Malaya Journal of Matematik, Vol. 8, No. 4, 1834-1841, 2020

https://doi.org/10.26637/MJM0804/0084

New results on existence of Atangana-Baleanu fractional differential equations with dependence on the Lipschitz first derivatives

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Abstract

We study on the existence and uniqueness of solutions for a Atangana-Baleanu fractional differential equations with dependence on the Lipschitz first derivative conditions. We develop a Gronwall inequality in the frame of Atangana-Baleanu fractional integral. An example is given to illustrate the main results and investigate the stability in the sense of Ulam.

Keywords

Fractional differential equations; Atangana-Baleanu fractional derivative; Lipschitz first derivatives; Gronwall inequality; Ulam-Hyer stability.

AMS Subject Classification

34A08, 34K37, 58C30.

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Article History: Received **21** August **2020**; Accepted **19** October **2020**

Contents

1. Introduction

In the past decades, there has been a growing interest in the study of the fractional differential equations due to the rigorous growth of the fractional calculus theory itself and its applications in various fields such as chemistry, physics, engineering, control theory, aerodynamics, electrodynamics of complex medium and control of dynamical systems and so on. In consequence, fractional differential equations is obtaining much significance and attention. For details, we refer readers to [12, 18, 19, 23] and references therein.

There are some approaches to the fractional derivatives such as Riemann-Liouville, Caputo, Weyl, Hadamard and Grunwald-Letnikov, etc. The most well known fractional operator are perhaps the Riemann-Liouville fractional integral and derivatives. Fractional operators are act as an magnificent tools for the mathematical modeling of the real world problems. Later, Atangana and Baleanu proposed two new fractional derivatives based on the Caputo and the Riemann-Liouville definitions of fractional-order derivatives. Other kinds of fractional derivative that look like the Riemann-Liouville and Caputo ones can be seen in [6, 21].

In recent years, many researchers paid much attention to ABC-derivative with several conditions in various spaces. The Atangana-Baleanu fractional derivative is familiar to followings nonsingularity as well as nonlocality of the kernel, which acquires the generalized Mittag-Leffler function. Some of the latest studies on ABC-derivatives such as, Jarad et al. investigated a ODE's in the form of AB-derivative [20]. Ravichandran et al. discussed in details the AB-fractional integro-differntial equations. Atangana and Koca find the chaos in a simple nonlinear system with AB-fractional derivatives [10].

More precisely in [11], Khan et al. investigated Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel. Sene discussed Stokes' first problem for heated flat plate with AB-derivative [31]. Owolabi studied the modelling and simulation of a dynamical system with the Atangana-Baleanu fractional derivative [30].

To be concise, in this paper we are concerned with the study of the existence and uniqueness of solutions of the Atangana-Baleanu fractional derivative equation in the sense of Caputo as follows

$$({}^{ABC}_{a}D^{\alpha}u)(t) = f(t,u(t),u'(t,u(t))), \quad 1<\alpha\le 2, \tag{1.1}$$

$$u(a) = u_0. \tag{1.2}$$

with $t\in C[a,b]$, where ${}^{ABC}_{a}D^{\alpha}$ is the left Caputo AB fractional derivative, $u(t), ({}^{ABC}_{a}D^{\alpha})u$, $f\in C[a,b], f(a,u(a),u'(a,u(a)))=0$. Consider $\mathfrak{D}u(t)=u'(t,u(t))$. Then (1.1) becomes

$$({}^{ABC}_{a}D^{\alpha}u)(t) = f(t,u(t),\mathfrak{D}u(t)), \quad 1<\alpha\le 2, \tag{1.3}$$

$$u(a) = u_0 \tag{1.4}$$

The rest of this paper is organized as follows: In Section 2, we review some useful properties, definitions, propositions and lemmas of fractional calculus. The existence and uniqueness of solutions for AB-fractional derivative results are proved in Section 3. In section 4 is devoted to illustrate an example numerically solved. Ulam-Hyer stability analysis is considered in section 5.

2. Preliminaries

In this section, we presents some definitions, lemmas and propositions of fractional calculus, which will be used throughout this paper.

