Malaya Journal of Matematik, Vol. 8, No. 4, 1818-1823, 2020

https://doi.org/10.26637/MJM0804/0081

The chain structure of intuitionistic level subgroups in cyclic groups of order pq

S. Divya Mary Daise[1], S. Deepthi Mary Tresa[2] and Shery Fernandez[3]*

Abstract

It is well known that the set of all level subgroups of any fuzzy subgroup of a finite group forms a chain. In this paper we prove that this result does not extend to Intuitionistic Fuzzy Subgroups (IFSGs) by providing a counter-example. For any two distinct prime numbers p and q, we prove that the cyclic group $\mathbb{Z}_{pq}$ has 36 non-isomorphic IFSGs. The Intuitionistic Level Subgroups (ILSGs) of only 28 of them form chains, while those of remaining 8 do not form chains. The list of all the 36 distinct IFSGs is also provided; and those whose ILSGs form a chain, and not, are identified. The case is illustrated using a specific example. We have also obtained a characterisation of IFSGs of $\mathbb{Z}_{pq}$, whose ILSGs form a chain.

Keywords

Intuitionistic fuzzy set, intuitionistic fuzzy subgroup, level subgroup, isomorphism, cyclic group.

AMS Subject Classification

03E72, 03F55, 20A99.

[1,2] *Department of Mathematics, St. Albert's College, Ernakulam (Affiliated to Mahatma Gandhi University), Cochin-682018, India.*

[3] *Department of Mathematics, Cochin University of Science and Technology, Cochin-682022, India.*

***Corresponding author**: [3] sheryfernandezaok@gmail.com; [1]divyamarydaises@gmail.com; [2]deepthimtresa@gmail.com

Article History: Received **28** August **2020**; Accepted **19** October **2020**

Contents

1. Introduction

In 1965 L. A. Zadeh [14] came up with the concept of the fuzzy subset of a non-empty set. According to him, a *Fuzzy Subset A* of a universal set X is represented by a membership function $A : X \rightarrow I$ where $I = [0,1]$. Now-a-days, fuzzy mathematics is an area of scrupulous research, with applications in diverse fields like engineering, computer science, medical diagnosis, social behavior studies, etc. Following the introduction of fuzzy sets, most of the abstract mathematical structures were generalised to the fuzzy context and were subjected to exhaustive research. Throughout the process of this evolution, many researchers were motivated to investigate the generalization of different concepts of abstract algebra in the fuzzy setting, since the abstract algebraic structures play a crucial role in mathematics and have innumerable implementations in several areas of research such as computer sciences, information sciences, cryptography, coding theory, etc. Rosenfeld[10] laid the foundation for this quest. He fuzzified the theory of groups by defining the concept of fuzzy subgroups of a group. Several works subsequently appeared in the literature, surrounding various fuzzy algebraic structures. Meanwhile, K. T. Atanassov[1] introduced the notion of intuitionistic fuzzy sets in 1983, as an extension of the fuzzy set theory. Later in 1989, Biswas[2] applied Atanassov's idea of intuitionistic fuzzy sets to the theory of groups and established the theory of intuitionistic fuzzy subgroups of a group. Many new results in this area of study are still emerging.

In 1981, P S Das[8] proved that, the level subgroups of fuzzy subgroups of a finite group form a chain. Later in 2006, eventhough Ahn et.al.[13] studied some properties of level subgroups of intuitionistic fuzzy subgroups of cyclic groups, there has been no attempt to test whether these level subgroups form a chain in the intuitionistic fuzzy framework also. Our work is aimed at investigating whether this finding can be translated into the domain of intuitionistic fuzzy subgroups of a group. In this paper, we try to extend the finding of P S Das[8] to intuitionistic fuzzy subgroups of cyclic groups of

order pq, where p and q are distinct primes.
This paper is organized into two main sections as follows. In the first section we outline the basic definitions and results needed to understand our work. In the second section we list out all the intuitionistic fuzzy subgroups of cyclic groups of order pq upto isomorphism and then identify those intuitionistic fuzzy subgroups whose Intuitionistic Level Subgroups form a chain. Some tables and figures are also provided in this section to illustrate our findings.

2. Preliminaries

The terms and results which are required for the proper understanding of results discussed in this paper are discussed in this section.
Throughout this paper we use the notations:

- I for the interval $[0,1]$ on the real line
- $\wedge$ for the min operator on I
- $\vee$ for the max operator on I
- G for an arbitrary finite multiplicative group unless otherwise stated
- $H \preceq G$ to denote the statement H is a subgroup of G
- $\langle a \rangle$ to denote the cyclic subgroup of G generated by $a \in G$
- $\mathbb{Z}_n$ to denote the abelian group $(\{0,1,2,...,n-1\},+_n)$

Definition 2.1. *[10] A fuzzy subset A of a group G is said to be a **Fuzzy Subgroup** (FSG) of G if, for all $x,y \in G$*
(1) $A(xy) \geq \wedge[A(x),A(y)]$
(2) $A(x^{-1}) = A(x)$.

