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On nearly recurrent Riemannian manifolds

B. PRASAD*[1] AND R.P.S. YADAV[1]

[1] *Department of Mathematics, S.M.M.T.P.G. College, Ballia-277001, Uttar Pradesh, India.*

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Abstract. The object of the present paper is to introduce a type of recurrent Riemannian manifold called nearly recurrent Riemannian manifold . The existence of nearly recurrent Riemannian manifold have been proved by non trivial example.

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Contents

1. Introduction

Recurrent spaces have been of great importance and were studied by a large number of authors such as Ruse [1], Patterson [2] , Walker [3], Singh and Khan([4] and [5]) etc. In 1991, De and Guha [6] introduced and studied generalized recurrent manifold whose curvature tensor $R(X,Y)Z$ of type (1,3) satisfies the condition:

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where A and B are two non-zero 1-forms and D denotes the operator of covariant differentiation with respect to metric tensor g. Such a space has been denoted by GK_n. In recent papers Bandyopadhyay [7], Prakasha and Yildiz [8], Khan [9] etc explored various geometrical properties by using generalized recurrent manifold on Sasakian manifold and Lorentzian α-Sasakian manifold.

Further one of the author Prasad [10] considered a non-flat Riemannian manifold $(M^n, g)(n > 3)$ whose curvature tensor R satisfies the following condition

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)g(Y,Z)X, \tag{1.2}$$

***Corresponding author.** Email address: **bhagwatprasad2010@rediffmail.com** (B. Prasad)

where A and B are two non-zero 1-forms and D has the meaning already mentioned. Such a manifold called by the author as semi-generalized recurrent manifold and denoted by $(SGK)_n$. Singh, Singh and Kumar[11],[12] and Chaudhary, Kumar and Singh [13] extended this notation to Lorentzian α-Sasakian manifold, P-Sasakian manifold and trans-Sasakian manifold.

The object of the present paper is to study a type of non-flat recurrent Riemannian manifold $(M^n, g)(n > 2)$ whose curvature tensor $R(X,Y)Z$ of the type (1,3) satisfies the condition

$$(D_U R)(X,Y)Z = [A(U) + B(U)]R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.3}$$

where A and B are two non-zero 1-forms and ρ_1 and ρ_2 are two vector fields such that

$$g(U, \rho_1) = A(U) \; and \; g(U, \rho_2) = B(U). \tag{1.4}$$

Such a manifold shall be called as a nearly recurrent Riemannian manifold and 1-forms A and B shall be called its associated 1-forms and n-dimensional recurrent manifold of this kind shall be denoted by $(NR)_n$.
If in particular $B = 0$, then the space reduced to a recurrent space according to Ruse [14] and Walker [3] which is denoted by K_n.
Moreover, in particular if $A = B = 0$ then (1.3) becomes $(D_U R)(X,Y)Z = 0$. That is , a Riemannian manifold is symmetric accordingly Kobayashi and Nomizu [15] and Desai and Amer [16]. The name nearly recurrent Riemannian manifold was chosen because if $B = 0$ in (1.3) then the manifold reduces to a recurrent manifold which is very close to recurrent space. This justifies the name $Nearly\ recurrent\ Riemannian\ manifold$ for the manifold defined by (1.3) and the use of the symbol $(NR)_n$ for it.

In this paper, after preliminaries, a necessary and sufficient condition for constant scaler curvature of $(NR)_n$ is obtained. Nearly recurrent manifold with cyclic Ricci tensor and Codazzi type Ricci tensor are studied. Finally, we give examples of $(NR)_n$.

