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On maximal, minimal μ-open and μ-closed sets

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Abstract

In this paper, we introduce and study cleanly μ-covered spaces along with two strong separation axioms in generalized topological spaces. Strong separation axioms are investigated by means of minimal μ-open and μ-closed sets of generalized topological spaces.

Keywords

μ-open set, μ-closed set, maximal μ-open set, minimal μ-open set, cleanly μ-covered.

AMS Subject Classification

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Contents

1. Introduction

On noting down the properties of some sets like semi-open sets [4], preopen sets [5] of topological spaces X, Császár [2] firstly introduced and studied γ-open sets in X. The properties of γ-open sets finally leads to introduce and study the μ-open sets in X by Császár [1].

A subcollection μ of the powerset of a nonempty set X is called a generalized topology on X if $\emptyset \in \mu$ and the union of arbitrary number of members of μ is again a member of μ. The ordered pair (X, μ) is called the generalized topological space. For brevity, a generalized topology (resp. generalized topological space) is expressed by GT (resp. GT space). A subset G of X is called μ-open in X if $G \in \mu$ and a subset E of X is called μ-closed if $X - E \in \mu$. The union of all μ-open sets in X contained in a subset A of X is called the generalized interior of A and it is denoted by $i_\mu(A)$. The intersection of all μ-closed sets in X containing a subset A of X is called the generalized closure of A and we write $c_\mu(A)$ to denote it. It can be shown easily that $i_\mu(A) = X - c_\mu(X - A)$. For a μ-open set (resp. μ-closed set) A of X, we write A is a proper μ-open (resp. proper μ-closed) set in X to mean $A \neq \emptyset$ (resp. $A \neq X$) and $A \neq X$ if $X \in \mu$ (resp. $A \neq \emptyset$ if $X \in \mu$). We also write $\mathbb{N}$ are denote the set of natural numbers.

2. Maximal, minimal μ-open and μ-closed sets

We recall some known definitions and results in the sequel.

Definition 2.1 (Roy and Sen [10]). *A proper μ-open set A of a GT space X is called a maximal μ-open set if there is no μ-open set $U(\neq A, X)$ such that $A \subset U \subset X$.*

Theorem 2.2 (Roy and Sen [10]). *If A is a maximal μ-open set and B is a μ-open set in a GT space X, then either $A \cup B = X$ or $B \subset A$. If B is also a maximal μ-open set distinct from A, then $A \cup B = X$.*

Definition 2.3 (Roy and Sen [10]). *A proper μ-closed set E of a GT space X is called a minimal μ-closed set if there is no μ-closed set $F(\neq \emptyset, E)$ such that $\emptyset \subset F \subset E$.*

Theorem 2.4 (Roy and Sen [10]). *If F is a minimal μ-closed set and E is a μ-closed set in a GT space X, then either $E \cap F = \emptyset$ or $F \subset E$. If E is also a minimal μ-closed set distinct from F, then $F \cap E = \emptyset$.*

Definition 2.5 (Roy [9]). *Let X be a GT space. X is called μ-locally finite if for each $x \in X$ there exists a finite μ-open set U such that $x \in U$.*

Definition 2.6 (S. Al Ghour et al. [3]). *A proper μ-open set U of X is said to be a minimal μ-open set if the only proper μ-open set which is contained in U is U.*

Theorem 2.7 (Mukharjee [6]). *If U is a minimal μ-open set and W is a μ-open set such that $U \cap W$ is a μ-open set, then either $U \cap W = \emptyset$ or $U \subset W$. If W is also a minimal μ-open set distinct from U, then $U \cap W = \emptyset$.*

Definition 2.8 (Mukharjee [6]). *A proper μ-closed set E in a GT space X is called a maximal μ-closed set if the only proper μ-closed sets which contains E is E.*

Theorem 2.9 (Mukharjee [6]). *If E is a maximal μ-closed set and F is any μ-closed set in a GT space X such that $E \cup F$ is a μ-closed set, then either $E \cup F = X$ or $F \subset E$.*

Theorem 2.10 (Roy and Sen [10]). *A proper μ-open set A in a GT space X is maximal μ-open iff $X - A$ is minimal μ-closed in X.*

Similarly, we see that a proper μ-closed set A in a GT space X is maximal μ-closed iff $X - A$ is minimal μ-open in X.

Definition 2.11 (Sarsak [11]). *A μ-space is called μ-compact if each μ-open cover of X has a finite subcover.*

We now introduce some new notions and obtain some of their properties.