The definition of Riemann-Liouville fractional integral and derivatives are given as follows:

- For $\alpha>0$, the left Riemann-Liouville fractional integral of order α is given as [20]

$$({}_aI^{\alpha}u)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds. \tag{2.1}$$

- For $0<\alpha<1$, the left Riemann-Liouville fractional derivative of order α is given as [20]

$$({}_aD^{\alpha}u)(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}u(s)ds\right) \tag{2.2}$$

- For $0\le\alpha\le 1$, the Caputo fractional derivative of order α is given as [20]

$$({}^{C}_{a}D^{\alpha}u)(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}u'(s)ds. \tag{2.3}$$

Definition 2.1. *[7] Let $u\in H^1(a,b)$, $a<b$ and α in [0,1]. The Caputo Atangana-Baleanu fractional derivative of u of order α is defined by*

$$({}^{ABC}_{a}D^{\alpha}u)(t) = \frac{B(\alpha)}{(1-\alpha)}\int_0^t u'(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right]ds. \tag{2.4}$$

where E_{α} is the Mittag-Leffler function defined by $E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}$ [25, 32] and $B(\alpha)>0$ is a normalizing function satisfying $B(0)=B(1)=1$. The Riemann Atangana-Baleanu fractional derivative of u of order α is defined by

$$({}^{ABR}_{a}D^{\alpha}u)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t u(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right]ds. \tag{2.5}$$

The associative fractional integral is defined by

$$({}^{AB}_{a}I^{\alpha}u)(t) = \frac{1-\alpha}{B(\alpha)}u(t)+\frac{\alpha}{B(\alpha)}({}_aI^{\alpha}u)(t) \tag{2.6}$$

where ${}_aI^{\alpha}$ is the left Riemann-Liouville fractional integral given in (2.1).

Lemma 2.2. *[7] Let $u\in H^1(a,b)$ and $\alpha\in[0,1]$. Then the following relation holds.*

$$({}^{ABC}_{a}D^{\alpha}u)(t) = ({}^{ABR}_{a}D^{\alpha}u)(t)-\frac{B(\alpha)}{1-\alpha}u(a)E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-a)^{\alpha}\right). \tag{2.7}$$

Lemma 2.3. *[20] Suppose that $\alpha>0$, $c(t)(1-\frac{1-\alpha}{B(\alpha)}d(t))^{-1}$ is a nonnegative, nondegreasing and locally integrable function on [a,b), $\frac{\alpha d(t)}{B(\alpha)}(1-\frac{1-\alpha}{B(\alpha)}d(t))^{-1}$ is non-negative and bounded on [a,b) and $u(t)$ is nonnegative and locally integrable [a,b) with*

$$u(t)\le c(t)+d(t)({}^{AB}_{a}I^{\alpha}u)(t), \tag{2.8}$$

then

$$u(t)\le\frac{c(t)B(\alpha)}{B(\alpha)-(1-\alpha)d(t)}E_{\alpha}\left(\frac{\alpha\ d(t)(t-a)^{\alpha}}{B(\alpha)-(1-\alpha)d(t)}\right). \tag{2.9}$$

Theorem 2.4. *(Ascoli-Arzela Theorem)([15]) Let S be a compact metric spaces.Then $M\subset C(\Omega)$ is relatively compact iff M is uniformly bounded and uniformly equicontinuous.*

Theorem 2.5. *(Krasnoselskii Fixed Point Theorem)([15]) Let S be a closed, bounded and convex subset of a real Banach space X and let T_1 and T_2 be operators on S satisfying the following conditions*

- *$T_1(s)+T_2(s)\subset S$*
- *T_1 is a strict contraction on S, i.e., there exist a $k\in[a,b)$ such that*
  $\|T_1(u)-T_1(v)\|\le k\|u-v\|\ \ \forall\, u,v\in S$
- *T_2 is continuous on S and $T_2(s)$ is a relatively compact subset of X.*

Then, there exist a $u\in S$ such that $T_1u+T_2u=u$

Proposition 2.6. *([4]) For $0\le\alpha\le 1$,*

$$\begin{aligned}&({}^{AB}_{a}I^{\alpha}({}^{ABC}_{a}D^{\alpha}u))(t)\\ &=u(t)-u(a)E_{\alpha}(\lambda t^{\alpha})-\frac{\alpha}{1-\alpha}u(a)E_{\alpha,\alpha+1}(\lambda t^{\alpha})\\ &=u(t)-u(a).\end{aligned}$$

