prod_{p \in P}(1-pz).$$

Definition 2.5. *[2] Given a function g on $\mathbb{R}^n$, the divided difference operator is defined as follows*

$$\partial_{p_i p_{i+1}}(g) = \frac{g(p_1, \cdots, p_i, p_{i+1}, \cdots, p_n)}{p_i - p_{i+1}} - \frac{g(p_1, \cdots, p_{i-1}, p_{i+1}, p_i, p_{i+2}, \cdots, p_n)}{p_i - p_{i+1}}.$$

Definition 2.6. *Let n be a positive integer and $P_2 = \{p_1, p_2\}$ be set of given variables, then, the n-th symmetric function $S_n(p_1 + p_2)$ is defined by*

$$S_n(P_2) = S_n(p_1 + p_2) = \frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2},$$

with

$$\begin{aligned} S_0(P_2) &= S_0(p_1+p_2) = 1, \\ S_1(P_2) &= S_1(p_1+p_2) = p_1 + p_2, \\ S_2(P_2) &= S_2(p_1+p_2) = p_1^2 + p_1 p_2 + p_2^2, \\ &\vdots \end{aligned}$$

Definition 2.7. *[3] Given an alphabet $P_2 = \{p_1, p_2\}$, the symmetrizing operator $\delta_{p_1 p_2}^k$ is defined by*

$$\delta_{p_1 p_2}^k g(p_1) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2}, \text{ for all } k \in \mathbb{N}_0. \tag{2.3}$$

If $g(p_1) = p_1$, the operator (2.3) gives us

$$\delta_{p_1 p_2}^k g(p_1) = \frac{p_1^{k+1} - p_2^{k+1}}{p_1 - p_2} = S_k(p_1 + p_2).$$

The following theorem is one of the key tools of the proof of our main results. It has been proved in [4]. For the completeness of the paper we state its proof here.

Theorem 2.8. *Given two alphabets $P_2 = \{p_1, p_2\}$ and $A_3 = \{a_1, a_2, a_3\}$, we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} S_n(A_3) \partial_{p_1 p_2}(p_1^{n+1}) z^n \\ &= \frac{S_0(-A_3) - p_1 p_2 S_2(-A_3) z^2 - p_1 p_2 S_3(-A_3) S_1(P_2) z^3}{\left(\sum_{n=0}^{\infty} S_n(-A_3) p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A_3) p_2^n z^n\right)} \end{aligned} \tag{2.4}$$

with $S_0(-A_3) = 1$, $S_2(-A_3) = a_1 a_2 + a_1 a_3 + a_2 a_3$, $S_3(-A_3) = -a_1 a_2 a_3$.

Proof. Let $\sum_{n=0}^{\infty} S_n(A_3) z^n$ and $\sum_{n=0}^{\infty} S_n(-A_3) z^n$ be two sequences such that $\sum_{n=0}^{\infty} S_n(A_3) z^n = \frac{1}{\sum_{n=0}^{\infty} S_n(-A_3) z^n}$. On one hand, since $g(p_1) = \sum_{n=0}^{\infty} S_n(A_3) p_1^n z^n$ and $g(p_2) = \sum_{n=0}^{\infty} S_n(A_3) p_2^n z^n$, we have

$$\begin{aligned} \delta_{p_1 p_2} g(p_1) &= \delta_{p_1 p_2}\left(\sum_{n=0}^{\infty} S_n(A_3) p_1^n z^n\right) \\ &= \frac{p_1 \sum_{n=0}^{\infty} S_n(A_3) p_1^n z^n - p_2 \sum_{n=0}^{\infty} S_n(A_3) p_2^n z^n}{p_1 - p_2} \\ &= \sum_{n=0}^{\infty} S_n(A_3) \left(\frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2}\right) z^n \\ &= \sum_{n=0}^{\infty} S_n(A_3) \partial_{p_1 p_2}(p_1^{n+1}) z^n. \end{aligned}$$

