



Neighbourhood V_4 –magic labeling of some subdivision graphs

K.P. Vineesh^{1*} and V. Anil Kumar²

Abstract

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0. A graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$, is said to be Neighbourhood V_4 -magic if there exists a labeling $f : V(G) \rightarrow V_4 \setminus \{0\}$ such that the sum $N_f^+(v) = \sum_{u \in N(v)} f(u)$ is a constant map. If this constant is p , where p is any non zero element in V_4 , then we say that f is a p -neighbourhood V_4 -magic labeling of G and G is said to be a p -neighbourhood V_4 -magic graph. If this constant is 0, then we say that f is a 0-neighbourhood V_4 -magic labeling of G and G is said to be a 0-neighbourhood V_4 -magic graph.

Keywords

Klein-4-group, a -neighbourhood V_4 -magic graphs and 0-neighbourhood V_4 -magic graphs.

AMS Subject Classification

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¹ Department of Mathematics, Sree Narayana Guru College, Chelannur, Kozhikode-673616, Kerala, India.

² Department of Mathematics, University of Calicut, Malappuram-670007, Kerala, India.

*Corresponding author: ¹ kpvineeshmaths@gmail.com; ² anil@uoc.ac.in

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1. Introduction

In this paper we consider graphs that are connected, finite, simple and undirected. For graph theory notations and terminology not directly defined in this paper, we refer readers to [1]. The Klein 4-group, denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$, with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. A graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$, is said to be Neighbourhood V_4 -magic if there exists a labeling $f : V(G) \rightarrow V_4 \setminus \{0\}$ such that the sum $N_f^+(v) = \sum_{u \in N(v)} f(u)$ is a constant map. If this constant is p , where p is any non zero element in V_4 , then we say that f is a p -neighbourhood V_4 -magic labeling of G and G is said to be a p -neighbourhood V_4 -magic graph. If this constant is 0, then we say that f is a 0-neighbourhood V_4 -magic labeling of G and G is said to be a 0-neighbourhood V_4 -magic graph.

The subdivision graph of a graph G is denoted by $S(G)$ and is obtained by inserting an additional vertex to each edge

of G . In this paper, we investigate Neighbourhood V_4 -magic labeling of subdivision graph of some graphs and we classify them into the following three categories:

- (i) $\Omega_a :=$ the class of all a -neighbourhood V_4 -magic graphs,
- (ii) $\Omega_0 :=$ the class of all 0-neighbourhood V_4 -magic graphs, and
- (iii) $\Omega_{a,0} := \Omega_a \cap \Omega_0$.

2. Main Results

Theorem 2.1. [2] $C_n \in \Omega_a$ if and only if $n \equiv 0 \pmod{4}$.

Theorem 2.2. For $n \geq 3$, $S(C_n) \in \Omega_a$ if and only if $n \equiv 0 \pmod{2}$.

Proof. We have $S(C_n) \simeq C_{2n}$. Then by Theorem 2.1, $S(C_n) \in \Omega_a$ if and only if $2n \equiv 0 \pmod{4}$, ie, if and only if $n \equiv 0 \pmod{2}$. This completes the proof. \square

Theorem 2.3. $S(C_n) \in \Omega_0$ for all $n \geq 3$.

Proof. By labeling all the vertices of $S(C_n)$ by a , we get $S(C_n) \in \Omega_0$. \square

Corollary 2.4. For $n \geq 3$, $S(C_n) \in \Omega_{a,0}$ if and only if $n \equiv 0 \pmod{2}$.

Proof. Proof directly follows from Theorem 2.2 and Theorem 2.3. \square

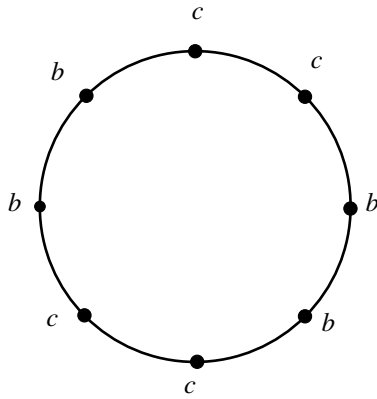


Figure 1. An a -neighbourhood V_4 -magic labeling of $S(C_4)$.

Definition 2.5. [3] *The friendship graph or the Dutch windmill graph, denoted by F_m (or $D_3^{(m)}$) is the graph obtained by taking m copies of C_3 with one vertex in common.*

Theorem 2.6. $S(F_m) \notin \Omega_a$ for any $m \geq 1$.