Proposition 2.7. *([22, 28]) $f'(u) \in D$ satisfy the Lipschitz condition.*
i.e.,There exist a constant $k > 0$ such that

$$\|f'(u) - f'(v)\| \leq k\,(\|u - v\|), \quad u, v \in D. \tag{2.10}$$

Definition 2.8. *A continuous function $u : [a,b] \to \Re$ is called a mild solution of the following Atangana-Baleanu fractional derivative equation in the sense of Caputo*

$$\begin{cases} ({}^{ABC}_{a}D^{\alpha}u)(t) = g(t), & 1 < \alpha \leq 2, \\ u(a) = u_0 \end{cases}$$

for each $t \in C[a,b]$, $u(t)$ satisfies the following integral equation

$$u(t) = u_o +^{AB}_{a} I^{\alpha} g(t)$$

3. Existence and Uniqueness

In this section, we prove the existence and uniqueness of (1.3) and (1.4).

We need the following assumptions to prove the existence and uniqueness results for the problem (1.3) and (1.4) by using the Banach contraction principle.

A_1 Let $u \in C[a,b]$ and $f \in (C[a,b] \times \Re \times \Re, \Re)$ is continuous function and there exist a positive constants $\mathfrak{M}_1, \mathfrak{M}_2$ and $\mathfrak{M}$ such that

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq \mathfrak{M}_1(\|u_1 - u_2\| + \|v_1 - v_2\|)$$

for all u_1, v_1, u_2, v_2 in Y, $\mathfrak{M}_2 = \max_{t \in \Re} \|f(t,0,0)\|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$. Let $Y = C[\Re, X]$ be the set continuous functions on $\Re$ with values in the Banach spaces X .

A_2 Let $u' \in C[a,b]$ satisfy the Lipschitz condition. i.e., There exist a positive constants $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}$ such that

$$\|\mathfrak{D}(t,u) - \mathfrak{D}(t,v)\| \leq \mathfrak{N}_1(\|u - v\|),$$

for all u, v in Y. $\mathfrak{N}_2 = \max_{t \in D} \|\mathfrak{D}(t,0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

A_3 For each $\lambda > 0$, Let $B_\lambda \in \{u \in Y : \|u\| \leq \lambda\} \subset Y$ where $\lambda = ((1-\rho)^{-1}\|u_0\|)$ and take ρ is $(\mathfrak{M}(\|u\| + \mathfrak{N}t\|u\|))$.

A_4 For each $\lambda_0 > 0$, Let $B_{\lambda_0} \in \{u \in C([a,b], \Re) : \|u\| \leq \lambda\}$ then B_{λ_0} is clearly bounded, closed and convex subset in $C([a,b], \Re)$.

Lemma 3.1. *If A_1 and A_2 are satisfied, then the estimate $\|\mathfrak{D}u(t)\| \leq t(\mathfrak{N}_1\|u\| + \mathfrak{N}_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \leq \mathfrak{N}t\|u - v\|$, are satisfied for any $t \in \Re$, and $u, v \in Y$.*

Theorem 3.2. *Let $u(t) \in C[a,b]$ such that $({}^{ABC}_{a}D^{\alpha}u)(t) \in C[a,b]$. Suppose that $f \in C([a,b] \times \Re \times \Re, \Re)$ satisfies $A_1 - A_3$. Then, if $f(a, u(a), \mathfrak{D}u(a)) = 0$ and*

$$\left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right) \leq 1$$

the problem (1.3) and (1.4) has an unique solution.

Proof. First, we have to prove that u(t) satisfies the problem (1.3) and (1.4) if and only if u(t) satisfies the integral equation

$$u(t) = u_0 +^{AB}_{a} I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \tag{3.1}$$

Let $u(t)$ satisfy (1.3). To apply the Atangana-Baleanu fractional integral to both sides of (1.3), we get

$$({}^{AB}_{a}I^{\alpha}({}^{ABC}_{a}D^{\alpha}u))(t) =^{AB}_{a} I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \tag{3.2}$$

Now, constructing use of Proposition 2.6, we get

$$u(t) - u(a) =^{AB}_{a} I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \tag{3.3}$$