Proposition 2.2. *[10] If A is FSG of a group G with identity element e, then $A(e) \geq A(x), \forall x \in G$.*

Definition 2.3. *[8] If A is a fuzzy subset of a non-empty set X and $t \in I$, then **t-cut** of A (or **Level Subset** of A at t), denoted by A_t, is defined as $A_t = \{x \in X : A(x) \geq t\}$.*

Proposition 2.4. *[8] In a group G, a Fuzzy Subset A will be a FSG of G if and only if A_t is a subgroup of G for $0 \leq t \leq A(e)$.*

Definition 2.5. *[8] For a FSG A of a group G, the subgroup A_t is called **Level Subgroup** of A at t, for $0 \leq t \leq A(e)$.*

Proposition 2.6. *[11] If A is a FSG of a group G then for all $t_1,t_2 \in I$ with $t_1 > t_2$, $A_{t_1} \subseteq A_{t_2}$.*

Proposition 2.7. *[8] Let A be a FSG of a finite group G with $Im(A) = \{t_i : i = 1,2,3,...,n\}$. Then the collection $\{A_{t_i} : i = 1,2,3,...,n\}$ contains all level subgroups of A.*
Moreover, if $t_1 > t_2 > t_3 > ... > t_n$, then all these level subgroups will form a chain $G_A = A_{t_1} \subsetneq A_{t_2} \subsetneq A_{t_3} \subsetneq ... \subsetneq A_{t_n} = G$, where $G_A = \{x \in G : A(x) = A(e)\}$.

Definition 2.8. *[1] An **Intuitionistic Fuzzy Subset** (IFS) of a set X is an object of the form $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in X\}$ where the functions $A^+, A^- : X \to I$ represent the degree of membership and degree of non membership of any element $x \in X$ and should satisfy the condition $0 \leq A^+(x) + A^-(x) \leq 1, \forall\, x \in X$.*

Definition 2.9. *[9] An IFS $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ of a group G is said to be an **Intuitionistic Fuzzy Subgroup** (IFSG) of G if*
(1) $A^+(xy) \geq \wedge[A^+(x), A^+(y)]$
(2) $A^+(x^{-1}) = A^+(x)$
(3) $A^-(xy) \leq \vee[A^-(x), A^-(y)]$, and
(4) $A^-(x^{-1}) = A^-(x)$.

Proposition 2.10. *[9] Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ be an IFSG of a group G with identity element e. Then, $A^+(e) \geq A^+(x)$ and $A^-(e) \leq A^-(x)$, $\forall\, x \in G$.*

Definition 2.11. *[12] Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ be an IFS of a set X and $\alpha,\ \beta \in I$. Then the **Intuitionistic Level Subset** (ILS) of A at $(\alpha,\ \beta)$ (or **$(\alpha,\ \beta)$–cut** of IFS A) is the crisp set $A_{\alpha,\beta} = \{x \in X : A^+(x) \geq \alpha$ and $A^-(x) \leq \beta\}$.*

Proposition 2.12. *[12] Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ be an IFSG of a group G. Then,*
(1) $A_{\alpha,\beta} = \phi$, for all $\alpha > A^+(e)$ and $\beta < A^-(e)$
(2) A is an IFSG of $G \Leftrightarrow A_{\alpha,\beta}$ is a subgroup of G for $0 \leq \alpha \leq A^+(e)$ and $A^-(e) \leq \beta \leq 1$.

Definition 2.13. *[4] Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ be an IFSG of a group G. Then the subgroup $A_{\alpha,\beta}$ (where $0 \leq \alpha \leq A^+(e)$ and $A^-(e) \leq \beta \leq 1$) of G is called **Intuitionistic Level Subgroup** (ILSG) of A at (α,β).*

Proposition 2.14. *[12] Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ be an IFSG of a group G and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I$ be such that $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$. Then $A_{\alpha_1,\beta_1} \subseteq A_{\alpha_2,\beta_2}$.*

Proposition 2.15. *[4] Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ be an IFSG of a finite group G, $Im(A^+) = \{t_i : i = 1,2,3,...,n\}$ and $Im(A^-) = \{s_j : j = 1,2,3,...,m\}$. Then the collection*

$$\{\ A_{t_i,s_j} : i = 1,2,3,...,n;\ j = 1,2,3,...,m\}$$

contains all ILSG's of G.

The above proposition states that the intuitionistic fuzzy analogue of the first part of proposition 2.7 holds true.

Definition 2.16. *[6] Let X be any non-empty finite set and $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in X\}$ be an IFS of X with $Im(A^+) = \{t_i : i = 1,2,3,...,n\}$ and $Im(A^-) = \{s_j : j = 1,2,3,...,m\}$ where $1 \geq t_1 > t_2 > ... > t_n \geq 0$ and $0 \leq s_1 < s_2 < ... < s_m \leq 1$. The finite sequence $\tilde{L}(A) = \{A_{t_1,s_1}, A_{t_1,s_2}, ..., A_{t_1,s_m}, A_{t_2,s_1}, A_{t_2,s_2}, ..., A_{t_2,s_m}, ..., A_{t_n,s_1}, A_{t_n,s_2}, ..., A_{t_n,s_m}\}$, consisting of all intuitionistic level subsets of A, is called the **Intuitionistic Level Representation (ILR)** of A.*

Being a finite *sequence*, the *members in $\tilde{L}(A)$ may repeat* and their *position is significant*.