Proof. Consider the Friendship graph F_m with vertex set $\{w, u_i, v_i : 1 \leq i \leq m\}$ and edge set $\{wu_i, wv_i, u_i v_i : 1 \leq i \leq m\}$. Let u'_i, v'_i, w_i be the vertices in $S(F_m)$ corresponding to the edges $wu_i, wv_i, u_i v_i$ of F_m . Suppose that $S(F_m) \in \Omega_a$ for some m with a labeling f . Then $N_f^+(w_1) = a$ implies that either $f(u_1) = b$ and $f(v_1) = c$ or $f(u_1) = c$ and $f(v_1) = b$. Without loss of generality, assume that $f(u_1) = b$ and $f(v_1) = c$. Now $N_f^+(u'_1) = a$ implies that $f(w) = c$, consequently $N_f^+(v'_1) = 0$, which is a contradiction. Hence $S(F_m) \notin \Omega_a$ for any m . \square

Theorem 2.7. $S(F_m) \in \Omega_0$ for all $m \geq 1$.

Proof. If we label all the vertices of $S(F_m)$ by a , we get $S(F_m) \in \Omega_0$. \square

Corollary 2.8. $S(F_m) \notin \Omega_{a,0}$ for any $m \geq 1$.

Proof. It directly follows from Theorem 2.6. \square

Definition 2.9. [5] *The helm H_n is the graph obtained from the wheel graph W_n by attaching a pendant edge at each vertex of the cycle C_n .*

Definition 2.10. [6] *The flower graph Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the central vertex of the helm.*

Theorem 2.11. $S(Fl_n) \notin \Omega_a$ for any $n \geq 3$.

Proof. Consider the flower graph Fl_n with vertex set $\{u, u_i, v_i : 1 \leq i \leq n\}$ and edge set $\{uu_i, uv_i, u_i v_i, u_i u_{i+1} : 1 \leq i \leq n\}$, where $i + 1$ is taken over modulo n . Let u'_i, v'_i, w'_i, u''_i be the vertices in $S(Fl_n)$ corresponding to the edges $uu_i, uv_i, u_i v_i, u_i u_{i+1}$ for $1 \leq i \leq n$, where $i + 1$ is taken over modulo n . Suppose that $S(Fl_n) \in \Omega_a$ for some n with a labeling f . Then $N_f^+(u'_1) = a = N_f^+(w'_1)$, implies that $f(u) = f(v_1)$. Therefore, $N_f^+(v'_1) = 0$, a contradiction. Hence $S(Fl_n) \notin \Omega_a$ for any n . \square

Theorem 2.12. $S(Fl_n) \in \Omega_0$ for all $n \geq 3$.

Proof. By labeling all the vertices of $S(Fl_n)$ by a , we get $S(Fl_n) \in \Omega_0$. \square

Corollary 2.13. $S(Fl_n) \notin \Omega_{a,0}$ for any $n \geq 3$.

Proof. Proof is obvious from Theorem 2.11. \square

Definition 2.14. [7] *The Sunflower SF_n is obtained from a wheel W_n with the central vertex w_0 and cycle $C_n = w_1 w_2 w_3 \dots w_n w_1$ and additional vertices $v_1, v_2, v_3, \dots, v_n$ where v_i is joined by edges to w_i and w_{i+1} , where $i + 1$ is taken over modulo n .*

Theorem 2.15. $S(SF_n) \notin \Omega_a$ for any $n \geq 3$.

Proof. Consider the sunflower SF_n with vertex set $V = \{w_0, w_i, v_i : 1 \leq i \leq n\}$ where w_0 is the central vertex, $w_1, w_2, w_3, \dots, w_n$ are vertices of the cycle and v_i is the vertex joined by edges to w_i and w_{i+1} where $i + 1$ is taken over modulo n . For $1 \leq i \leq n$, let w'_i, v'_i, v''_i, u'_i be the vertices in $S(SF_n)$ corresponding to the edges $w_0 w_i, w_i v_i, v_i w_{i+1}, w_i w_{i+1}$ of SF_n , where $i + 1$ is taken over modulo n . Assume that $S(SF_n) \in \Omega_a$ for some n with a labeling f . Then $N_f^+(v'_1) = a = N_f^+(v''_1)$, implies that $f(w_1) = f(w_2)$, consequently $N_f^+(u'_1) = 0$. This is a contradiction. Hence $S(SF_n) \notin \Omega_a$ for any n . \square

Theorem 2.16. For $n \geq 3$, $S(SF_n) \in \Omega_0$ if $n \equiv 0 \pmod{2}$.