Since $u(a) = u_0$ from (1.4) and $f(a, u(a), \mathfrak{D}u(a)) = 0$, (3.1) is satisfied. If $u(t)$ satisfies (3.1), then by using that $f(a, u(a), \mathfrak{D}u(a)) = 0$ it is obvious that $u(a) = u_0$.
To apply the Riemann-Liouville Atangana-Baleanu fractional derivative to both sides of (3.1) and utilize that $({}^{AB}_{a}D^{\alpha}({}^{AB}_{a}I^{\alpha}u))(t) = u(t)$. We get

$$\begin{aligned}({}^{ABR}_{a}D^{\alpha}u)(t) &= u_0({}^{ABR}_{a}D^{\alpha}1)(t) \\ &\quad + ({}^{ABR}_{a}D^{\alpha}({}^{AB}_{a}I^{\alpha}))(t) f(t, u(t), \mathfrak{D}u(t))\end{aligned} \tag{3.4}$$

Thus, we have

$$\begin{aligned}({}^{ABR}_{a}D^{\alpha}u)(t) &= u_0 E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-a)^{\alpha}\right) \\ &\quad + f(t, u(t), \mathfrak{D}u(t))\end{aligned} \tag{3.5}$$

Then, the result is acquired by benefiting from theorem 3.2 in [7]. Now, we define the operator

$$Tu(t) = u_0 +^{AB}_{a} I^{\alpha} f(t, u(t), \mathfrak{D}u(t)).$$

Then, by A_3, $\|u\| \leq \lambda$ we get

$$
\begin{aligned}
&\|Tu(t)\| \leq \|u_0\| + \|{}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t))\| \\
&\leq \|u_0\| + \left\| \frac{1-\alpha}{B(\alpha)} f(t,u(t),\mathfrak{D}u(t)) \right\| \\
&+ \frac{\alpha}{B(\alpha)} \|{}_a I^{\alpha} f(t,u(t),\mathfrak{D}u(t))\| \\
&\leq \|u_0\| + \frac{1-\alpha}{B(\alpha)} (\mathfrak{M}(\|u\|) + \mathfrak{N}t\|u\|) \\
&+ \frac{\alpha}{B(\alpha)} (\mathfrak{M}(\|u\|) + \mathfrak{N}t\|u\|)({}_a I^{\alpha})(t) \\
&\leq \|u_0\| + \frac{1-\alpha}{B(\alpha)} (\mathfrak{M}(\|u\|) + \mathfrak{N}t\|u\|) \\
&+ \frac{\alpha}{B(\alpha)} (\mathfrak{M}(\|u\|) + \mathfrak{N}t\|u\|) \frac{t^{\alpha}-a}{\Gamma(\alpha+1)} \\
&\leq \|u_0\| + \frac{1-\alpha}{B(\alpha)} \rho\|u\| + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \rho\|u\| \\
&\leq \|u_0\| + \rho\left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \right)\|u\| \\
&\leq \|u_0\| + \rho\|u\| \\
&\leq \lambda(1-\rho) + \rho\lambda \\
&\leq \lambda
\end{aligned}
$$

i.e., $\|Tu(t)\| \leq \lambda$. Now to prove uniqueness

$$
\begin{aligned}
&\|T(u) - T(v)\| \\
&= \|{}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)) - {}_a^{AB}I^{\alpha} f(t,v(t),\mathfrak{D}v(t))\| \\
&= \Big\| \frac{1-\alpha}{B(\alpha)} f(t,u(t),\mathfrak{D}u(t)) - \frac{1-\alpha}{B(\alpha)} f(t,v(t),\mathfrak{D}v(t)) \\
&+ \frac{\alpha}{B(\alpha)} {}_a I^{\alpha} f(t,u(t),\mathfrak{D}u(t)) \\
&- \frac{\alpha}{B(\alpha)} {}_a I^{\alpha} f(t,v(t),\mathfrak{D}v(t)) \Big\| \\
&\leq \frac{1-\alpha}{B(\alpha)} \|f(t,u(t),\mathfrak{D}u(t)) - f(t,v(t),\mathfrak{D}v(t))\| \\
&+ \frac{\alpha}{B(\alpha)} ({}_a I^{\alpha}) \|f(t,u(t),\mathfrak{D}u(t)) - f(t,v(t),\mathfrak{D}v(t))\| \\
&\leq \frac{1-\alpha}{B(\alpha)} (\mathfrak{M}(\|u-v\| + \mathfrak{N}t\|u-v\|)) \\
&+ \frac{\alpha}{B(\alpha)} (\mathfrak{M}(\|u-v\| + \mathfrak{N}t\|u-v\|))({}_a I^{\alpha})(t) \\
&\leq \rho\left(\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)} \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \right)\|u-v\| \\
&\leq \rho\left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \right)\|u-v\| \\
&\leq \|u-v\|
\end{aligned}
$$