Remark 2.17 (Geometric Representation of IFS). *Let X be any non-empty set and $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in X\}$ be an IFS of X. For geometric representation of A, A^+ is taken along x-axis and A^- along y-axis. Then an element x of X is represented as an element of A by the point $(A^+(x), A^-(x))$ in the coordinate plane. In this representation all elements of A will lie inside the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$.*

Example 2.18. *Let $X = \{a, b, c\}$ be a non-empty set and $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in X\}$ be an IFS of X, where A^+ and A^- are defined by the following table:*

	A^+	A^-
a	0.7	0.2
b	0.5	0.3
c	0.2	0.3

This can be represented geometrically by figure 1. Also the ILS's of A are: $A_{0.7,0.2} = A_{0.5,0.2} = A_{0.2,0.2} = A_{0.7,0.3} = \{a\}, A_{0.5,0.3} = \{a, b\}, A_{0.2,0.3} = \{a, b, c\}$, as can be clearly seen from figure 1. Hence the ILR of A is $\tilde{Ł}(A) = \{\{a\}, \{a\}, \{a\}, \{a, b\}, \{a\}, X\}$.

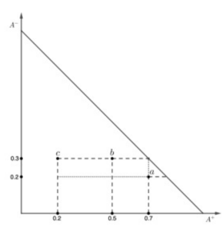

Figure 1. IFS of the example 2.18

Definition 2.19 (Isomorphic Intuitionistic Fuzzy Subsets). *[6] Let X be any non-empty set and $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in X\}$ and $B = \{\langle x, B^+(x), B^-(x)\rangle : x \in X\}$ be two IFS's of X. We say that A is isomorphic to B, denoted by $A \cong B$, if for all $x, y \in X$*
(I_1) $A^+(x) < A^+(y) \Leftrightarrow B^+(x) < B^+(y)$
(I_2) $A^+(x) = A^+(y) \Leftrightarrow B^+(x) = B^+(y)$
(I_3) $A^-(x) < A^-(y) \Leftrightarrow B^-(x) < B^-(y)$
(I_4) $A^-(x) = A^-(y) \Leftrightarrow B^-(x) = B^-(y)$

Theorem 2.20. *[6] Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in X\}$ and $B = \{\langle x, B^+(x), B^-(x)\rangle : x \in X\}$ be two IFS's of a non-empty set X and $\tilde{Ł}(A)$, $\tilde{Ł}(B)$ be the ILRs of A and B respectively. Then, $A \cong B$, if and only if, $\tilde{Ł}(A) = \tilde{Ł}(B)$.*

It may be noted that, if two IFS's A and B of a non-empty set X are isomorphic, then the degrees of membership and non-membership of various elements of X w.r.t. A and B will have the same hierarchical ordering, but differ in values.

3. Main Results

Now we investigate whether the intuitionistic fuzzy analogue of second part of proposition 2.7 holds true. In our earlier works, we have already investigated it for finite non-cyclic groups and for cyclic groups of prime order and prime power order. We have proved in [4] that the above result does not always hold true for finite non-cyclic groups and in [7] we have proved the result affirmatively for cyclic groups of prime and prime power orders. We now do this exploration in the case of cyclic groups of order pq, where p and q are distinct primes. Since every cyclic group of order pq is isomorphic (in the group theoretic sense) to $\mathbb{Z}_{pq}$, we restrict our efforts to $\mathbb{Z}_{pq}$. Thus, we will inspect whether the distinct ILSG's of an IFSG of $\mathbb{Z}_{pq}$ form a chain or not. During this investigation, we also characterise all the IFSG's of $\mathbb{Z}_{pq}$ upto isomorphism. Throughout this section we take p and q to be two distinct primes.

Proposition 3.1. *[5] In any IFSG of a finite cyclic group, all generators will have the minimum membership value and maximum non-membership value.*

We proceed to prove that if A is an IFSG of G, then for any cyclic subgroup of G, any two generators have the same membership and non-membership degrees.

Proposition 3.2. *Let G be any group and $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in G\}$ be an IFSG of G. Let $a, b \in G$ be such that $\langle a\rangle = \langle b\rangle$. Then $A^+(a) = A^+(b)$ and $A^-(a) = A^-(b)$.*

Proof. $\langle a\rangle = \langle b\rangle \Rightarrow a \in \langle b\rangle$ and $b \in \langle a\rangle$
$\Rightarrow a = b^k$ and $b = a^l$ for some $k, l \in \mathbb{Z}$
$\Rightarrow A^+(a) \geq A^+(b)$ and $A^+(b) \geq A^+(a)$
(by first axiom in definition 2.9)
$\Rightarrow A^+(a) = A^+(b)$
Similarly it can be proved that $A^-(a) = A^-(b)$. □

Proposition 3.3. *Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in \mathbb{Z}_{pq}\}$ be an IFSG of $\mathbb{Z}_{pq}$. Then $A^+(p) = A^+(kp)$ and $A^-(p) = A^-(kp)$, $\forall k = 1, 2, 3, \ldots, q-1$.*

Proof. (Throughout this proof we will use arithmetic modulo pq.)
We have $\langle p\rangle = \{0, p, 2p, 3p, \ldots, (q-1)p\}$ and $\langle kp\rangle = \{0, kp, 2kp, 3kp, \ldots, (q-1)kp\}$. Clearly $\langle kp\rangle \preceq \langle p\rangle$ and $|\langle p\rangle| = q$.
Now suppose $ikp = jkp$ where $0 < j < i < q$
$\Rightarrow (i-j)kp$ is a multiple of pq
$\Rightarrow (i-j)k$ is a multiple of q
$\Rightarrow (i-j)$ or k is a multiple of q, which is a contradiction.
Hence the elements of $\langle kp\rangle$ are all distinct and $|\langle kp\rangle| = q$, which means that $\langle p\rangle = \langle kp\rangle, \forall k = 1, 2, 3, \ldots, q-1$. Now the required result follows from proposition 3.2. □