Definition 2.12. *A cover $\mathscr{U}$ of a μ-space X is said to be a minimal cover if for each $U \in \mathscr{U}, \mathscr{U} - \{U\}$ is not a cover of X. $\mathscr{U}$ is said to be a minimal μ-open (resp. μ-closed) cover if each member of $\mathscr{U}$ is μ-open (resp. μ-closed).*

We see that a μ-open cover $\mathscr{U}$ can not be a minimal μ-open cover of X, if there are two distinct μ-open sets $U, V \in \mathscr{U}$ such that $V \subset U$. It easily follows that each μ-compact space has a finite minimal μ-open cover. By a nontrivial cover of a GT space X, we mean a cover of X which is not equal to $\{X\}$.

We introduce the following idea on the requirement of at least two proper μ-open sets in a μ-open cover of μ-spaces.

Definition 2.13. *A μ-space X is said to be cleanly μ-covered if each nontrivial μ-open cover of X has a minimal μ-open subcover consisting of exactly two μ-open sets.*

So every cleanly μ-covered μ-space is a μ-compact space. It is very easy to see that a μ-compact space may fail to be a cleanly μ-covered space.

Example 2.14. *Let $X = \mathbb{N}$ and $\mu = \{\emptyset\} \bigcup \{G \subset \mathbb{N} \mid X - G$ is finite $\}$. The μ-space (X, μ) is not cleanly μ-covered but μ-compact.*

Example 2.15. *Let $X = [-1, 1]$ and $\mu = \{\emptyset, X, [-1, 0), (0, 1], [-1, 0], [0, 1], X - \{0\}\}$. The GT space (X, μ) is an example of cleanly μ-covered space.*

Theorem 2.16. *If each μ-open cover of a μ-space X contains a maximal μ-open set, then X is cleanly μ-covered.*

Proof. Let G be a maximal μ-open set belonging to a μ-open cover $\mathscr{U}$ of X. If there exists a distinct maximal μ-open set $H \in \mathscr{U}$, then we have $G \cup H = X$ by Theorem 2.2. Hence $\{G, H\}$ is a subcover of $\mathscr{U}$ for X.

Now we suppose that there is only one maximal μ-open set G in $\mathscr{U}$. There might exists another μ-open set $H \in \mathscr{U}$ distinct from G in order to cover X by $\mathscr{U}$. The μ-open cover $\mathscr{U}$ becomes a trivial μ-open cover if $H = X$. That is why we suppose $H \neq X$. So we have $H \subsetneq G$ or $H \cup G = X$. If $U \subsetneq G$ for all $U \in \mathscr{U}$, then $\mathscr{U}$ can not be a μ-open cover of X. It means that there exists a μ-open set $H \in \mathscr{U}$ distinct from G such that $G \cup H = X$. □

Definition 2.17. *A cover $\mathscr{U}$ of a μ-space X is said to be μ-disconnected if for each $U \in \mathscr{U}$, there exists a $V \in \mathscr{U}$ such that $U \cap V = \emptyset$.*

Theorem 2.18. *If a minimal μ-open cover of a μ-space contains a minimal μ-open set and the intersection of any two μ-open sets is μ-open then the minimal μ-open cover is μ-disconnected.*

Proof. Suppose a minimal μ-open cover $\mathscr{U}$ contains a minimal μ-open set U. There exists one more μ-open set $V \in \mathscr{U}$ different from U to cover X by $\mathscr{U}$. So we have $U \cap V = \emptyset$ or $U \subsetneq V$ by Theorem 2.7. Here $U \subsetneq V$ is not possible as $\mathscr{U}$ is a minimal μ-open cover of X. □

Corollary 2.19. *If a minimal μ-open cover of a μ-space contains only minimal μ-open sets then the minimal μ-open cover is μ-disconnected.*

Definition 2.20 (Deb Ray and Bhowmick [8]). *A collection $\mathscr{U}$ of subsets of a μ-space X is called μ-locally finite if each $x \in X$ belongs to a μ-open set meeting only finitely many members of $\mathscr{U}$.*

Note that μ-locally finite collection of a GT space and μ-locally finite space are two distinct notions.

Theorem 2.21 (Deb Ray and Bhowmick [8]). *If $\mathscr{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ is a μ-locally finite collection of sets in X then $c_\mu(\bigcup_{\lambda \in \Lambda} U_\lambda) = \bigcup_{\lambda \in \Lambda} c_\mu(U_\lambda)$.*

Theorem 2.22. *If $\{U_\lambda \mid \lambda \in \Lambda\}$ is a collection of distinct minimal μ-open sets in a μ-locally finite space X such that $U_{\lambda_1} \cap U_{\lambda_2}$ is μ-open for $\lambda_1, \lambda_2 \in \Lambda$, then $c_\mu(\bigcup_{\lambda \in \Lambda} U_\lambda) = \bigcup_{\lambda \in \Lambda} c_\mu(U_\lambda)$.*