Proof. Consider the sunflower SF_n with vertex set $V = \{w_0, w_i, v_i : 1 \leq i \leq n\}$ where w_0 is the central vertex, $w_1, w_2, w_3, \dots, w_n$ are vertices of the cycle and v_i is the vertex joined by edges to w_i and w_{i+1} where $i + 1$ is taken over modulo n . For $1 \leq i \leq n$, let w'_i, v'_i, v''_i, u'_i be the vertices in $S(SF_n)$ corresponding to the edges $w_0 w_i, w_i v_i, v_i w_{i+1}, w_i w_{i+1}$ of SF_n , where $i + 1$ is taken over modulo n . Suppose that $n \equiv 0 \pmod{2}$. We define $f : V[S(SF_n)] \rightarrow V_4 \setminus \{0\}$ as :

$$f(w_0) = f(w_i) = f(w'_i) = f(v_i) = f(v'_i) = f(v''_i) = a \quad \text{for } 1 \leq i \leq n,$$

$$f(u'_i) = \begin{cases} b & \text{if } i \equiv 1 \pmod{2} \\ c & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

Clearly f is an a -neighbourhood V_4 -magic labeling of $S(SF_n)$. This completes the proof of the theorem. \square

Corollary 2.17. $S(SF_n) \notin \Omega_{a,0}$ for any $n \geq 3$.



Proof. It directly follows from Theorem 2.15. □

Definition 2.18. [8] A complete bipartite graph of the form $K_{1,n}$ is called a star graph. A star graph $K_{1,n}$ is sometimes called an n -star.

Definition 2.19. [9] The Bistar $B_{m,n}$ is the graph obtained by joining the central vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.

Theorem 2.20. $S(B_{m,n}) \in \Omega_a$ if and only if both m and n are even.

Proof. Consider the bistar $B_{m,n}$ with vertex set $\{u, v, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ where $u_i (1 \leq i \leq m)$ and $v_j (1 \leq j \leq n)$ are pendant vertices adjacent to u and v respectively. Let w, u'_i and v'_j be vertices in $S(B_{m,n})$ corresponding to the edges uv, uu_i and vv_j of $B_{m,n}$. Suppose that both m and n are even. Define $f : V[S(B_{m,n})] \rightarrow V_4 \setminus \{0\}$ as:

$f(u) = f(v_j) = c$ for $1 \leq j \leq n$
 $f(v) = f(u_i) = b$ for $1 \leq i \leq m$
 $f(w) = f(u'_i) = f(v'_j) = a$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.
 Then f is an a -neighbourhood V_4 -magic labeling of $S(B_{m,n})$. Hence $S(B_{m,n}) \in \Omega_a$. Conversely, suppose that not both m and n are even. Then either m or n is odd. Without loss of generality take m is odd. If possible, let $S(B_{m,n}) \in \Omega_a$ with a labeling f . Then $N_f^+(u_i) = a$ implies that $f(u'_i) = a$ for $1 \leq i \leq m$. Then $N_f^+(u) = a$ implies that $ma + f(u) = a$, consequently $f(u) = 0$. This is a contradiction. Hence $S(B_{m,n}) \notin \Omega_a$. Which completes the proof of the theorem. □

Theorem 2.21. $S(B_{m,n}) \notin \Omega_0$ for any m and n .

Proof. Proof is obvious, due to the presence of pendant vertices in $S(B_{m,n})$. □

Corollary 2.22. $S(B_{m,n}) \notin \Omega_{a,0}$ for any m and n .

Proof. Proof directly follows from Theorem 2.21. □

Definition 2.23. [9] Jelly fish graph $J(m,n)$ is obtained from a 4-cycle $w_1w_2w_3w_4w_1$ by joining w_1 and w_3 with an edge and appending the central vertex of $K_{1,m}$ to w_2 and appending the central vertex of $K_{1,n}$ to w_4 .

Theorem 2.24. $S(J(m,n)) \notin \Omega_a$ for any m and n .

Proof. Consider the jelly fish $J(m,n)$ with vertex set $V = V_1 \cup V_2$ where $V_1 = \{w_1, w_2, w_3, w_4\}$, $V_2 = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E = E_1 \cup E_2$, where $E_1 = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3\}$, $E_2 = \{w_2u_i, w_4v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Let u'_i, v'_j be the new vertices corresponding to the edges w_2u_i, w_4v_j and $w'_1, w'_2, w'_3, w'_4, w'_5$ be the new vertices corresponding to the edges $w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3$ respectively in $S(J(m,n))$. Suppose that $S(J(m,n)) \in \Omega_a$ for some m and n with a labeling function f . Then $N_f^+(w'_3) = a = N_f^+(w'_4)$ implies that $f(w_1) = f(w_3)$. Therefore, $N_f^+(w'_5) = 0$, which is a contradiction. Hence $S(J(m,n)) \notin \Omega_a$ for any m and n . □

Theorem 2.25. $S(J(m,n)) \notin \Omega_0$ for any m and n .