2. Preliminaries

Let S and r denote the Ricci tensor of type (0,2) and scalar curvature respectively and Q denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, i.e.

$$S(X,Y) = g(QX,Y), \tag{2.1}$$

for any vector field X and Y.
From(1.3), we get

$$(D_U S)(Y,Z) = [A(U) + B(U)]S(Y,Z) + (n-1)B(U)g(Y,Z). \tag{2.2}$$

Contracting (2.2), we have

$$dr(U) = Ur = [A(U) + B(U)]r + n(n-1)B(U). \tag{2.3}$$

3. Nature of the 1-forms A and B on a nearly recurrent space

From (2.3) suppose $r = 0$,then

$$B(U) = 0$$

which is not possible. Hence we have the following theorem:

Theorem 3.1. *The scalar curvature tensor of $(NR)_n$ can not be zero.*

Now we consider $(NR)_n$ is of constant scalar curvature then from (2.3), we have

$$[A(U)+B(U)]r+n(n-1)B(U)=0. \tag{3.1}$$

Again if (3.1) holds, then from (2.3), we get

$$dr(U)=0,$$
$$r=constant$$

Hence, we can state the following theorem:

Theorem 3.2. *A $(NR)_n$ is of constant curvature if and only if (3.1) holds.*

Now, taking covariant derivative of (3.1) with respect to V, we get

$$[(D_V A)(U)+(D_V B)(U)]r+n(n-1)(D_V B)U=0. \tag{3.2}$$

Interchanging U and V in (3.2) and then subtracting, we get

$$[(dA(U,V)+dB(U,V)]r+n(n-1)dB(U,V)=0. \tag{3.3}$$

Thus we have the following theorem:

Theorem 3.3. *In a nearly recurrent space of non-zero constant scalar curvature r, if the 1-forms B is closed then A is closed , if A is closed then B is also closed.*

From (1.3), we have

$$(D_V R)(X,Y)Z=[A(V)+B(V)]R(X,Y)Z+B(V)[g(Y,Z)X-g(X,Z)Y].$$

This gives

$$\begin{aligned}(D_U D_V R)(X,Y)Z=&[(D_U A)(V)+A(D_U V)+(D_U B)(V)+B(D_U V)]R(X,Y)Z\\&+[A(U)+B(U)][A(V)+B(V)]R(X,Y)Z+\\&[A(V)+B(V)]B(U)[g(Y,Z)X-g(X,Z)Y]\end{aligned} \tag{3.4}$$

Therefore from(3.4), we have

$$\begin{aligned}(D_V D_U R)(X,Y)Z=&[(D_V A)(U)+A(D_V U)+(D_V B)(U)+B(D_V U)]R(X,Y)Z\\&+[A(U)+B(U)][A(V)+B(V)]R(X,Y)Z+\\&[A(U)+B(U)]B(V)[g(Y,Z)X-g(X,Z)Y]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(D_{[U,V]}R)(X,Y)Z=&\left[A\left([U,V]\right)+B\left([U,V]\right)\right]R(X,Y)Z+\\&B\left([U,V]\right)[g(Y,Z)X-g(X,Z)Y].\end{aligned} \tag{3.6}$$

Now, subtracting (3.5) and (3.6) from (3.4), we get

$$\begin{aligned}(R(U,V).R)(X,Y)Z=&[(dA(U,V)+dB(U,V)]R(X,Y)Z+\\&dB(U,V)[g(Y,Z)X-g(X,Z)Y]+\\&[A(V)B(U)-A(U)B(V)].\end{aligned} \tag{3.7}$$

Thus, we can state the following theorem:

Theorem 3.4. *In a $(NR)_n$ with constant scalar curvature , R(X,Y).R=0 if and only if*

$$[(dA(U,V)+dB(U,V)]R(X,Y)Z+dB(U,V)[g(Y,Z)X-g(X,Z)Y]+$$
$$[A(V)B(U)-A(U)B(V)]=0.$$

Next, we consider the case when the scalar curvature r is not constant.
From (2.3) it follows that

$$VUr=(D_VA)(U)r+A(U)(Vr)+n(n-1)(D_VB)(U). \tag{3.8}$$

Interchanging U and V in (3.8) and then subtracting, we get

$$[(D_VA)(U)-(D_UA)(V)+(D_VB)(U)-(D_UB)(V)]r+$$
$$n(n-1)\{(D_VB)(U)-(D_UB)(V)\}+[r+n(n-1)][A(U)B(V)-A(V)B(U)]=0.$$

which gives

$$\begin{aligned}&[dA(V,U)+dB(V,U)]r+n(n-1)dB(V,U)+\\&[r+n(n-1)][A(U)B(V)-A(V)B(U)]=0.\end{aligned} \tag{3.9}$$