On the other part, since $g(p_1) = \frac{1}{\sum_{n=0}^{\infty} S_n(-A_3) p_1^n z^n}$ and $g(p_2) =$

$\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A_3)p_2^n z^n}$, we have

$$\delta_{p_1p_2}g(p_1) = \frac{\frac{p_1}{\sum\limits_{n=0}^{\infty} S_n(-A_3)p_1^n z^n} - \frac{p_2}{\sum\limits_{n=0}^{\infty} S_n(-A_3)p_2^n z^n}}{p_1 - p_2}$$
$$= \frac{p_1 \sum\limits_{n=0}^{\infty} S_n(-A_3)p_2^n z^n - p_2 \sum\limits_{n=0}^{\infty} S_n(-A_3)p_1^n z^n}{(p_1-p_2)\left(\sum\limits_{n=0}^{\infty} S_n(-A_3)p_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A_3)p_2^n z^n\right)}$$
$$= \frac{\sum\limits_{n=0}^{\infty} S_n(-A_3)\frac{p_1p_2^n - p_2p_1^n}{p_1-p_2} z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A_3)p_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A_3)p_2^n z^n\right)}$$
$$= \frac{S_0(-A_3) - p_1p_2S_2(-A_3)z^2 - p_1p_2S_3(-A_3)S_1(P_2)z^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-A_3)p_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A_3)p_2^n z^n\right)}.$$

This completes the proof. □

3. The generating functions for some well-known generalized numbers and polynomials of third-order linear recurrence sequences

In this part, we now derive the new generating functions of generalized Gaussian Padovan numbers, generalized trivariate Fibonacci polynomials and generalized Gaussian Padovan polynomials. The technique used is based on the theory of the so called symmetric functions.

- For the case $A_3 = \{a_1, a_2, a_3\}$ and $P_2 = \{1, 0\}$ in theorem (2.8) we deduce the following lemma.

Lemma 3.1. *Given an alphabet $A_3 = \{a_1, a_2, a_3\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A_3)z^n = \frac{1}{(1-a_1z)(1-a_2z)(1-a_3z)}. \quad (3.1)$$

with

$$(1-a_1z)(1-a_2z)(1-a_3z) = 1-(a_1+a_2+a_3) + (a_1a_2+a_1a_3+a_2a_3)z^2 - a_1a_2a_3z^3.$$

Proposition 3.2. *Given an alphabet $A_3 = \{a_1, a_2, a_3\}$, we have*

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)z^n = \frac{z}{(1-a_1z)(1-a_2z)(1-a_3z)}. \quad (3.2)$$

Proof. By applying the operator $\delta^0_{a_1a_2}$ to the identity

$$\sum_{n=0}^{\infty} a_1^n z^n = \frac{1}{1-a_1z},$$

we get

$$\delta^0_{a_1a_2}\sum_{n=0}^{\infty} a_1^n z^n = \delta^0_{a_1a_2}\frac{1}{1-a_1z}$$

$$\iff \sum_{n=0}^{\infty} \delta^0_{a_1a_2}a_1^n z^n = \frac{z}{(1-a_1z)(1-a_2z)}$$
$$\iff \sum_{n=0}^{\infty} S_{n-1}(a_1+a_2)z^n = \frac{z}{(1-a_1z)(1-a_2z)}.$$

By applying the operator $\delta_{a_2a_3}$ to the identity

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+a_2)z^n = \frac{z}{(1-a_1z)(1-a_2z)},$$

we obtain

$$\delta_{a_2a_3}\sum_{n=0}^{\infty} S_{n-1}(a_1+a_2)z^n = \delta_{a_2a_3}\frac{z}{(1-a_1z)(1-a_2z)}$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \delta_{a_2a_3}S_{n-1}(a_1+a_2)z^n = \delta_{a_2a_3}\frac{z}{(1-a_1z)(1-a_2z)}$$
$$\Leftrightarrow \sum_{n=0}^{\infty} S_{n-1}(a_1+a_2+a_3)z^n = \frac{z}{(1-a_1z)}\delta_{a_2a_3}\frac{1}{(1-a_2z)}$$
$$\Leftrightarrow \sum_{n=0}^{\infty} S_{n-1}(a_1+a_2+a_3)z^n = \frac{z}{(1-a_1z)}\frac{1}{\prod\limits_{i=2}^{3}(1-a_iz)}.$$