Theorem 3.4. *[3] Given integers a and b, not both of which are zero, there exist integers x and y such that $gcd(a, b) = ax + by$.*

Proposition 3.5. *For distinct primes p and q, there exist integers a and b with $1 = ap + bq$, where $ap, bq \in \mathbb{Z}_{pq}$.*

Proof. Let $n = pq$. Since p and q are distinct primes, $gcd(p, q) = 1$ and hence by Theorem 3.4, there exist integers x and y

such that $1 = xp + yq$. If xp or $yq \notin \mathbb{Z}_{pq}$, then there exist integers l and k such that $0 \le xp + ln, yq + kn < n$. In that case, $xp + ln = (x + lq)p = ap$ and similarly $yq + kn = bq$ where a and b are integers. Clearly $ap, bq \in \mathbb{Z}_{pq}$ (since $0 \le xp + ln, yq + kn < n$). Also, $1 = xp + yq = ap + bq$ (since $ln = kn = 0$ in $\mathbb{Z}_{pq}$). □

The following result is very significant. It says that any IFSG of $\mathbb{Z}_{pq}$ can have atmost three membership and non-membership levels.

Theorem 3.6. *Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in \mathbb{Z}_{pq}\}$ be an IFSG of $\mathbb{Z}_{pq}$. Then $|Im(A^+)| \le 3$ and $|Im(A^-)| \le 3$.*

Proof. The elements of $\mathbb{Z}_{pq}$ can be categorised into 4 as: (i) the identity element 0, (ii) the multiples of p, (iii) the multiples of q and (iv) the numbers which are relatively prime to pq. Then, $A^+(0)$ is maximum (by proposition 2.10), all numbers in category (ii) and (iii) have same membership degrees as that of p and q respectively (by proposition 3.3) and all numbers in category (iv) have same membership value as that of 1 (which is the minimum), as they are all generators of $\mathbb{Z}_{pq}$ (by proposition 3.1). This implies,

$$A^+(0) \ge A^+(p), A^+(q) \ge A^+(1) \tag{3.1}$$

Now we will proceed to find the relationship between $A^+(p)$ and $A^+(q)$. By proposition 3.5 there exist integers a and b with $1 = ap + bq$, where $ap, bq \in \mathbb{Z}_{pq}$. Then, $A^+(ap + bq) \ge \wedge\{A^+(ap), A^+(bq)\}$ (by first axiom of IFSG), and hence by proposition 3.3 we get

$$A^+(1) \ge \wedge\{A^+(p), A^+(q)\} \tag{3.2}$$

Case (i): $A^+(p) > A^+(q)$
(3.2) $\Rightarrow A^+(1) \ge A^+(q)$ and hence (3.1) $\Rightarrow A^+(1) = A^+(q)$. So in this case we get $A^+(0) \ge A^+(p) > A^+(q) = A^+(1)$.

Case (ii): $A^+(p) < A^+(q)$
Proceeding as in *case (i)* we get $A^+(0) \ge A^+(q) > A^+(p) = A^+(1)$.

Case (iii): $A^+(p) = A^+(q)$
As in the above cases we get $A^+(0) \ge A^+(q) = A^+(p) = A^+(1)$.

In all these possible cases, the maximum possible number of distinct membership degrees is 3, which means that $|Im(A^+)| \le 3$.
The proof for A^- is similar. □

Using the above theorem, in the next proposition we prove that for distinct primes p and q, $\mathbb{Z}_{pq}$ can have only 36 non-isomorphic IFSGs. We also give a list of these 36 IFSGs.

Proposition 3.7. *Given $t_1, t_2, t_3, s_1, s_2, s_3 \in I$ with $1 \ge t_1 \ge t_2 \ge t_3 \ge 0$ and $0 \le s_1 \le s_2 \le s_3 \le 1$, there exist exactly 36 distinct (non-isomorphic) IFSG's $A_k = \{\langle x, A_k^+(x), A_k^-(x)\rangle : x \in \mathbb{Z}_{pq}\}$ $(k = 1, 2, 3, ...36)$ in $\mathbb{Z}_{pq}$ with $Im(A_k^+) = \{t_1, t_2, t_3\}$ and $Im(A_k^-) = \{s_1, s_2, s_3\}$ for all $k = 1, 2, 3, ...36$.*

Proof. Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in \mathbb{Z}_{pq}\}$ be any IFSG of $\mathbb{Z}_{pq}$ with $Im(A^+) = \{t_1, t_2, t_3\}$ and $Im(A^-) = \{s_1, s_2, s_3\}$. Then as stated in the proof of theorem 3.6, the only possible hierarchy of membership degrees are:

$$\begin{aligned} &A^+(0) \ge A^+(p) > A^+(q) = A^+(1) \\ &A^+(0) \ge A^+(q) > A^+(p) = A^+(1) \\ &A^+(0) \ge A^+(p) = A^+(q) = A^+(1) \end{aligned} \tag{3.3}$$

and that of non-membership degrees are:

$$\begin{aligned} &A^-(0) \le A^-(p) < A^-(q) = A^-(1) \\ &A^-(0) \le A^-(q) < A^-(p) = A^-(1) \\ &A^-(0) \le A^-(p) = A^-(q) = A^-(1) \end{aligned} \tag{3.4}$$