Thus we have the following theorem:

Theorem 3.5. *In a nearly recurrent space of non-zero constant scalar curvature r, the 1-forms A and B are closed if and only if the 1-forms A and B are co-directional.*

4. $(NR)_n$ with cyclic Ricci tensor

In this section we consider a $(NR)_n$ in which the Ricci tensor is a cyclic tensor, i.e.

$$(D_XS)(Y,Z)+(D_YS)(Z,X)+(D_ZS)(X,Y)=0, \tag{4.1}$$

which implies

$$dr(X)=0. \tag{4.2}$$

From (1.3), we have

$$dr(X)=[A(X)+B(X)]r+n(n-1)B(X). \tag{4.3}$$

Therefore from (4.2) and (4.3), we get

$$[A(X)+B(X)]r+n(n-1)B(X)=0. \tag{4.4}$$

From (4.1), we have

$$[A(X)+B(X)]S(Y,Z)+[A(Y)+B(Y)]S(Z,X)+[A(Z)+B(Z)]S(X,Y)$$
$$+(n-1)[B(X)g(Y,Z)+B(Y)g(X,Z)+B(Z)g(X,Y)]=0,$$

which yields on contraction

$$\begin{aligned}A(QX)+B(QX)&=\frac{r}{n}[A(X)+B(X)]\\ or\ \ S(X,\rho_1)+S(X,\rho_2)&=\frac{r}{n}[g(X,\rho_1)+g(X,\rho_2)]\\ or\ \ S(X,\rho_1+\rho_2)&=\frac{r}{n}[g(X,\rho_1+\rho_2)\end{aligned}$$

Above can be written as

$$S(X,\mu) = \frac{r}{n} g(X,\mu), \tag{4.5}$$

where $\mu = \rho_1 + \rho_2$.
Hence we have the following theorem:

Theorem 4.1. *If $(NR)_n$ has cyclic Ricci tensor, then $\frac{r}{n}$ is an eigen value of Ricci tensor S and μ is an eigen vector corresponding to the eigen value.*

5. $(ER)_n$ with Codazzi type of Ricci tensor

In this section, we consider an $(NR)_n$ in which the Ricci tensor is a Codazzi type of Ricci tensor Ferus [17]

$$(D_X S)(Y,Z) = (D_Z S)(Y,X). \tag{5.1}$$

By view of Bianchi identity and (5.1), we have

$$(divR)(X,Y)Z = 0. \tag{5.2}$$

In view of (1.3), we get on contraction

$$(divR)(X,Y)Z = A(R(X,Y)Z) + B(R(X,Y)Z) + B(X)g(Y,Z) - B(Y)g(X,Z). \tag{5.3}$$

Now using (5.2) in (5.3), we get

$$A(R(X,Y)Z) + B(R(X,Y)Z) + B(X)g(Y,Z) - B(Y)g(X,Z) = 0. \tag{5.4}$$

In view of (5.4), we get

$$A(QX) + B(QX) = -(n-1)B(X). \tag{5.5}$$

From (2.2) and (5.1), we have

$$\begin{aligned}&[A(X)+B(X)]S(Y,Z) - [A(Z)+B(Z)]S(Y,X)\\&+(n-1)[B(X)g(Y,Z) - B(Z)g(X,Y)] = 0.\end{aligned} \tag{5.6}$$