Therefore

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)z^n = \frac{z}{(1-a_1z)(1-a_2z)(1-a_3z)}.$$

Hence, we obtain the desired result. □

Proposition 3.3. *Given an alphabet $A_3 = \{a_1, a_2, a_3\}$, we have*

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)z^n = \frac{z^2}{(1-a_1z)(1-a_2z)(1-a_3z)}. \quad (3.3)$$

Proof. By applying the operator $\delta^0_{a_2a_3}$ to the identity

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+a_2)z^n = \frac{z}{(1-a_1z)(1-a_2z)},$$

we have

$$\delta^0_{a_2a_3}\sum_{n=0}^{\infty} S_{n-1}(a_1+a_2)z^n = \delta^0_{a_2a_3}\frac{z}{(1-a_1z)(1-a_2z)}$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \delta^0_{a_2a_3}S_{n-1}(a_1+a_2)z^n = \delta^0_{a_2a_3}\frac{z}{(1-a_1z)(1-a_2z)}$$
$$\Leftrightarrow \sum_{n=0}^{\infty} S_{n-2}(a_1+a_2+a_3)z^n = \frac{z}{(1-a_1z)}\delta^0_{a_2a_3}\frac{1}{(1-a_2z)}$$
$$\Leftrightarrow \sum_{n=0}^{\infty} S_{n-2}(a_1+a_2+a_3)z^n == \frac{z}{(1-a_1z)}\frac{z}{\prod\limits_{i=2}^{3}(1-a_iz)}.$$

Therefore

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)z^n = \frac{z^2}{(1-a_1z)(1-a_2z)(1-a_3z)}.$$

This completes the proof. □

This part consists of three cases.

Case 1. The substitution of

$$\begin{cases} a_1+a_2+a_3=0 \\ a_1a_2+a_1a_3+a_2a_3=-a \\ a_1a_2a_3=1 \end{cases} \tag{3.4}$$

in the relationships (3.1), (3.2) and (3.3), we obtain

$$\sum_{n=0}^{\infty} S_n(A_3)z^n = \frac{1}{1-az^2-z^3}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)z^n = \frac{z}{1-az^2-z^3}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)z^n = \frac{z^2}{1-az^2-z^3}, \tag{3.7}$$

respectively.

Multiplying the equation (3.5) by $(1+bi)$ and adding it to the equation obtained by (3.6) multiplying by $(1+i)$ and adding it to the equation obtained by (3.7) multiplying by $(1-a+i(1-ab))$, then we obtain

$$\sum_{n=0}^{\infty}\left(\begin{array}{c}(1+bi)S_n(A_3)+(1+i)S_{n-1}(A_3)\\ +(1-a+i(1-ab))S_{n-2}(A_3)\end{array}\right)z^n = \frac{1+bi+(1+i)z+(1-a+i(1-ab))z^2}{1-az^2-z^3},$$

and we have the following theorem.

Theorem 3.4. *For $n\in\mathbb{N}$, the new generating function of generalized Gaussian Padovan numbers GN_n is given by*

$$\sum_{n=0}^{\infty} GN_n z^n = \frac{1+bi+(1+i)z+(1-a+i(1-ab))z^2}{1-az^2-z^3}, \tag{3.8}$$

with

$$GN_n = (1+bi)S_n(A_3)+(1+i)S_{n-1}(A_3)+ (1-a+i(1-ab))S_{n-2}(A_3).$$

Proof. The generalized Gaussian Padovan numbers can be considered as the coefficients of the formal power series