Since $\rho \leq 1$, we have $\|Tu - Tv\| \leq \|u-v\|$. Hence, the operator $Tu(t), t \in B_{\lambda}$ proved the existence and uniqueness conditions and has a fixed point by Banach contraction principle in Banach spaces X. □

Next, we investigate the problem (1.3) and (1.4) has a fixed point by using another fixed point technique, namely Krasnoselskii's fixed point theorem.

Theorem 3.3. *If* $\mathbf{A_1} - \mathbf{A_4}$ *are satisfied and*

$$q(t_2 - t_1) = [\mathfrak{M}(\|u(t_2) - u(t_1)\| + \mathfrak{N}t\|u(t_2) - u(t_1)\|)],$$

then the problem (1.3) *and* (1.4) *has a solution.*

Proof. For any constant $\lambda_0 > 0$ and $u \in B_{\lambda_0}$, defined two operator T_1 and T_2 on B_{λ_0} as follows

$$(T_1 u)(t) = u_0 \tag{3.6}$$

$$(T_2 u)(t) = {}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)). \tag{3.7}$$

Obviously, u is a solution of (1.3) and (1.4) iff the operator $T_1 u + T_2 u = u$ has a solution $u \in B_{\lambda_0}$
Our proof will be divided into three steps.
Step 1. $\|T_1 u + T_2 u\| \leq \lambda_0$ whenever $u \in B_{\lambda_0}$.
For every $u \in B_{\lambda_0}$, we have

$$
\begin{aligned}
&\|(T_1 u)(t) + (T_2 u)(t)\| \\
&\leq \|u_0\| + \|{}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t))\| \\
&\leq \|u_0\| + \frac{1-\alpha}{B(\alpha)} (\mathfrak{M}(\|u\|) + \mathfrak{N}t\|u\|) \\
&+ \frac{\alpha}{B(\alpha)} (\mathfrak{M}(\|u\|) + \mathfrak{N}t\|u\|) \frac{(t^{\alpha}-a)}{\Gamma(\alpha+1)} \\
&\leq \|u_0\| + \frac{1-\alpha}{B(\alpha)} \rho\|u\| + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \rho\|u\| \\
&\leq \|u_0\| + \rho\Big(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \Big)\|u\| \\
&\leq \|u_0\| + \rho\|u\| \\
&\leq \lambda(1-\rho) + \rho\lambda \\
&\leq \lambda
\end{aligned}
$$

Hence, $\|T_1 u + T_2 u\| \leq \lambda_0$ for every $u \in B_{\lambda_0}$.
Step 2. T_1 is a contraction on B_{λ_0} for any $u, v \in B_{\lambda_0}$, according to $\mathbf{A_4}$ and (3.6), we have

$$\|(T_1 u)(t) - (T_1 v)(t\| \leq \|u_0 - v_0\| = R\|u_0 - v_0\|$$

which implies that $\|T_1 u - T_1 v\| \leq R\|u_0 - v_0\|$, since $R = 1$, T_1 is a contraction.
Step 3. T_2 is completely continuous operator.