On contracting of (5.6), we have

$$[A(X)+B(X)]r = [A(QX)+B(QX)] - (n-1)^2 B(X). \tag{5.7}$$

Using (5.5) and (5.7) in (2.3), we have

$$dr(X) = 0. \tag{5.8}$$

Again it is known [18] that in a Riemannian manifold $(M^n, g)(n > 3)$

$$\begin{aligned}(divC)(X,Y)Z =&\frac{n-3}{n-2}[(D_X S)(Y,Z) - (D_Z S)(Y,X)]+\\&\frac{1}{2(n-1)}[g(X,Y)dr(Z) - g(Y,Z)dr(X)],\end{aligned} \tag{5.9}$$

where C denotes the conformal curvature.
As a consequences of (5.1) and (5.8), (5.9) reduces to

$$(divC)(X,Y)Z = 0,$$

which shows that the tensor is conservative [19].
Hence we can state the following theorem:

Theorem 5.1. *If in a $(NR)_n$ the Ricci tensor is a Codazzi type tensor then its conformal curvature tensor is conservative.*

6. Nearly recurrent with concurrent vector field

In this section first we suppose that the $(NR)_n$ admits a concurrent unit vector fields $\widetilde{V}$,

$$D_X\widetilde{V} = \rho X, \tag{6.1}$$

where ρ is a non-zero constant .
By Ricci-identity

$$R(X,Y)\widetilde{V} = 0. \tag{6.2}$$

Taking covariant derivative of (6.2), we get

$$(D_W R)(X,Y)\widetilde{V} = -\rho R(X,Y)W \tag{6.3}$$

Also by definition of $(NR)_n$,we find

$$(D_W R)(X,Y)\widetilde{V} = [A(W)+B(W)]R(X,Y)\widetilde{V} + B(W)[g(Y,\widetilde{V})X - g(X,\widetilde{V})Y]. \tag{6.4}$$

In view of (6.2),(6.3)and (6.4),we get

$$-\rho R(X,Y)W = B(W)[g(Y,\widetilde{V})X - g(X,\widetilde{V})Y].$$

On contraction, we find

$$-\rho S(Y,W) = (n-1)B(W)g(Y,\widetilde{V}). \tag{6.5}$$

Again on contraction of (6.5), we get

$$-\rho r = (n-1)B(\widetilde{V}) = (n-1)g(\rho_2,\widetilde{V}), \tag{6.6}$$

Since $\rho \neq 0$ and $r \neq 0$,then from (6.6),we get

$$g(\rho_2,\widetilde{V}) \neq 0. \tag{6.7}$$

Hence we have the following theorem:

Theorem 6.1. *If a $(NR)_n$ the associated vector field ρ_2 cannot concurrent vector field .*

7. Example

Example (7.1) Let us consider $M^4 = \left\{(x^1,x^2,x^3,x^4) \in R^4\right\}$ be an open subset of R^4 endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{3}{2}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \tag{7.1}$$

where $i,j = 1,2,3,4$.
Then the only non-vanishes components of the Christoffel symbols and curvature tensor are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{3}{4(x^4)}, \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{3}{4}(x^4)^{\frac{1}{2}}$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{3}{16(x^4)^{\frac{1}{2}}} \tag{7.2}$$

The non-vanishing components of the Ricci tensor are

$$R_{11} = R_{22} = R_{33} = -\frac{3}{16(x^4)^{\frac{1}{2}}}, \quad R_{44} = -\frac{3}{16(x^4)^2}$$

and scalar curvature is

$$R = g^{ii}R_{ii} = -\frac{3}{16(x^4)^2}$$

Taking covariant derivative of (7.2), we get

$$R_{1441,4} = \frac{3}{32(x^4)^{\frac{3}{2}}}, \ \ R_{2442,4} = \frac{3}{32(x^4)^{\frac{3}{2}}}, \ \ R_{3443,4} = \frac{3}{32(x^4)^{\frac{3}{2}}} \tag{7.3}$$