$$g(z)=\sum_{n=0}^{\infty} GN_n z^n.$$

Using the initial condition, we get

$$\begin{aligned} g(z) &= GN_0+GN_1z+GN_2z^2+\sum_{n=3}^{\infty} GN_n z^n \\ &= GN_0+GN_1z+GN_2z^2+\sum_{n=3}^{\infty} aGN_{n-2}z^n \\ &\quad +\sum_{n=3}^{\infty} GN_{n-3}z^n \\ &= GN_0+GN_1z+GN_2z^2+az^2\sum_{n=1}^{\infty} GN_n z^n \\ &\quad +z^3\sum_{n=0}^{\infty} GN_n z^n \\ &= GN_0+GN_1z+GN_2z^2-aGN_0z^2 \\ &\quad +az^2\sum_{n=0}^{\infty} GN_n z^n+z^3\sum_{n=0}^{\infty} GN_n z^n \\ &= 1+bi+(1+i)z+(1-a+i(1-ab))z^2 \\ &\quad +\left(az^2+z^3\right)g(z). \end{aligned}$$

Hence, we obtain

$$\begin{aligned}\left(1-az^2-z^3\right)g(z) &= 1+bi+(1+i)z \\ &\quad +(1-a+i(1-ab))z^2.\end{aligned}$$

Therefore

$$g(z)=\frac{1+bi+(1+i)z+(1-a+i(1-ab))z^2}{1-az^2-z^3}.$$

Thus, this completes the proof. □

- By putting $a=1$ and $b=0$ in the relationship (3.8), we obtain the following corollary.

Corollary 3.5. *[9] For $n\in\mathbb{N}$, the generating function of Gaussian Padovan numbers GP_n is given by*

$$\sum_{n=0}^{\infty} GP_n z^n = \frac{1+(1+i)z+iz^2}{1-z^2-z^3},$$

with

$$GP_n = S_n(A_3)+(1+i)S_{n-1}(A_3)+iS_{n-2}(A_3).$$

- Put $a=2$ and $b=-1$ in the relationship (3.8), we can state the following corollary.

Corollary 3.6. *[9] For $n\in\mathbb{N}$, the generating function of Gaussian Pell Padovan numbers GR_n is given by*

$$\sum_{n=0}^{\infty} GR_n z^n = \frac{1-i+(1+i)z+(-1+3i)z^2}{1-2z^2-z^3},$$

with

$$\begin{aligned} GR_n &= (1-i)S_n(A_3)+(1+i)S_{n-1}(A_3) \\ &\quad +(-1+3i)S_{n-2}(A_3). \end{aligned}$$

Case 2. The setting of

$$\begin{cases} a_1+a_2+a_3=x \\ a_1a_2+a_1a_3+a_2a_3=-y \\ a_1a_2a_3=t \end{cases}$$

in the relationships (3.1), (3.2) and (3.3), we obtain

$$\sum_{n=0}^{\infty} S_n(A_3)z^n = \frac{1}{1-xz-yz^2-tz^3}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)z^n = \frac{z}{1-xz-yz^2-tz^3}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)z^n = \frac{z^2}{1-xz-yz^2-tz^3}, \tag{3.11}$$

respectively.

Multiplying the equation (3.9) by (a) and adding it to the equation obtained by (3.10) multiplying by $(b+(c-a)x)$ and adding it to the equation obtained by (3.11) multiplying by $((d-b)x+(e-a)y+(f-c)x^2)$, then we obtain

$$\sum_{n=0}^{\infty}\left(\begin{array}{c} aS_n(A_3)+(b+(c-a)x)S_{n-1}(A_3) \\ +((d-b)x+(e-a)y+(f-c)x^2)S_{n-2}(A_3) \end{array}\right)z^n$$

$$=\frac{a+(b+(c-a)x)z+((d-b)x+(e-a)y+(f-c)x^2)z^2}{1-xz-yz^2-tz^3},$$

and we have the following theorem.