In order to define the IFSG A in $\mathbb{Z}_{pq}$, A^+ can be chosen from (3.3) in $3C_1$ possible ways and A^- can be chosen from (3.4) in $3C_1$ possible ways. Thus, A can be defined in 9 different ways in each of the following cases: (i) strict inequality with both $A^+(0)$ and $A^-(0)$, (ii) equality with both $A^+(0)$ and $A^-(0)$, (iii) strict inequality with $A^+(0)$ and equality with $A^-(0)$, and (iv) strict inequality with $A^-(0)$ and equality with $A^+(0)$. In total, A can be defined in 36 different ways so that $Im(A^+) = \{t_1, t_2, t_3\}$ and $Im(A^-) = \{s_1, s_2, s_3\}$. Therefore, there are exactly 36 distinct IFSGs of $\mathbb{Z}_{pq}$ with a given set of membership and non-membership degrees. □

Any other IFSG of $\mathbb{Z}_{pq}$ will differ from the 36 IFSG's mentioned in proposition 3.7 only in the values of $t_1, t_2, t_3, s_1, s_2, s_3$. That is, any other IFSG of $\mathbb{Z}_{pq}$ will be isomorphic to one among the 36 IFSG's mentioned in proposition 3.7.

Example 3.8. *Consider the group $\mathbb{Z}_{35} = \{0, 1, 2, ..., 34\}$ and any set of membership and non-membership degrees given by $t_1, t_2, t_3, s_1, s_2, s_3 \in I$ with $1 \ge t_1 \ge t_2 \ge t_3 \ge 0$ and $0 \le s_1 \le s_2 \le s_3 \le 1$. Then by theorem 3.7, exactly 36 IFSG's can be defined on $\mathbb{Z}_{35}$ with these membership and non-membership degrees. Also, these IFSG's can be obtained from table 2 and table 3, just by replacing p and q by 5 and 7 respectively.*

Remark 3.9. *Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in \mathbb{Z}_{35}\}$ be an IFSG of $\mathbb{Z}_{35}$, where A^+ and A^- are defined by the following table:*

Elements of $\mathbb{Z}_{35}$	A^+	A^-
0	*0.8*	*0.1*
1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 16, 17, 18 19, 22, 23, 24, 26, 27, 29, 31, 32, 33, 34	*0.4*	*0.3*
5, 10, 15, 20, 25, 30	*0.4*	*0.3*
7, 14, 21, 28	*0.4*	*0.3*

This IFSG corresponds to A_9 in table 2. Here $A^+(5) = A^+(7)$ and $A^-(5) = A^-(7)$, but $\langle 5\rangle \ne \langle 7\rangle$. This shows that, the converse of proposition 3.2 does not hold true.

Proposition 3.10. *Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in \mathbb{Z}_{pq}\}$ be an IFSG of $\mathbb{Z}_{pq}$. If $A^+(p) > A^+(q)$ and $A^-(p) > A^-(q)$ or if $A^+(p) < A^+(q)$ and $A^-(p) < A^-(q)$, then the ILSG's of A will not form a chain.*

Proof. Suppose $A^+(p) < A^+(q)$ and $A^-(p) < A^-(q)$. Then from the previous discussions we get: $A^+(0) \geq A^+(q) > A^+(p) = A^+(1)$ and $A^-(0) \leq A^-(p) < A^-(q) = A^-(1)$. Hence there exist real numbers $t_1 \geq t_2 > t_3$ and $s_1 \leq s_2 < s_3$ in I, such that $A^+(0) = t_1, A^+(q) = t_2, A^+(p) = A^+(1) = t_3$ and $A^-(0) = s_1, A^-(p) = s_2, A^-(q) = A^-(1) = s_3$. Then ILR of A is:

$$\begin{aligned}\tilde{Ł}(A) &= \{A_{t_1,s_1}, A_{t_1,s_2}, A_{t_1,s_3}, A_{t_2,s_1}, A_{t_2,s_2}, A_{t_2,s_3}, A_{t_3,s_1}, A_{t_3,s_2}, A_{t_3,s_3}\} \\ &= \{\{0\},\{0\},\{0\},\{0\},\{0\},\langle p\rangle,\{0\},\langle q\rangle,\mathbb{Z}_{pq}\}\end{aligned}$$

Hence, the distinct ILSG's of A are: $\{0\}, \langle p\rangle, \langle q\rangle, \mathbb{Z}_{pq}$. Since p and q are distinct primes neither $\langle p\rangle \subseteq \langle q\rangle$ nor $\langle q\rangle \subseteq \langle p\rangle$. Hence the ILSG's does not form a chain.
The case when $A^+(p) > A^+(q)$ and $A^-(p) > A^-(q)$ can be proved similarly. □

Proposition 3.11. *Let $A = \{\langle x, A^+(x), A^-(x)\rangle : x \in \mathbb{Z}_{pq}\}$ be an IFSG of $\mathbb{Z}_{pq}$. If $A^+(p) \geq A^+(q)$ and $A^-(p) \leq A^-(q)$ or if $A^+(p) \leq A^+(q)$ and $A^-(p) \geq A^-(q)$, then the ILSG's of A form a chain.*

