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On derivations in CI-algebras

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Abstract
Derivation Map plays an important role in the study of different algebras. In this paper we introduce such a map for CI-algebras, with some of its examples and properties. Also we investigate how to extend a derivation map of a BE-algebra to that of a CI-algebra.

Keywords
CI-algebra, BE-algebra, Derivation.

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Contents

1. Introduction

Y. Imai and K. Iseki introduced two classes of abstract algebras, BCK-algebras and BCI- algebras ([**2**,**3**]). It is known that the class of BCK -algebras is a proper subclass of the class of BCI-algebras. As a generalization of these algebras, different authors initiated different algebras like BCH([**1**]), BH([**4**]), d([**8**]), Dual BCK algebras ([5]) etc. H. S. Kim and Y. H. Kim defined and studied an important algebraic structure known as BE-algebras ([6]) in 2006. In 2010, B. L. Meng ([7]) introduced the idea of CI-algebras as a generalization of BE-algebras and also studied many new concepts associated with CI-algebras. The concept of Cartesian product has been developed in 2013 ([9]) by us which plays a key role in the development of this CI- algebras. Some new concepts like, homomorphisms ([10]), multipliers ([11]), dual multipliers ([12]) were also introduced by us. In recent times derivation map ([13]) also plays an important role in the study of different algebras. In this paper we introduce derivation map for a CI-algebra with some of its examples and properties. Also we investigate how to extend a derivation map of a BE-algebra to that of a CI-algebra.

2. Preliminaries

Definition 2.1 ([6]). *An algebra $(P;*,1)$ consisting of a non-empty set P, a binary operation $*$ and a fixed element 1 is said to be a BE-algebra if the following postulates are satisfied:*

(P1) $v*v=1$.

(P2) $v*1=1$.

(P3) $1*v=v$.

(P4) $v*(w*r)=w*(v*r)$ *for all* $v,w,r\in P$.

Definition 2.2 ([7]). *An algebra $(P;*,1)$ consisting of a non-empty set P, a binary operation $*$ and a fixed element 1 is said to be a CI-algebra if if the following postulates are satisfied:*

(P1) $v*v=1$.

(P2) $1*v=v$.

(P3) $v*(w*r)=w*(v*r)$ *for all* $v,w,r\in P$.

Example 2.3 ([7]). *Let $P=1,p,q$ and let the binary operation $*$ be given by the Cayley table*

*	1	p	q
1	1	p	q
p	1	1	1
q	1	1	1

It is easy to see that $(P;*,1)$ is a CI-algebra. A binary relation $\leq$ in P can be defined by $v\leq w$ iff $v*w=1$.

Definition 2.4 ([7]). *A non-empty subset S of a CI-algebra P is said to be a sub-algebra of P if* $\mathrm{v} \in \mathrm{S}, \mathrm{w} \in \mathrm{S}$ *imply* $\mathrm{v} * \mathrm{w} \in \mathrm{S}$.

Theorem 2.5. *Let* $(P; *, 1)$ *be a BE-algebra and let* $a \notin P$. *A binary operation* $\circ$ *is defined on* $\mathrm{P} \cup \{\mathrm{a}\}$ *as follows:*
For any $\mathrm{v}, \mathrm{w} \in \mathrm{P} \cup \{\mathrm{a}\}$

$$v \circ w = \begin{cases} \mathrm{v} * \mathrm{w} \ \textit{if} \ \mathrm{v}, \mathrm{w} \in \mathrm{P} \\ \mathrm{a} \ \textit{if} \ \mathrm{v} = \mathrm{a}, \mathrm{w} \neq \mathrm{a} \\ \mathrm{a} \ \textit{if} \ \mathrm{v} \neq \mathrm{a}, \mathrm{w} = \mathrm{a} \\ 1 \ \textit{if} \ \mathrm{v} = \mathrm{w} = \mathrm{a}. \end{cases}$$

Then $(\mathrm{P} \cup \{\mathrm{a}\}; \circ, 1)$ *is a CI-algebra.*

Note 2.6. *The above result provides a method to extend a BE-algebra into a CI-algebra, by adjoining an element not in the given BE-algebra.*

3. Derivations in CI-Algebras

Let $(\mathrm{P}; *, 1)$ be a BE/Cl -algebra.