Theorem 3.7. *For $n\in\mathbb{N}$, the new generating function of generalized trivariate Fibonacci polynomials $W_n(x,y,t)$ is given by*

$$\sum_{n=0}^{\infty} W_n(x,y,t)z^n$$

$$=\frac{a+(b+(c-a)x)z+((d-b)x+(e-a)y+(f-c)x^2)z^2}{1-xz-yz^2-tz^3}, \tag{3.12}$$

with

$$W_n(x,y,t)=aS_n(A_3)+(b+(c-a)x)S_{n-1}(A_3)$$

$$+((d-b)x+(e-a)y+(f-c)x^2)S_{n-2}(A_3).$$

Proof. The generalized trivariate Fibonacci polynomials can be considered as the coefficients of the formal power series

$$g(x,y,t,z)=\sum_{n=0}^{\infty} W_n(x,y,t)z^n.$$

Using the initial condition, we get

$$\begin{aligned} g(x,y,t,z) &= W_0(x,y,t)+W_1(x,y,t)z+W_2(x,y,t)z^2 \\ &\quad +\sum_{n=3}^{\infty} W_n(x,y,t)z^n \\ &= W_0(x,y,t)+W_1(x,y,t)z+W_2(x,y,t)z^2 \\ &\quad +\sum_{n=3}^{\infty}(xW_{n-1}(x,y,t)+yW_{n-2}(x,y,t))z^n \\ &\quad +\sum_{n=3}^{\infty} tW_{n-3}(x,y,t)z^n \\ &= W_0(x,y,t)+W_1(x,y,t)z+W_2(x,y,t)z^2 \\ &\quad +xz\sum_{n=2}^{\infty} W_n(x,y,t)z^n+yz^2\sum_{n=1}^{\infty} W_n(x,y,t)z^n \\ &\quad +tz^3\sum_{n=0}^{\infty} W_n(x,y,t)z^n \\ &= W_0(x,y,t)+W_1(x,y,t)z+W_2(x,y,t)z^2 \\ &\quad -x(W_0(x,y,t)+W_1(x,y,t)z)z-yW_0(x,y,t)z^2 \\ &\quad +\left(xz+yz^2+tz^3\right)\sum_{n=0}^{\infty} W_n(x,y,t)z^n \\ &= a+(b+(c-a)x)z+((d-b)x+(e-a)y \\ &\quad +(f-c)x^2)z^2+\left(xz+yz^2+tz^3\right)g(x,y,t,z). \end{aligned}$$

Hence, we obtain

$$\left(1-xz-yz^2-tz^3\right)g(x,y,t,z)=a+(b+(c-a)x)z+$$

$$((d-b)x+(e-a)y+(f-c)x^2)z^2.$$

Therefore

$$g(x,y,t,z)=\frac{a+(b+(c-a)x)z}{1-xz-yz^2-tz^3}$$

$$+\frac{((d-b)x+(e-a)y+(f-c)x^2)z^2}{1-xz-yz^2-tz^3}.$$

Thus, this completes the proof. □

- By setting $a=c=e=f=0$ and $b=d=1$ in the relationship (3.12), we obtain the following corollary.

Corollary 3.8. *[9] For $n\in\mathbb{N}$, the generating function of trivariate Fibonacci polynomials $H_n(x,y,t)$ is given by*

$$\sum_{n=0}^{\infty} H_n(x,y,t)z^n=\frac{z}{1-xz-yz^2-tz^3},$$

with

$$H_n(x,y,t)=S_{n-1}(A_3).$$

- Put $a=3$, $b=d=0$, $c=f=1$ and $e=2$ in the relationship (3.12), we can state the following corollary.