Proof. For $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$, we have $U_{\lambda_1} \cap U_{\lambda_2} = \emptyset$. We choose $x \in X$. Since X is a μ-locally finite space, we obtain a finite μ-open set U such that $x \in U$. From the fact that U is finite and $U_{\lambda_1} \cap U_{\lambda_2} = \emptyset$ whenever $\lambda_1 \neq \lambda_2$, we see that U can intersect only finite number of $U_\lambda, \lambda \in \Lambda$. It implies that $\{U_\lambda \mid \lambda \in \Lambda\}$ is μ-locally finite. So by Theorem 2.21, we get $c_\mu(\bigcup_{\lambda \in \Lambda} U_\lambda) = \bigcup_{\lambda \in \Lambda} c_\mu(U_\lambda)$. □

Analogous to Theorem 2.16, Theorem 2.18, Corollary 2.19 and Theorem 2.22, we have Theorem 2.23, Theorem

2.24, Corollary 2.25 and Theorem 2.26 respectively. The proofs of the them are omitted as proofs are very much similar to corresponding results already established.

Theorem 2.23. *If each nontrivial μ-closed cover $\mathscr{F}$ of a μ-space X contains a maximal μ-closed set E and the intersection of any two μ-closed sets in X is μ-closed, then there exists an $F \in \mathscr{F}$ such that $E \cup F = X$.*

Theorem 2.24. *If a minimal μ-closed cover of a μ-space contains a minimal μ-closed set then the is μ-disconnected.*

Corollary 2.25. *If a minimal μ-closed cover of a μ-space contains only a minimal μ-closed set then the space is μ-disconnected.*

Theorem 2.26. *If $\{U_\lambda \mid \lambda \in \Lambda\}$ is a collection of distinct minimal μ-closed sets in a μ-locally finite space X with $U_{\lambda_1} \cap U_{\lambda_2} = \emptyset$ for any two $\lambda_1 \neq \lambda_2$, then $c_\mu(\bigcup_{\lambda\in\Lambda} U_\lambda) = \bigcup_{\lambda\in\Lambda}(U_\lambda)$.*

3. Two strong separation axioms

We introduce the following two strong separation axioms in a GT space on noting down that two distinct minimal μ-open sets in a GT space are disjoint [6].

Definition 3.1. *A GT space X is said to be strongly μ-regular if for each $x \in X$ and each μ-closed set F with $x \notin F$, there exist disjoint minimal μ-open sets U, V such that $x \in U$ and $F \subset V$.*

Clearly, a strongly μ-regular GT space is μ-regular.

Definition 3.2. *A GT space X is said to be strongly μ-normal if for each pair of disjoint μ-closed sets E, F there exist disjoint minimal μ-open sets U, V such that $E \subset U$ and $F \subset V$.*

Example 3.3. *Let $X = [-1,1]$ and μ be the generalized topology on X where $\mu = \{\emptyset, X, [-1,0), \{0\}, (0,1], [-1,0], [0,1], X - \{0\}\}$. This GT space (X,μ) is both μ-regular and μ-normal. But this space is neither strongly μ-regular nor strongly μ-normal. So we conclude that a μ-regular (resp. μ-normal) space may fail to be be a strongly μ-regular (resp. strongly μ-normal) space.*

Example 3.4. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a\}, \{b,c\}, X\}$. This GT space (X,μ) is an example of strongly μ-regular as well as strongly μ-normal space.*

Definition 3.5. *For a subset A of a GT space X, we define*

$$MinI_\mu(A) = \begin{cases} \emptyset \text{ if } A \text{ contains no minimal } \mu\text{-open set} \\ \bigcup\{G \mid G \text{ is a minimal} \\ \qquad \mu\text{-open set contained in } A\}. \end{cases}$$

and

$$MaxC_\mu(A) = \begin{cases} X \text{ if } A \text{ contained in no maximal} \\ \qquad \mu\text{-closed set} \\ \bigcap\{E \mid E \text{ is a maximal} \\ \qquad \mu\text{-closed set containing } A\}. \end{cases}$$

We see that $MinI_\mu(G) = G$ if G is a minimal μ-open set and $MaxC_\mu(E) = E$ if E is a maximal μ-closed set in X. If G, H are two distinct minimal μ-open sets, then G, H are proper μ-open sets distinct from $G \cup H$ and $G, H \subset G \cup H$ which means that $G \cup H$ is not a minimal μ-open set. So we conclude that the union of even finitely many distinct minimal μ-open sets is not minimal μ-open. Again if E, F are two distinct maximal μ-closed sets, then E, F are proper μ-closed sets distinct from $E \cap F$ and contain $E \cap F$ which means that $E \cap F$ is not a maximal μ-closed set. So we conclude that the intersection of even finitely many distinct maximal μ-closed sets is not maximal μ-closed. Thus it follows that $MinI_\mu(A)$ (resp. $MaxC_\mu(A)$) may not be minimal μ-open (resp. maximal μ-closed).