Proof. Proof is obvious, since $S(J(m,n))$ has pendant vertex in it. □

Corollary 2.26. $S(J(m,n)) \notin \Omega_{a,0}$ for any m and n .

Proof. Proof directly follows from Theorem 2.24. □

Definition 2.27. The graph $P_2 \times P_n$ is called a Ladder. It is denoted by L_n .

Theorem 2.28. $S(L_n) \in \Omega_a$ for all $n > 1$.

Proof. Consider the ladder L_n with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Let u'_i, v'_i be the new vertices corresponding to the edges u_iu_{i+1}, v_iv_{i+1} for $1 \leq i \leq n-1$ and w_i be the vertices corresponding to the edges u_iv_i for $1 \leq i \leq n$ in $S(L_n)$. Now define $f : V[S(L_n)] \rightarrow V_4 \setminus \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \pmod{2} \\ c & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if } i \equiv 1 \pmod{2} \\ b & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

$$f(u'_i) = f(v'_i) = c \text{ for } 1 \leq i \leq n-1$$

$$f(w_i) = a \text{ for } 1 < i < n$$

$$f(w_1) = f(w_n) = b.$$

Then $N_f^+ \equiv a$. Hence $S(L_n) \in \Omega_a$. □

Definition 2.29. [10] The open ladder $O(L_n)$ is the graph obtained from two paths of lengths $n-1$ with $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i < n\} \cup \{u_iv_i : 1 < i < n\}$.

Theorem 2.30. $S(O(L_n)) \in \Omega_a$ for all $n > 2$.

Proof. Consider the open ladder $O(L_n)$ with vertex set $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i < n\} \cup \{u_iv_i : 1 < i < n\}$. Let u'_i, v'_i be the new vertices corresponding to the edges u_iu_{i+1}, v_iv_{i+1} for $1 \leq i < n$ and w_i be the vertices corresponding to u_iv_i for $1 < i < n$ in $S(O(L_n))$. Now define $f : V[S(O(L_n))] \rightarrow V_4 \setminus \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \pmod{2} \\ c & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if } i \equiv 1 \pmod{2} \\ b & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

$$f(u'_i) = f(v'_i) = a \text{ for } 1 \leq i \leq n-1$$

$$f(w_i) = a \text{ for } 1 < i < n$$

Then f is an a -neighbourhood V_4 -magic labeling of $S(O(L_n))$. □



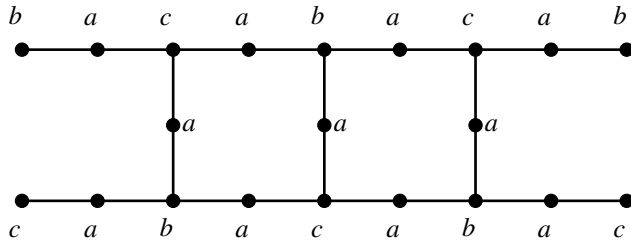


Figure 2. An a -neighbourhood V_4 -magic labeling of $S(OL_5)$.

Theorem 2.31. $S(O(L_n)) \notin \Omega_0$ for any $n > 2$.

Proof. Proof is obvious, since $S(O(L_n))$ has pendant vertices in it. \square

Corollary 2.32. $S(O(L_n)) \notin \Omega_{a,0}$ for any $n > 2$.

Proof. Proof directly follows from Theorem 2.31. \square

Definition 2.33. The Corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 , which has p_1 vertices, and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 .

Definition 2.34. The Corona $P_n \odot K_1$ is called the comb graph CB_n .

Theorem 2.35. $S(CB_n) \notin \Omega_a$ for any $n \geq 2$.

Proof. Consider CB_n with vertex set $\{u_i, v_i : 1 \leq i \leq n\}$ and edge set $\{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$. Let v'_i and u'_j be vertices in $S(CB_n)$ corresponding to the edges $u_i v_i, u_j u_{j+1}$ for $1 \leq i \leq n$ and $1 \leq j \leq n-1$ of CB_n . Suppose that $S(CB_n) \in \Omega_a$ for some n with a labeling f . Then $N_f^+(v_i) = a$ implies that $f(v'_i) = a$ for $1 \leq i \leq n$. Also, $N_f^+(u_1) = a$ implies that $f(v'_1) + f(u'_1) = a$, implies that $f(u'_1) = 0$. This is a contradiction. Hence $S(CB_n) \notin \Omega_a$ for any n . \square

Theorem 2.36. $S(CB_n) \notin \Omega_0$ for any n .