Proof. Suppose $A^+(p) \geq A^+(q)$ and $A^-(p) \leq A^-(q)$. Then from the previous discussions we get: $A^+(0) \geq A^+(p) \geq A^+(q) = A^+(1)$ and $A^-(0) \leq A^-(p) \leq A^-(q) = A^-(1)$. Hence there exist real numbers $t_1 \geq t_2 > t_3$ and $s_1 \leq s_2 < s_3$ in I, such that $A^+(0) = t_1, A^+(p) = t_2, A^+(q) = A^+(1) = t_3$ and $A^-(0) = s_1, A^-(p) = s_2, A^-(q) = A^-(1) = s_3$. Then ILR of A is:

$$\begin{aligned}\tilde{Ł}(A) &= \{A_{t_1,s_1}, A_{t_1,s_2}, A_{t_1,s_3}, A_{t_2,s_1}, A_{t_2,s_2}, A_{t_2,s_3}, A_{t_3,s_1}, A_{t_3,s_2}, A_{t_3,s_3}\} \\ &= \{\{0\},\{0\},\{0\},\{0\},\langle p\rangle,\langle p\rangle,\{0\},\langle p\rangle,\mathbb{Z}_{pq}\}\end{aligned}$$

Hence, the distinct ILSG's of A are: $\{0\}, \langle p\rangle, \mathbb{Z}_{pq}$ which form the chain $A_{t_1,s_1} \subseteq A_{t_2,s_2} \subseteq A_{t_3,s_3}$.
In the case when $A^+(p) \leq A^+(q)$ and $A^-(p) \geq A^-(q)$, proceeding similarly as above, we will obtain the distinct ILSG's of A as $\{0\}, \langle q\rangle, \mathbb{Z}_{pq}$, which will form the chain $\{0\} \subseteq \langle q\rangle \subseteq \mathbb{Z}_{pq}$. □

Combining the above two propositions, we get the following *characterisation of IFSGs of a cyclic group of order pq, whose ILSGs form a chain.*

Theorem 3.12. *Let A be an IFSG of $\mathbb{Z}_{pq}$. Then the ILSGs of A form a chain if, and only if, either $A^+(p) \geq A^+(q)$ and $A^-(p) \leq A^-(q)$, or $A^+(p) \leq A^+(q)$ and $A^-(p) \geq A^-(q)$.*

Proposition 3.13. *The probability that the ILSG's corresponding to a randomly defined IFSG of $\mathbb{Z}_{pq}$ forms a chain is 7/9.*

Proof. As stated in proposition 3.7, 36 distinct IFSG's can be defined on $\mathbb{Z}_{pq}$ (upto isomorphism). By propositions 3.10 and 3.11, ILSG's corresponding to exactly 8 among them will not form a chain. Hence the proportion of IFSG's in which the ILSG's form a chain is 7/9. □

4. Tables and Figures

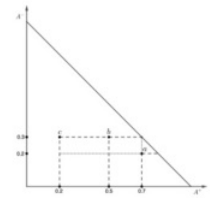

Figure 2. ILSG's in the proof of proposition 3.10

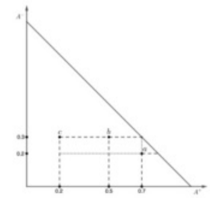

Figure 3. ILSG's in the proof of proposition 3.11

$A_2^+(0) > A_2^+(p) > A_2^+(q) = A_2^+(1)$ $A_2^-(0) < A_2^-(q) < A_2^-(p) = A_2^-(1)$	$A_{11}^+(0) = A_{11}^+(p) > A_{11}^+(q) = A_{11}^+(1)$ $A_{11}^-(0) = A_{11}^-(q) < A_{11}^-(p) = A_{11}^-(1)$
$A_4^+(0) > A_4^+(q) > A_4^+(p) = A_4^+(1)$ $A_4^-(0) < A_4^-(p) < A_4^-(q) = A_4^-(1)$	$A_{13}^+(0) = A_{13}^+(q) > A_{13}^+(p) = A_{13}^+(1)$ $A_{13}^-(0) = A_{13}^-(p) < A_{13}^-(q) = A_{13}^-(1)$
$A_{20}^+(0) > A_{20}^+(p) > A_{20}^+(q) = A_{20}^+(1)$ $A_{20}^-(0) = A_{20}^-(q) < A_{20}^-(p) = A_{20}^-(1)$	$A_{29}^+(0) = A_{29}^+(p) > A_{29}^+(q) = A_{29}^+(1)$ $A_{29}^-(0) < A_{29}^-(q) < A_{29}^-(p) = A_{29}^-(1)$
$A_{22}^+(0) > A_{22}^+(q) > A_{22}^+(p) = A_{22}^+(1)$ $A_{22}^-(0) = A_{22}^-(p) < A_{22}^-(q) = A_{22}^-(1)$	$A_{31}^+(0) = A_{31}^+(q) > A_{31}^+(p) = A_{31}^+(1)$ $A_{31}^-(0) < A_{31}^-(p) < A_{31}^-(q) = A_{31}^-(1)$