First we have to prove that T_2 is continuous on B_{λ_0}. For any $u_n, u \subset B_{\lambda_0}$, $n = 1,2,3....$ with $\lim_{n\to u} \|u_n - u\| = 0$, we get $\lim_{n\to u} u_n(t) = u(t)$, for $t \in [a,b]$.
Thus by $\mathbf{A_1}$, we have

$$\lim_{n\to\infty} f(t,u_n(t),\mathfrak{D}u_n(t)) = f(t,u(t),\mathfrak{D}u(t))$$

for $t \in [a,b]$.
We can conclude that

$$\sup_{s\in[a,b]} \|f(t,u_n(t),\mathfrak{D}u_n(t)) - f(t,u(t),\mathfrak{D}u(t))\| \to 0 \text{ as } n \to \infty$$

On other hand, for $t \in [a,b]$

$$\begin{aligned}
&\|(T_2u_n)(t)-(T_2u)(t)\| \\
&\le \|{}_a^{AB}I^{\alpha} f(t,u_n(t),\mathfrak{D}u_n(t)) - {}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t))\| \\
&\le \frac{1-\alpha}{B(\alpha)}\|{}_a^{AB}I^{\alpha} f(t,u_n(t),\mathfrak{D}u_n(t)) \\
&- {}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t))\| \\
&+\frac{\alpha}{B(\alpha)}\|{}_a^{AB}I^{\alpha} f(t,u_n(t),\mathfrak{D}u_n(t)) \\
&- {}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t))\|{}_aI^{\alpha}(t) \\
&\le \frac{1-\alpha}{B(\alpha)}\sup_{s\in[a,b]}\|f(t,u_n(t),\mathfrak{D}u_n(t))-f(t,u(t),\mathfrak{D}u(t))\| \\
&+\frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\sup_{s\in[a,b]}\|f(t,u_n(t),\mathfrak{D}u_n(t)) \\
&-f(t,u(t),\mathfrak{D}u(t))\| \\
&\le \Big(\frac{1-\alpha}{B(\alpha)}-\frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big) \\
&\sup_{s\in[a,b]}\|f(t,u_n(t),\mathfrak{D}u_n(t))-f(t,u(t),\mathfrak{D}u(t))\|
\end{aligned}$$

Hence $\|(T_2u_n)(t)-(T_2u)(t)\| \to 0$ as $n\to\infty$. Therefore T_2 is continuous on B_{λ_0}.

Now, we have to show that $T_2u, u\in B_{\lambda_0}$ is relatively compact which is sufficient to prove that the function $T_2u, u\in B_{\lambda_0}$ uniformly bounded and equicontinuous, and $\forall\, t\in[a,b]$

$\|T_2u\|\le \lambda_0$, for any $u\in B_{\lambda_0}$, therefore $(T_2u)(t), u\in B_{\lambda_0}$ is bounded uniformly.

Now, we prove that $(T_2u)(t), u\in B_{\lambda_0}$ is a equicontinuous. For any $u\in B_{\lambda_0}$ and $a\le t_1\le t_2\le t$, we get

$$\begin{aligned}
&\|(T_2u)(t_2)-(T_2u)(t_1)\| \\
&\le \|{}_a^{AB}I^{\alpha} f(t_2,u(t_2),\mathfrak{D}u(t_2)) - {}_a^{AB}I^{\alpha} f(t_1,u(t_1),\mathfrak{D}u(t_1))\| \\
&\le \frac{1-\alpha}{B(\alpha)}\|f(t_2,u(t_2),\mathfrak{D}u(t_2))-f(t_1,u(t_1),\mathfrak{D}u(t_1))\| \\
&+\frac{\alpha}{B(\alpha)}{}_aI^{\alpha}\|f(t_2,u(t_2),\mathfrak{D}u(t_2))-f(t_1,u(t_1),\mathfrak{D}u(t_1))\| \\
&\le \frac{1-\alpha}{B(\alpha)}(\mathfrak{M}(\|u(t_2)-u(t_1)\|+\mathfrak{N}t\|u(t_2)-u(t_1)\|)) \\
&+\frac{\alpha}{B(\alpha)}(\mathfrak{M}(\|u(t_2)-u(t_1)\|+\mathfrak{N}t\|u(t_2)-u(t_1)\|)) \\
&({}_aI^{\alpha})(t_2-t_1) \\
&\le \frac{1-\alpha}{B(\alpha)}q(t_2-t_1)+\frac{\alpha}{B(\alpha)}q(t_2-t_1)\frac{(t_2-t_1)^{\alpha}}{\alpha\Gamma(\alpha)} \\
&\le \Big(\frac{1-\alpha}{B(\alpha)}-\frac{(t_2-t_1)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)(t_2-t_1)
\end{aligned}$$

$\|(T_2u)(t_2)-(T_2u)(t_1)\|\to 0$ as $t_2\to t_1$. Therefore, the operator T_2 is a equicontinuous on B_{λ_0}. Hence, which implies T_2 is relatively compact on B_{λ_0}.