Proof. Proof is obvious, since $S(CB_n)$ has pendant vertices in it. \square

Corollary 2.37. $S(CB_n) \notin \Omega_{a,0}$ for any n .

Proof. Proof directly follows from Theorem 2.36. \square

Definition 2.38. [11] A Crown graph C_n^* is obtained from C_n by attaching a pendant edge at each vertex of the cycle C_n .

Theorem 2.39. For $n \geq 3$, $S(C_n^*) \in \Omega_a$ if and only if $n \equiv 0 \pmod{2}$.

Proof. Consider the crown C_n^* with vertex set $\{u_i, v_i : 1 \leq i \leq n\}$, where $u_1, u_2, u_3, \dots, u_n$ are vertices on the cycle C_n and $v_1, v_2, v_3, \dots, v_n$ are pendant vertices adjacent to the vertices $u_1, u_2, u_3, \dots, u_n$ respectively. Let v'_i ($1 \leq i \leq n$) be the vertices in $S(C_n^*)$ corresponding to the edges $u_i v_i$ ($1 \leq i \leq n$) and u'_i

be the vertices corresponding to the edges $u_i u_{i+1}$ ($1 \leq i \leq n$), where $i+1$ is taken over modulo n . Suppose $S(C_n^*) \in \Omega_a$ with a labeling f . Then $N_f^+(v'_1) = a$ gives either $f(u_1) = b$ or $f(u_1) = c$. Without loss of generality, take $f(u_1) = b$. Then $N_f^+(u'_1) = a$ implies that $f(u_2) = c$. Also $N_f^+(u'_2) = a$ implies that $f(u_3) = b$. Proceeding like this, we should have

$$f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \pmod{2} \\ c & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

Now, $N_f^+(u'_n) = a$ implies that $f(u_n) = c$. Therefore, $n \equiv 0 \pmod{2}$. Conversely suppose that $n \equiv 0 \pmod{2}$. Define $f : V[S(C_n^*)] \rightarrow V_4 \setminus \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \pmod{2} \\ c & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if } i \equiv 1 \pmod{2} \\ b & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

$$f(u'_i) = f(v'_i) = a \text{ for } 1 \leq i \leq n.$$

Then f is an a -neighbourhood V_4 -magic labeling of $S(C_n^*)$. This completes the proof of the theorem. \square

Theorem 2.40. $S(C_n^*) \notin \Omega_0$ for any $n \geq 3$.

Proof. Proof is obvious, since $S(C_n^*)$ has pendant vertices in it. \square

Corollary 2.41. $S(C_n^*) \notin \Omega_{a,0}$ for any $n \geq 3$.

Proof. Proof directly follows from Theorem 2.40. \square

References

- [1] Chartrand G, Zhang P, *Introduction to Graph Theory*, McGraw-Hill, Boston; 2005.
- [2] K. P. Vineesh and V. Anil Kumar, Neighbourhood V_4 -magic labeling of some cycle related graphs, *Far East Journal of Mathematical Sciences* 111(2)(2019), 263-272.
- [3] Vandana P. T. and Anil Kumar V., V_4 -magic labelings of some graphs, *British Journal of Mathematics and Computer Science*, 11(5)(2015), 1-20.
- [4] D.Sinha and J.Kaur, Full friendly index set-I, *Discrete Appl. Math.*, 161(2013), 1262-1274.
- [5] Seoud M. A. and Youssef M. Z., *Harmonious labellings of helms and related graphs*, unpublished.
- [6] Vandana P. T. and Anil Kumar V., V_4 -magic labelings of wheel related graphs, *British Journal of Mathematics and Computer Science*, 8(3)(2015), 189-219.
- [7] Ponraj R., Radio mean labeling of a graph, *AKCE International Journal of Graphs and Combinatorics*, 12(2015), 224-228.



- [8] R Balakrishnan and K Ranganathan, *A Text Book of Graph Theory*, Springer-Verlag, New York, 2012.
- [9] Joseph A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 21(2018), 1–10.
- [10] P. Sumathi, A. Rathi, and A. Mahalakshmi, Quotient labeling of corona of ladder graphs, *International Journal of Innovative Research in Applied Sciences and Engineering*, 1(3)(2017), 1–12
- [11] Vaithilingam K. and Meena S., Prime labeling for some crown related graphs, *International Journal of Scientific and Technology Research*, 2(2013), 1–08.

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