Table 1. The 8 IFSG's mentioned in proposition 3.10

Inequality with both $A^+(0)$ and $A^-(0)$	Equality with both $A^+(0)$ and $A^-(0)$
$A_1^+(0) > A_1^+(p) > A_1^+(q) = A_1^+(1)$ $A_1^-(0) < A_1^-(p) < A_1^-(q) = A_1^-(1)$	$A_{10}^+(0) = A_{10}^+(p) > A_{10}^+(q) = A_{10}^+(1)$ $A_{10}^-(0) = A_{10}^-(p) < A_{10}^-(q) = A_{10}^-(1)$
$A_2^+(0) > A_2^+(p) > A_2^+(q) = A_2^+(1)$ $A_2^-(0) < A_2^-(q) < A_2^-(p) = A_2^-(1)$	$A_{11}^+(0) = A_{11}^+(p) > A_{11}^+(q) = A_{11}^+(1)$ $A_{11}^-(0) = A_{11}^-(q) < A_{11}^-(p) = A_{11}^-(1)$
$A_3^+(0) > A_3^+(p) > A_3^+(q) = A_3^+(1)$ $A_3^-(0) < A_3^-(p) = A_3^-(q) = A_3^-(1)$	$A_{12}^+(0) = A_{12}^+(p) > A_{12}^+(q) = A_{12}^+(1)$ $A_{12}^-(0) = A_{12}^-(p) = A_{12}^-(q) = A_{12}^-(1)$
$A_4^+(0) > A_4^+(q) > A_4^+(p) = A_4^+(1)$ $A_4^-(0) < A_4^-(p) < A_4^-(q) = A_4^-(1)$	$A_{13}^+(0) = A_{13}^+(q) > A_{13}^+(p) = A_{13}^+(1)$ $A_{13}^-(0) = A_{13}^-(p) < A_{13}^-(q) = A_{13}^-(1)$
$A_5^+(0) > A_5^+(q) > A_5^+(p) = A_5^+(1)$ $A_5^-(0) < A_5^-(q) < A_5^-(p) = A_5^-(1)$	$A_{14}^+(0) = A_{14}^+(q) > A_{14}^+(p) = A_{14}^+(1)$ $A_{14}^-(0) = A_{14}^-(q) < A_{14}^-(p) = A_{14}^-(1)$
$A_6^+(0) > A_6^+(q) > A_6^+(p) = A_6^+(1)$ $A_6^-(0) < A_6^-(p) = A_6^-(q) = A_6^-(1)$	$A_{15}^+(0) = A_{15}^+(q) > A_{15}^+(p) = A_{15}^+(1)$ $A_{15}^-(0) = A_{15}^-(p) = A_{15}^-(q) = A_{15}^-(1)$
$A_7^+(0) > A_7^+(p) = A_7^+(q) = A_7^+(1)$ $A_7^-(0) < A_7^-(p) < A_7^-(q) = A_7^-(1)$	$A_{16}^+(0) = A_{16}^+(p) = A_{16}^+(q) = A_{16}^+(1)$ $A_{16}^-(0) = A_{16}^-(p) < A_{16}^-(q) = A_{16}^-(1)$
$A_8^+(0) > A_8^+(p) = A_8^+(q) = A_8^+(1)$ $A_8^-(0) < A_8^-(q) < A_8^-(p) = A_8^-(1)$	$A_{17}^+(0) = A_{17}^+(p) = A_{17}^+(q) = A_{17}^+(1)$ $A_{17}^-(0) = A_{17}^-(q) < A_{17}^-(p) = A_{17}^-(1)$
$A_9^+(0) > A_9^+(p) = A_9^+(q) = A_9^+(1)$ $A_9^-(0) < A_9^-(p) = A_9^-(q) = A_9^-(1)$	$A_{18}^+(0) = A_{18}^+(p) = A_{18}^+(q) = A_{18}^+(1)$ $A_{18}^-(0) = A_{18}^-(p) = A_{18}^-(q) = A_{18}^-(1)$

Table 2. The 36 IFSG's mentioned in proposition 3.7 (*i*)

Inequality with $A^+(0)$ and equality with $A^-(0)$	Equality with $A^+(0)$ and inequality with $A^-(0)$
$A^+_{19}(0) > A^+_{19}(p) > A^+_{19}(q) = A^+_{19}(1)$ $A^-_{19}(0) = A^-_{19}(p) < A^-_{19}(q) = A^-_{19}(1)$	$A^+_{28}(0) = A^+_{28}(p) > A^+_{28}(q) = A^+_{28}(1)$ $A^-_{28}(0) < A^-_{28}(p) < A^-_{28}(q) = A^-_{28}(1)$
$A^+_{20}(0) > A^+_{20}(p) > A^+_{20}(q) = A^+_{20}(1)$ $A^-_{20}(0) = A^-_{20}(q) < A^-_{20}(p) = A^-_{20}(1)$	$A^+_{29}(0) = A^+_{29}(p) > A^+_{29}(q) = A^+_{29}(1)$ $A^-_{29}(0) < A^-_{29}(q) < A^-_{29}(p) = A^-_{29}(1)$
$A^+_{21}(0) > A^+_{21}(p) > A^+_{21}(q) = A^+_{21}(1)$ $A^-_{21}(0) = A^-_{21}(p) = A^-_{21}(q) = A^-_{21}(1)$	$A^+_{30}(0) = A^+_{30}(p) > A^+_{30}(q) = A^+_{30}(1)$ $A^-_{30}(0) < A^-_{30}(p) = A^-_{30}(q) = A^-_{30}(1)$
$A^+_{22}(0) > A^+_{22}(q) > A^+_{22}(p) = A^+_{22}(1)$ $A^-_{22}(0) = A^-_{22}(p) < A^-_{22}(q) = A^-_{22}(1)$	$A^+_{31}(0) = A^+_{31}(q) > A^+_{31}(p) = A^+_{31}(1)$ $A^-_{31}(0) < A^-_{31}(p) < A^-_{31}(q) = A^-_{31}(1)$
$A^+_{23}(0) > A^+_{23}(q) > A^+_{23}(p) = A^+_{23}(1)$ $A^-_{23}(0) = A^-_{23}(q) < A^-_{23}(p) = A^-_{23}(1)$	$A^+_{32}(0) = A^+_{32}(q) > A^+_{32}(p) = A^+_{32}(1)$ $A^-_{32}(0) < A^-_{32}(q) < A^-_{32}(p) = A^-_{32}(1)$
$A^+_{24}(0) > A^+_{24}(q) > A^+_{24}(p) = A^+_{24}(1)$ $A^-_{24}(0) = A^-_{24}(p) = A^-_{24}(q) = A^-_{24}(1)$	$A^+_{33}(0) = A^+_{33}(q) > A^+_{33}(p) = A^+_{33}(1)$ $A^-_{33}(0) < A^-_{33}(p) = A^-_{33}(q) = A^-_{33}(1)$
$A^+_{25}(0) > A^+_{25}(p) = A^+_{25}(q) = A^+_{25}(1)$ $A^-_{25}(0) = A^-_{25}(p) < A^-_{25}(q) = A^-_{25}(1)$	$A^+_{34}(0) = A^+_{34}(p) = A^+_{34}(q) = A^+_{34}(1)$ $A^-_{34}(0) < A^-_{34}(p) < A^-_{34}(q) = A^-_{34}(1)$
$A^+_{26}(0) > A^+_{26}(p) = A^+_{26}(q) = A^+_{26}(1)$ $A^-_{26}(0) = A^-_{26}(q) < A^-_{26}(p) = A^-_{26}(1)$	$A^+_{35}(0) = A^+_{35}(p) = A^+_{35}(q) = A^+_{34}(1)$ $A^-_{35}(0) < A^-_{35}(q) < A^-_{35}(p) = A^-_{35}(1)$
$A^+_{27}(0) > A^+_{27}(p) = A^+_{27}(q) = A^+_{27}(1)$ $A^-_{27}(0) = A^-_{27}(p) = A^-_{27}(q) = A^-_{27}(1)$	$A^+_{36}(0) = A^+_{36}(p) = A^+_{36}(q) = A^+_{36}(1)$ $A^-_{36}(0) < A^-_{36}(p) = A^-_{36}(q) = A^-_{36}(1)$