Therefore T_2 is relatively compact subset of X by theorem 2.4 and, by theorem 2.5 we can conclude that T_2 has atleast one fixed point. Therefore the operator T has a fixed point u which is the solution of (1.3) and (1.4). □

4. Ulam-Hyer stability

In this section, we study the Ulam-Hyer stability of (1.3) and (1.4). Now, we present the definition of Ulam-Hyer stability.

Definition 4.1. *Equation* (1.3) *is Ulam-Hyer stable, if for all* $v(t)$ *satisfying the inequality*

$$|{}_a^{ABC}D^{\alpha}v(t)-f(t,v(t),\mathfrak{D}v(t))|<\varepsilon, \tag{4.1}$$

there exist a solution $u(t)$ *of* (1.3) *and* (1.4) *satisfying*

$$|v(t)-u(t)|<h_f.\varepsilon\ , \quad h_f\in\mathfrak{R}. \tag{4.2}$$

Theorem 4.2. *Suppose that the hypothesis for existence of solutions to* (1.3) *and* (1.4) *are satisfied with*

$$\mathfrak{M}(1+\mathfrak{N})\le\frac{B(\alpha)}{1-\alpha}.$$

Then (1.3) *and* (1.4) *is Ulam-Hyer stable.*

Proof. If $v(t)$ satisfies (4.1), there exists a function $\xi(t)$ satisfying $|\xi(t)|<\varepsilon$ such that

$${}_a^{ABC}D^{\alpha}v(t)-f(t,v(t),\mathfrak{D}v(t))=\xi(t), \tag{4.3}$$

which is satisfies to

$$v(t)-v(a)-{}_a^{AB}I^{\alpha}f(t,v(t),\mathfrak{D}v(t))={}_a^{AB}I^{\alpha}\xi(t). \tag{4.4}$$

Therefore, we have

$$\begin{aligned}
&|v(t)-v(a)-{}_a^{AB}I^{\alpha}f(t,v(t),\mathfrak{D}v(t))| \\
&=|{}_a^{AB}I^{\alpha}\xi(t)| \\
&=\Big|\frac{1-\alpha}{B(\alpha)}\xi(t)+\frac{\alpha}{B(\alpha)}{}_aI^{\alpha}\xi(t)\Big| \\
&\le\frac{1-\alpha}{B(\alpha)}|\xi(t)|+\frac{\alpha}{B(\alpha)}|{}_aI^{\alpha}\xi(t)| \\
&\le\frac{1-\alpha}{B(\alpha)}|\xi(t)|+\frac{\alpha}{B(\alpha)}|\xi(t)|({}_aI^{\alpha}1)(t) \\
&\le\varepsilon\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right)
\end{aligned}$$

Now let $u(t)$ be the solution of (1.3) satisfies $u(a)=v(a)$. Then, we have

$$u(t)=v(a)+{}_a^{AB}I^{\alpha}f(t,u(t),\mathfrak{D}u(t)). \tag{4.5}$$

Note that the existence and uniqueness of $u(t)$ is guaranteed by theorem 3.2. we have

$$\begin{aligned}|v(t)-u(t)| &= |v(t) -{}_a^{AB}I^{\alpha} f(t,v(t),\mathfrak{D}v(t)) \\ &\quad +{}_a^{AB}I^{\alpha} f(t,v(t),\mathfrak{D}v(t)) - v(a) \\ &\quad -{}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t))| \\ &\leq |v(t)-v(a){}_a^{AB}I^{\alpha} f(t,v(t),\mathfrak{D}v(t))| \\ &\quad +|{}_a^{AB}I^{\alpha} f(t,v(t),\mathfrak{D}v(t)) \\ &\quad -{}_a^{AB}I^{\alpha} f(t,u(t),u'(t))| \\ &\leq \varepsilon\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right) \\ &\quad +{}_a^{AB}I^{\alpha}|f(t,v(t),\mathfrak{D}v(t)) \\ &\quad -f(t,u(t),\mathfrak{D}u(t))| \\ &\leq \varepsilon\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right) \\ &\quad +\mathfrak{M}(1+\mathfrak{N}){}_a^{AB}I^{\alpha}|v(t)-u(t)|\end{aligned}$$