Consequently, the manifold under consideration is not recurrent .
Let us choose the associated 1-form as

$$A_i = \begin{cases} \frac{3}{32} \cdot \frac{64(x^4)^2-1}{(x^4)^3-4(x^4)^2}, & i=4 \\ 0, & \text{otherwise} \end{cases} \tag{7.4}$$

$$B_i = \begin{cases} \frac{3}{32} \cdot \frac{1}{(x^4)^3-4(x^4)^2}, & i=4 \\ 0, & \text{otherwise} \end{cases} \tag{7.5}$$

From (1.3), we have

$$R_{hiih,i} = (A_i + B_i)R_{hiih} + B_i[g_{ii}g_{hh} - g_{hi}g_{ih}] \tag{7.6}$$

By virtue of (7.2), (7.3),(7.4) and (7.5), it can be easily seen that the Riemannian manifold satisfies relation (7.6). Hence the manifold under consideration is a nearly recurrent Riemannian manifold (M^4, g), which is neither recurrent nor symmetric.
This leads to the following theorem:

Theorem 7.1. *There exist a nearly recurrent Riemannian manifold (M^4, g), which is neither recurrent nor symmetric.*

Example (7.2) Let us consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are standard co-ordinate of R^3.
We choose the vector fields

$$e_1 = \frac{1}{2}\frac{\partial}{\partial y}, \ e_2 = \frac{\partial}{\partial x} - z\frac{\partial}{\partial y}, \ e_3 = \frac{\partial}{\partial z} \tag{7.7}$$

which is linearly independently at each point of M.
Let g be the Riemannian metric denoted by

$$g(e_i, e_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \tag{7.8}$$

Let D be the Levi-Civita connection with respect to metric g. Then from equation (7.7), we have

$$[e_1, e_2] = 0, [e_1, e_3] = 2e_1, [e_2, e_3] = 0. \tag{7.9}$$

The Riemannian connection D of the metric g is given by

$$\begin{aligned} 2g\,(D_X Y, Z) =& Xg\,(Y,Z) + Yg\,(X,Z) - Zg\,(X,Y) - g\,(X,[Y,Z]) \\ & - g\,(Y,[X,Z]) + g\,(Z,[X,Y])\,, \end{aligned} \tag{7.10}$$

which is known as Koszul's formula. Using (7.8) and (7.9) in (7.10), we get

$$\begin{aligned} &D_{e_1}e_3 = -e_2, \quad D_{e_1}e_2 = e_3, \quad D_{e_1}e_1 = 0, \\ &D_{e_2}e_3 = e_1, \quad\ \ D_{e_2}e_2 = 0, \quad\ \ D_{e_2}e_1 = -e_3, \\ &D_{e_3}e_3 = 0, \quad D_{e_3}e_2 = -e_1, \quad D_{e_3}e_1 = e_2. \end{aligned} \tag{7.11}$$

The curvature tensor is given by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{7.12}$$

Using (7.9) and (7.11) in (7.12) , we get

$$\begin{aligned}
&R(e_1,e_2)e_1 = e_2, \quad R(e_1,e_2)e_2 = e_1, \quad R(e_1,e_2)e_3 = 0 \\
&R(e_2,e_3)e_1 = 0, \quad R(e_2,e_3)e_2 = 3e_3, \quad R(e_2,e_3)e_3 = e_2 \\
&R(e_1,e_3)e_1 = -e_3, \quad R(e_1,e_3)e_2 = 0 \qquad R(e_1,e_3)e_3 = -e_1 \\
&R(e_1,e_1)e_1 = R(e_1,e_1)e_2 = R(e_1,e_1)e_3 = 0 \\
&R(e_2,e_2)e_1 = R(e_2,e_2)e_2 = R(e_2,e_2)e_3 = 0 \\
&R(e_3,e_3)e_1 = R(e_3,e_3)e_2 = R(e_3,e_3)e_3 = 0.
\end{aligned} \tag{7.13}$$