Table 3. The 36 IFSG's mentioned in proposition 3.7 (*ii*)

5. Conclusion

In the course of fuzzification of the abstract algebraic concepts and theories, it has been confirmed that the level subgroups of a fuzzy subgroup of any group form a chain. The case of IFSGs still remains unsettled. Our research was aimed at investigating whether this result can be extended to intuitionistic fuzzy subgroups. We have proved that this result of Das [8] does not extend to intuitionistic fuzzy case. Any cyclic group of order pq, where p and q are distinct primes, is proved to have some IFSGs whose ILSGs do not form chains. We have proved that the generators of cyclic subgroups of any group G have the same membership and non-membership degrees. In the case of $\mathbb{Z}_{pq}$, we have proved that, any IFSG can have at most three membership and non-membership levels. We have also proved that, $\mathbb{Z}_{pq}$ has 36 non-isomorphic IFSGs. Of these, the ILSGs of only 28 IFSGs form chains, while the ILSGs of the remaining 8 do not form chains. We have given the list of the 36 distinct IFSGs of $\mathbb{Z}_{pq}$, and identified those whose IFSGs form, and do not form, chains. The case is illustrated using the example of $\mathbb{Z}_{35}$. During the process, we have also obtained a characterisation of the IFSGs of cyclic groups of order pq, whose ILSGs form a chain.

References

[1] Atanassov K. T., Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, 20(1) (1986), 87-96.

[2] Biswas R., Intuitionistic Fuzzy Subgroups, *Mathematical Forum*, X (1996), 39 – 44

[3] Burton. David M., *Elementary Number Theory*, Seventh Edition, Mc Graw Hill, New York, (2011)

[4] Daise S. D. M. and Tresa S. D. M., On Level Subgroups of Intuitionistic Fuzzy Groups, *J. Comp. Math. Sci.*, 7(11) (2016), 606 - 612

[5] Daise S. D. M., Tresa S. D. M., Fernandez S., Intuitionistic Level Subgroups of Intuitionistic Fuzzy Groups, *Proceedings of International Conference of Mathematics (ICM 2018)*, (2018), 78-82

[6] Daise S. D. M., Tresa S. D. M., Fernandez S., On Isomorphism of Intuitionistic Fuzzy Sets (Communicated).

[7] Daise S. D. M., Tresa S. D. M., Fernandez S., Intuitionistic Level Subgoups in Cyclic Groups of Order p^n (Communicated).

[8] Das P. S., Fuzzy Groups and Level Subgroups, *J. Math. Anal. Appl.*, 84 (1981), 264 – 269.

[9] Palaniappan N, Naganathan S, Arjunan K, A Study on Intuitionistic L-Fuzzy Subgroups, *Applied Mathematical Sciences*, 3(53)(2009), 2619 – 2624

[10] Rosenfeld A., Fuzzy Groups, *J. Math. Anal. Appl.*, 35 (1971), 512 – 517.

[11] Sebastian S., Babu Sunder, Fuzzy groups and group homomorphisms, *Fuzzy Sets and Systems*, 81(1996), 397 - 401.

[12] Sharma P. K., Intuitionistic Fuzzy Groups, *International Journal of Data Warehousing and Mining*, 1(1)(2011), 1–10.

[13] Tae-Chon. Ahn, Hur. Kul, Jang. Kyung-Won, Intuitionistic Fuzzy Subgroups and Level Subgroups, *International Journal of Fuzzy Logic and Intelligent Systems*, 6(3)(2006), 240 - 246.

[14] Zadeh L. A., Fuzzy Sets, *Information and Control*, 8 (1965), 338-353.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