Now, by using the Gronwall inequality in lemma 2.3, we get

$$\begin{aligned}&|v(t)-u(t)| \\ &\leq \varepsilon\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right) \\ &\frac{B(\alpha)}{B(\alpha)-(1-\alpha)\mathfrak{M}(1+\mathfrak{N})} \\ &E_{\alpha}\left(\frac{\alpha\mathfrak{M}(1+\mathfrak{N})(t-a)^{\alpha}}{B(\alpha)-(1-\alpha)\mathfrak{M}(1+\mathfrak{N})}\right) \\ &\leq \varepsilon\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right) \\ &\frac{B(\alpha)}{B(\alpha)-(1-\alpha)\mathfrak{M}(1+\mathfrak{N})} \\ &E_{\alpha}\left(\frac{\alpha\mathfrak{M}(1+\mathfrak{N})(b-a)^{\alpha}}{B(\alpha)-(1-\alpha)\mathfrak{M}(1+\mathfrak{N})}\right).\end{aligned}$$

Therfore $|v(t)-u(t)| \leq h_f.\varepsilon$, where

$$\begin{aligned}h_f = &\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)}\frac{(b-a)^{\alpha}}{\Gamma(\alpha)}\right) \\ &\frac{B(\alpha)}{B(\alpha)-(1-\alpha)\mathfrak{M}(1+\mathfrak{N})} \\ &E_{\alpha}\left(\frac{\alpha\mathfrak{M}(1+\mathfrak{N})(b-a)^{\alpha}}{B(\alpha)-(1-\alpha)\mathfrak{M}(1+\mathfrak{N})}\right).\end{aligned}$$

Hence (1.1) is Ulam-Hyer stable. □

5. Example

Consider the following problem

$$\begin{aligned}&({}_0^{ABC}D^{\frac{3}{2}}u)(t) = \frac{t}{3\sqrt{(\pi)}}sin(u(t)+u'(t)),\ t\in[1,2], \\ &B(\alpha)=1\end{aligned} \tag{5.1}$$

$$u(0)=1 \tag{5.2}$$

Notice that

$$f(0,u(0),\mathfrak{D}u(0))=0$$

and

$$u'(t)\in C[1,2]$$

satisfy the Lipschitz conditions.
Let

$$f(t,u,v)=\frac{t}{3\sqrt{(\pi)}}sin(u+v),\ t\in[1,2].$$

It is easy to see that

$$\begin{aligned}&|f(t,u_1,v_1)-f(t,u_2,v_2))| \\ &\leq \frac{t}{3\sqrt{\pi}}(|u_1-u_2|+|v_1-v_2|), \\ &\text{for all } \ t\in[1,2],u_1,u_2,v_1,v_2\in\mathfrak{R} \\ &\leq \frac{1}{3\sqrt{\pi}}|u-v|\end{aligned}$$

Thus $\rho=\frac{1}{3\sqrt{\pi}}$, Now

$$\begin{aligned}&\rho\left(\frac{1-\alpha}{B(\alpha)}+\frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right) \\ &= \frac{1}{3\sqrt{\pi}}\left(1-\alpha+\frac{1}{\Gamma(\alpha)}\right)<1\end{aligned}$$

By theorem 3.2, (5.1) and (5.2) has a unique solution. It can be written as

$$u(t)=\lim_{n\to\infty}u_n(t),$$

where

$$\begin{aligned}&u_n(t)=1+\frac{1}{3\sqrt{\pi}}{}_0^{AB}I^{\alpha}(tu_{n-1}(t)),\ n=0,1,2,... \\ &\quad or \\ &u_n(t)=1+(1-\alpha)tu_{n-1}(t)+\alpha\ {}_0I^{\alpha}(tu_{n-1}(t)) \\ &\quad =1+(1-\alpha)tu_{n-1}(t) \\ &\quad +\frac{\alpha}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s\,u_{n-1}(s)ds,\ n=1,2,3,...\end{aligned}$$

Solving (5.1) and (5.2), we apply the method proposed by Mekkaoui and Atangana in [29], utilizing from the two-step Lagrange polynomial interpolation.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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