Definition 3.1. *A map* $\mathrm{d}: \mathrm{P} \to \mathrm{P}$ *is said to be*

(i) left - right (briefly, (l, r)) derivation at $\mathrm{v} \in \mathrm{P}$ *if it satisfies the condition*

$$\mathrm{d}(\mathrm{v} * \mathrm{w}) = (\mathrm{dv} * \mathrm{w}) + (\mathrm{v} * \mathrm{dw}),$$

for all $w \in P$, *where* $v + w$ *means* $(v * w) * w$.

(ii) right - left (briefly, (r, I)) derivation at $\mathrm{v} \in \mathrm{P}$ *if it satisfies the condition*

$$\mathrm{d}(\mathrm{v} * \mathrm{w}) = (\mathrm{v} * \mathrm{dw}) + (\mathrm{dv} * \mathrm{w}),$$

for all $w \in P$.

Definition 3.2. *A map* $d : P \to P$ *is called a derivation at* $v \in P$ *if it is both* $(1, r)$ *and* $(r, 1)$ *derivation at* $\mathrm{v} \in \mathrm{P}$.

Definition 3.3. *A map* $\mathrm{d}: \mathrm{P} \to \mathrm{P}$ *is called a derivation on* P *if it is a derivation at every point* $\mathrm{v} \in \mathrm{P}$.

Example 3.4. *(a) The identity map* $i(v) = v$ *on a* $CI-$ *algebra P is a derivation map.*
(b) The unit map $1^{\sim}(\mathrm{v}) = 1$ *is not a derivation map on a CI-algebra P.*
For,

(a) we have $i(v * w) = v * w$,

$$\begin{aligned} &(\mathrm{i}(\mathrm{v}) * \mathrm{w}) + (\mathrm{v} * \mathrm{i}(\mathrm{w})) \\ &\quad = (\mathrm{v} * \mathrm{w}) + (\mathrm{v} * \mathrm{w}) \\ &\quad = ((\mathrm{v} * \mathrm{w}) * (\mathrm{v} * \mathrm{w})) * (\mathrm{v} * \mathrm{w}) \\ &\quad = 1 * (\mathrm{v} * \mathrm{w}) = (\mathrm{v} * \mathrm{w}). \end{aligned}$$

and $(\mathrm{v} * \mathrm{i}(\mathrm{w})) + (\mathrm{i}(\mathrm{v}) * \mathrm{w}) = \mathrm{v} * \mathrm{w}$.
So i is a derivation.

(b) we have $1^{\sim}(v * w) = 1$

$$\begin{aligned} (1^{\sim}(\mathrm{v}) * \mathrm{w}) + (\mathrm{v} * 1^{\sim}(\mathrm{w})) &= (1 * \mathrm{w}) + (\mathrm{v} * 1) \\ &= \mathrm{w} + (\mathrm{v} * 1) \\ &= (\mathrm{w} * (\mathrm{v} * 1)) * (\mathrm{v} * 1). \end{aligned}$$

and

$$\begin{aligned} \big(\mathrm{v} * 1^{-}(\mathrm{w})\big) + \big(1^{-}(\mathrm{v}) * \mathrm{w}\big) &= (\mathrm{v} * 1) + (1 * \mathrm{w}) \\ &= (\mathrm{v} * 1) + \mathrm{w} \\ &= ((\mathrm{v} * 1) * \mathrm{w}) * \mathrm{w}. \end{aligned}$$

In general $(3.4.1), (3.4.2)$ *and* $(3.4.3)$ *are not equal. So* 1^{-} *is not a derivation on P. The following result regarding derivation in BE-algebra is very useful.*

Lemma 3.5. *If d is a derivation on a* $BE-$ *algebra* $(P; *, 1)$ *then*

(a) $d(1) = 1$.

(b) the kernel of d, *defined as*

$$\textit{ker}\ \mathrm{d} = \{\mathrm{v} \in \mathrm{P} : \mathrm{d}(\mathrm{v}) = 1\}.$$

is a sub algebra of P.