Corollary 3.9. *[9] For $n \in \mathbb{N}$, the generating function of trivariate Lucas polynomials $K_n(x,y,t)$ is given by*

$$\sum_{n=0}^{\infty} K_n(x,y,t) z^n = \frac{3-2xz-yz^2}{1-xz-yz^2-tz^3},$$

with

$$K_n(x,y,t) = 3S_n(A_3) - 2xS_{n-1}(A_3) - yS_{n-2}(A_3).$$

Case 3. The substitution of

$$\begin{cases} a_1+a_2+a_3=0 \\ a_1a_2+a_1a_3+a_2a_3=-ax \\ a_1a_2a_3=1 \end{cases}$$

in the relationships (3.1), (3.2) and (3.3), we obtain

$$\sum_{n=0}^{\infty} S_n(A_3) z^n = \frac{1}{1-axz^2-z^3}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A_3) z^n = \frac{z}{1-axz^2-z^3}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A_3) z^n = \frac{z^2}{1-axz^2-z^3}, \tag{3.15}$$

respectively.

Multiplying the equation (3.13) by $(1+bi)$ and adding it to the equation obtained by (3.14) multiplying by $(1+i)$ and adding it to the equation (3.15) multiplying by $(1-ax+i(1-abx))$, then we obtain

$$\sum_{n=0}^{\infty} \left(\begin{array}{c} (1+bi) S_n(A_3) + (1+i) S_{n-1}(A_3) \\ +(1-ax+i(1-abx)) S_{n-2}(A_3) \end{array} \right) z^n$$

$$= \frac{1+bi+(1+i)z+(1-ax+i(1-abx))z^2}{1-axz^2-z^3},$$

and we have the following theorem.

Theorem 3.10. *For $n \in \mathbb{N}$, the new generating function of generalized Gaussian Padovan polynomials $GM_n(x)$ is given by*

$$\sum_{n=0}^{\infty} GM_n(x)z^n = \frac{1+bi+(1+i)z+(1-ax+i(1-abx))z^2}{1-axz^2-z^3}, \tag{3.16}$$

with

$$GM_n(x) = (1+bi) S_n(A_3) + (1+i)S_{n-1}(A_3) + (1-ax+i(1-abx))S_{n-2}(A_3).$$

Proof. The generalized Gaussian Padovan polynomials can be considered as the coefficients of the formal power series

$$g(x,z) = \sum_{n=0}^{\infty} GM_n(x) z^n.$$

Using the initial condition, we get

$$g(x,z) = GM_0(x) + GM_1(x)z + GM_2(x)z^2 + \sum_{n=3}^{\infty} GM_n(x) z^n$$

$$= GM_0(x) + GM_1(x)z + GM_2(x)z^2 + \sum_{n=3}^{\infty} (axGM_{n-2}(x) + GM_{n-3}(x)) z^n$$

$$= GM_0(x) + GM_1(x)z + GM_2(x)z^2 + axz^2 \sum_{n=1}^{\infty} GM_n(x) z^n + z^3 \sum_{n=0}^{\infty} GM_n(x) z^n$$

$$= GM_0(x) + GM_1(x)z + GM_2(x)z^2 - axGM_0(x)z^2 + axz^2 \sum_{n=0}^{\infty} GM_n(x) z^n + z^3 \sum_{n=0}^{\infty} GM_n(x) z^n$$

$$= 1+bi+(1+i)z+(1-ax+i(1-abx))z^2 + \left(axz^2+z^3\right) g(x,z).$$

Hence, we obtain

$$\left(1-axz^2-z^3\right) g(x,z) = 1+bi+(1+i)z + (1-ax+i(1-abx))z^2.$$

Therefore

$$g(x,z) = \frac{1+bi+(1+i)z+(1-ax+i(1-abx))z^2}{1-axz^2-z^3}.$$

Thus, this completes the proof. □

- By taking $a=1$ and $b=0$ in the relationship (3.16), we obtain the following corollary.