From Example 3.3, we have $MinI_\mu([-1,1/2)) = [-1,0]$ which is not a minimal μ-open set and $MaxC\mu(\{0\}) = \{0\}$ which is not a maximal μ-closed set.

Theorem 3.6. *For each $A \subseteq X$, $X - MinI_\mu(A) = MaxC_\mu(X - A)$.*

Proof. First of all, we see that if $MinI_\mu(A) = \emptyset$, then there is no minimal μ-open set in A. In this case, we prove $X - A$ is not contained in a maximal μ-closed set by contradiction. Let E be a maximal μ-closed set such that $X - A \subset E$. Here $X - E$ is minimal μ-open with $X - E \subset A$, a contradiction. So we see $MaxC_\mu(X - A) = X$. It means $X - MinI_\mu(A) = MaxC_\mu(X - A)$.

Now we suppose that $MinI_\mu(A) \neq \emptyset$. So there exists a minimal μ-open set $G \subset A$. Then $X - G$ is a maximal μ-closed set that contains $X - A$. It means $MaxC_\mu(X - A) \neq X$. Clearly, if $\{G\}$ is the family of all minimal μ-open sets in A, then $\{X - G\}$ is the family of all maximal μ-closed sets that contain $X - A$ and conversely. Hence

$$\begin{aligned} X - MinI_\mu(A) &= X - \bigcup\{G \mid G \text{ is minimal } \mu\text{-open set}\} \\ &= \bigcap\{X - G \mid X - G \text{ is maximal} \\ &\qquad \mu\text{-closed containing } X - A\} \\ &= MaxC_\mu(X - A). \end{aligned}$$

□

Theorem 3.7. *For each $A \subseteq X$, $MinI_\mu(A)$ is minimal μ-open if there is one and only one minimal μ-open set in A and vice-versa.*

Proof. Firstly, let $MinI_\mu(A)$ be minimal μ-open and G, H be two minimal μ-open sets contained in A. So we get $G \cup H \subset MinI_\mu(A) = G \cup H \subset A$. As $G, H \subset G \cup H$ and $G \cup H = MinI_\mu(A)$ is a minimal μ-open set by assumption, we have $G = G \cup H$ as well as $H = G \cup H$. From $G \cup H = G$ we get $H \subset G$. Again from $G \cup H = H$ we have $G \subset H$. Hence $G = H$. Thus there is only one minimal μ-open set in $MinI_\mu(A)$. The converse is easy to follow. □

Theorem 3.8 is a dual of Theorem 3.7. The proof of the theorem is omitted as the proof is similar to the Theorem 3.7.

Theorem 3.8. *For any $A \subseteq X$, $MaxC_\mu(A)$ is maximal μ-closed if there is one and only one maximal μ-closed set that contains A and vice-versa.*

Theorem 3.9. *A GT space X is strongly μ-regular if for each μ-open set G and each $x \in G$, there exist a minimal μ-open set U and a maximal μ-closed set E such that $x \in U \subset MaxC_\mu(U) \subset E \subset G$ and conversely.*

Proof. At first, we suppose that the GT space X is strongly μ-regular and G be μ-open. Then for any $x \in G$, there exists two disjoint minimal μ-open sets U, V such that $x \in U, X - G \subset V$. Since $U \cap V = \emptyset$, we have $U \subset X - V$. As $X - V$ is maximal μ-closed, we get $MaxC_\mu(X - V) = X - V$. So we get $MaxC_\mu(U) \subset MaxC_\mu(X - V) = X - V \subset G$. We set $E = X - V$, then E is maximal μ-closed. It also follows that $x \in U \subset MaxC_\mu(U) \subset E \subset G$.

To prove the converse, let E be μ-closed and $x \in X$ be such that $x \notin E$. We see that $X - E$ is μ-open and $x \in X - E$. So we obtain a minimal μ-open set U and a maximal μ-closed set F such that $x \in U \subset MaxC_\mu(U) \subset F \subset X - E$. Putting $V = X - F$, we see that V is minimal μ-open such that $E \subset V$ and $U \cap V = \emptyset$. □

Theorem 3.10. *A GT space X is strongly μ-normal if for a μ-closed set E and for a μ-open set G with $E \subset G$, there exist a minimal μ-open set U and a maximal μ-closed F such that $E \subset U \subset MaxC_\mu(U) \subset F \subset G$ and conversely.*

Proof. Similar to the proof of Theorem 3.9. □

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