Proof. (a) Let $\mathrm{d}(1) = \mathrm{e} \in \mathrm{P}$. Then

$$\begin{aligned} \mathrm{e} = \mathrm{d}(1) &= \mathrm{d}(1 * 1) = (\mathrm{d}1 * 1) + (1 * \mathrm{d}1) \\ &= (\mathrm{e} * 1) + (1 * \mathrm{e}) = 1 + \mathrm{e} = (1 * \mathrm{e}) * \mathrm{e} \\ &= \mathrm{e} * \mathrm{e} = 1. \end{aligned}$$

Also $\mathrm{e} + 1 = (\mathrm{e} * 1) * 1 = 1 * 1 = 1$. Hence, $\mathrm{e} = 1$.
(b) Since $\mathrm{d}(1) = 1, 1 \in \ker \mathrm{d}$. Let $\mathrm{m}, \mathrm{n} \in \ker \mathrm{d}$. Then $dm = 1$ and $\mathrm{dn} = 1$. Now

$$\begin{aligned} \mathrm{d}(\mathrm{m} * \mathrm{n}) &= (\mathrm{dm} * \mathrm{n}) + (\mathrm{m} * \mathrm{dn}) \\ &= (1 * \mathrm{n}) + (\mathrm{m} * 1) \\ &= \mathrm{n} + 1 = (\mathrm{n} * 1) * 1 = 1 * 1 = 1. \end{aligned}$$

Also,

$$\begin{aligned} (\mathrm{m} * \mathrm{dn}) + (\mathrm{dm} * \mathrm{n}) &= (\mathrm{m} * 1) + (1 * \mathrm{n}) \\ &= 1 + n = (1 * n) * n \\ &= n * n = 1. \end{aligned}$$

So $\mathrm{m} * \mathrm{n} \in \ker \mathrm{d}$. Thus $\mathrm{m}, \mathrm{n} \in \ker \mathrm{d}$ implis $\mathrm{m} * \mathrm{n} \in \ker \mathrm{d}$. Hence the result. □

Theorem 3.6. *Let* $(P; *, 1)$ *be a* $BE-$ *algebra and let* $d : P \to P$ *be a derivation on P. Let* $Q = \mathrm{PU}\{\mathrm{t}\}, \mathrm{t} \notin \mathrm{P}$. *Then* $(\mathrm{Q}; \mathrm{o}, 1)$ *is a* $\mathrm{CI}-$ *algebra. We extend* d *to* $\mathrm{d}^1 : \mathrm{Q} \to \mathrm{Q}$ *as*

$$\mathrm{d}^1(\mathrm{v}) = \mathrm{d}(\mathrm{v}) \ \textit{if} \ \mathrm{v} \in \mathrm{P} \ \textit{and} \ \mathrm{d}^1(\mathrm{t}) = \mathrm{t}.$$

Then d^1 *is a derivation on* Q.

Proof: From the above lemma (3.5) and the definition given in the theorem we see that $d^1(1) = d(1) = 1$ Now we observe that if $v \in P$ then

(i) $d^1(v \circ t) = d^1(t) = t$.

(ii) $\left(d^1(v) \circ t\right) + \left(v \circ d^1(t)\right) = (d(v) \circ t) + (v \circ t)$

$$= t + t = (t * t) * t = 1 * t = t.$$

(iii) $\left(v \circ d^1(t)\right) + \left(d^1(v) \circ t\right) = (v \circ t) + (d(v) \circ t) = t + t = t.$

So the definition of derivation at v remains invariant. Again, for any $v \in P$, we have

(a) $d^1(t \circ v) = d^1(t) = t$.

(b) $\left(d^1(t) \circ v\right) + \left(tod^1(v)\right) = (t \circ v) + (t \circ d(v))$

$$= t + t = t.$$

(c) $\left(t \circ d^1(v)\right) + \left(d^1(t) \circ v\right) = (t \circ d(v)) + (t \circ v)$

$$= t + t = t.$$

Also,

(a^1) $d^1(t \circ t) = d^1(1) = 1$.

(b^1) $\left(d^1(t) o t\right) + \left(t \circ d^1(t)\right) = (tot) + (t \circ t)$

$$= 1 + 1 = 1.$$

(c^1) $\left(t \circ d^1(t)\right) + \left(d^1(t) \circ t\right) = (t \circ t) + (t \circ t)$

$$= 1 + 1.$$

From the above we see that d^1 is a derivation on Q.

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