Corollary 3.11. *[9] For $n \in \mathbb{N}$, the generating function of Gaussian Padovan polynomials $GP_n(x)$ is given by*

$$\sum_{n=0}^{\infty} GP_n(x) z^n = \frac{1+(1+i)z+(1-x+i)z^2}{1-xz^2-z^3},$$

with

$$GP_n(x) = S_n(A_3) + (1+i) S_{n-1}(A_3) + (1-x+i) S_{n-2}(A_3).$$

- Put $a=2$ and $b=-1$ in the relationship (3.16), we can state the following corollary.

Corollary 3.12. *[9] For $n \in \mathbb{N}$, the generating function of Gaussian Pell Padovan polynomials $GR_n(x)$ is given by*

$$\sum_{n=0}^{\infty} GR_n(x) z^n = \frac{1-i+(1+i)z+(1-2x+i(1+2x))z^2}{1-2xz^2-z^3},$$

with

$$\begin{aligned} GR_n(x) &= (1-i)S_n(A_3)+(1+i)S_{n-1}(A_3)+ \\ &\quad (1-2x+i(1+2x))S_{n-2}(A_3). \end{aligned}$$

4. Construction of generating functions of some well-known generalized Vieta polynomials

In this part, we now derive the generating functions of generalized Vieta-Jacobsthal polynomials, generalized Vieta-Jacobsthal-Lucas polynomials, generalized Vieta-Pell polynomials, generalized Vieta-Pell-Lucas polynomials and generalized Vieta-modified Pell polynomials.

- For the case $A_3 = \{a_1, -a_2, 0\}$ in the relationships (3.1) and (3.2) we deduce the following corollaries.

Corollary 4.1. *Given an alphabet $A_2 = \{a_1, -a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2]) z^n = \frac{1}{1-(a_1-a_2)z-a_1a_2z^2}. \quad (4.1)$$

Corollary 4.2. *Given an alphabet $A_2 = \{a_1, -a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2]) z^n = \frac{z}{1-(a_1-a_2)z-a_1a_2z^2}. \quad (4.2)$$

This part consists of two cases.
Case 1. Assuming that

$$\begin{cases} a_1-a_2 = 2^k x \\ a_1a_2 = -1 \end{cases}$$

in the relationships (4.1) and (4.2), we obtain

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2]) z^n = \frac{1}{1-2^k xz+z^2}, \quad (4.3)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2]) z^n = \frac{z}{1-2^k xz+z^2}, \quad (4.4)$$

respectively, and we have the following corollary.

Corollary 4.3. *For $n \in \mathbb{N}$, the generating function of generalized Vieta-Pell polynomials $P_{k,n}(x)$ is given by*

$$\sum_{n=0}^{\infty} P_{k,n}(x) z^n = \frac{z}{1-2^k xz+z^2}, \quad (4.5)$$

with

$$P_{k,n}(x) = S_{n-1}(a_1+[-a_2]).$$

Multiplying the equation (4.3) by (2) and adding it to the equation obtained by (4.4) multiplying by $(-2^k x)$, then we have the following proposition and corollary.

Proposition 4.4. *For $n \in \mathbb{N}$, the generating function of generalized Vieta-Pell-Lucas polynomials $Q_{k,n}(x)$ is given by*

$$\sum_{n=0}^{\infty} Q_{k,n}(x) z^n = \frac{2-2^k xz}{1-2^k xz+z^2}. \quad (4.6)$$

Corollary 4.5. *The following identity holds true:*

$$Q_{k,n}(x) = 2S_n(a_1+[-a_2]) - 2^k x S_{n-1}(a_1+[-a_2]).$$

- Based on the relationships (4.5) and (4.6) and with $k=1$, we obtain the following corollaries.

Corollary 4.6. *For $n \in \mathbb{N}$, the generating function of Vieta-Pell polynomials $t_n(x)$ is given by*

$$\sum_{n=0}^{\infty} t_n(x) z^n = \frac{z}{1-2xz+z^2},$$

with

$$t_n(x) = S_{n-1}(a_1+[-a_2]).$$

Corollary 4.7. *For $n \in \mathbb{N}$, the generating function of Vieta-Pell-Lucas polynomials $s_n(x)$ is given by*

$$\sum_{n=0}^{\infty} s_n(x) z^n = \frac{2-2xz}{1-2xz+z^2},$$

with

$$s_n(x) = 2S_n(a_1+[-a_2]) - 2xS_{n-1}(a_1+[-a_2]).$$

Multiplying the equation (4.4) by $(-2^{k-1}x)$ and adding it to the equation (4.3), then we have the following proposition and corollary.

Proposition 4.8. *For $n \in \mathbb{N}$, the generating function of generalized Vieta-modified Pell polynomials $q_{k,n}(x)$ is given by*

$$\sum_{n=0}^{\infty} q_{k,n}(x) z^n = \frac{1-2^{k-1}xz}{1-2^k xz+z^2}. \quad (4.7)$$

Corollary 4.9. *The following identity holds true:*

$$q_{k,n}(x) = S_n(a_1+[-a_2]) - 2^{k-1} x S_{n-1}(a_1+[-a_2]).$$

Case 2. By taking

$$\begin{cases} a_1-a_2 = 1 \\ a_1a_2 = -2^k x \end{cases}$$

in the relationships (4.1) and (4.2), we obtain

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2]) z^n = \frac{1}{1-z+2^k xz^2}, \quad (4.8)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2]) z^n = \frac{z}{1-z+2^k xz^2}, \quad (4.9)$$

respectively, and we have the following corollary.

Corollary 4.10. *For $n \in \mathbb{N}$, the generating function of generalized Vieta-Jacobsthal polynomials $G_{k,n}(x)$ is given by*

$$\sum_{n=0}^{\infty} G_{k,n}(x) z^n = \frac{z}{1-z+2^k xz^2}, \tag{4.10}$$

with

$$G_{k,n}(x) = S_{n-1}(a_1 + [-a_2]).$$

Multiplying the equation (4.8) by (2) and adding it to the equation obtained by (4.9) multiplying by (-1), then we have the following proposition.

Proposition 4.11. *For $n \in \mathbb{N}$, the generating function of generalized Vieta-Jacobsthal-Lucas polynomials $g_{k,n}(x)$ is given by*

$$\sum_{n=0}^{\infty} g_{k,n}(x) z^n = \frac{2-z}{1-z+2^k xz^2}, \tag{4.11}$$

with

$$g_{k,n}(x) = 2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2]).$$

- Based on the relationships (4.10) and (4.11) and with $k = 1$, we obtain the following corollaries.

Corollary 4.12. *For $n \in \mathbb{N}$, the generating function of Vieta-Jacobsthal polynomials $G_n(x)$ is given by*

$$\sum_{n=0}^{\infty} G_n(x) z^n = \frac{z}{1-z+2xz^2},$$

with

$$G_n(x) = S_{n-1}(a_1 + [-a_2]).$$

Corollary 4.13. *For $n \in \mathbb{N}$, the generating function of Vieta-Jacobsthal-Lucas polynomials $g_n(x)$ is given by*

$$\sum_{n=0}^{\infty} g_n(x) z^n = \frac{2-z}{1-z+2xz^2},$$

with

$$g_n(x) = 2S_n(a_1 + [-a_2]) - S_{n-1}(a_1 + [-a_2]).$$

5. Conclusion

In this paper, we have generalized the work of Saba, Boussayoud and Abderrezzak [9] by introduced the generalizations of some numbers and polynomials. Some important generalizations of generating functions are produced. By making use of theorem (2.8), we have obtained propositions and corollaries which is led to generating function for a class of generalized vieta polynomials.

The results obtained in this work are promising, but there are other perspectives to follow in the field. Future work should be based on the extension of the generating functions of binary products of Gaussian generalized Tribonacci numbers with generalized polynomials of second-order linear recurrence sequences.

Acknowledgment

Authors are grateful to the Editor-In-Chief of the Journal and the anonymous reviewers for their constructive comments which improved the quality and the presentation of the paper.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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