The Ricci tensor is given by

$$S(e_i,e_i) = \sum_{i=1}^{3} g(R(e_i,X)Y,e_i) \tag{7.14}$$

From (7.13) and (7.14), we get

$$S(e_1,e_1) = 0, \quad S(e_2,e_2) = 2, \quad S(e_3,e_3) = 0 \tag{7.15}$$

and the scalar curvature is 2.
Since $\{e_1,e_2,e_3\}$ forms a basis of Riemannian manifold any vector field $X,Y,Z \in \chi(M)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3, \;\; Y = a_2e_1 + b_2e_2 + c_2e_3, \;\; Z = a_3e_1 + b_3e_2 + c_3e_3,$$

where $a_i, b_i, c_i \in R^+$ (the set of all positive real numbers), $i = 1,2,3$.
Hence

$$R(X,Y)Z = l_1e_1 + m_1e_2 + n_1e_3 \tag{7.16}$$

$$g(Y,Z)X - g(X,Z)Y = l_2e_1 + m_2e_2 + n_2e_3 \tag{7.17}$$

By view of (7.16), we get

$$(D_{e_i}R)(X,Y)Z = u_ie_1 + v_ie_2 + w_ie_3 \;\; for\ i = 1,2,3. \tag{7.18}$$

where

$$\begin{aligned}
l_1 &= a_1b_2b_3 + a_2c_1c_3 - c_1c_2c_3, \\
m_1 &= a_1b_2a_3 + a_3b_1b_2 - b_1a_2a_3 + b_1c_2c_3, \\
n_1 &= 3b_1b_3c_2 - 3b_3c_1b_2 - a_1a_3c_2, \\
l_2 &= a_1b_2b_3 + a_1c_2c_3 - a_2b_1b_3 - a_2c_1c_3, \\
m_2 &= a_2a_3b_1 + b_1c_2c_3 - a_1a_3b_2 - b_2c_1c_3, \\
n_2 &= a_2a_3c_1 + b_2b_3c_1 - a_1a_3c_2 - b_1b_3c_2, \\
u_1 &= a_2a_3c_1 - a_2b_3c_1 - a_1b_3c_2, \\
v_1 &= 2b_2b_3c_1 - 2b_1b_3c_2 + a_1a_3c_2 - a_2a_3c_1, \\
w_1 &= 2a_2a_3b_1 - 2a_1a_3b_2 - a_3b_1b_2 + 2b_1c_2c_3, \\
u_2 &= 4b_1b_3c_2 - 3b_2b_3c_1 - 2a_1a_3c_2 - c_1c_2c_3 + a_2a_3c_1, \\
v_2 &= -2a_1b_2c_3 + 2a_3b_1c_2 + a_2b_1c_3 + a_2b_2c_3, \\
w_2 &= -4a_1b_2b_3 - 2a_2c_1c_3 + c_1c_2c_3 + 3a_2b_2b_3 - a_3c_1c_2 - 3a_3b_2c_1 + a_1a_2c_3, \\
u_3 &= -2a_1a_3b_2 - a_3b_1b_2 + 2a_2a_3b_1 - 2b_1c_2c_3 - 2a_1a_2b_3 + b_2c_1c_3, \\
v_3 &= 2a_1b_2b_3 + a_2c_1c_3 - c_1c_2c_3 - a_1a_2c_3 + a_3c_1c_3 + a_2b_1b_3, \\
w_3 &= -3a_1b_3c_2 - a_3b_1c_2 + 3a_2b_3c_1 + 2a_3b_2c_1.
\end{aligned}$$

Consequently, the manifold under consideration is not recurrent. Let us now consider 1-form non vanishes

$$A(e_i) = \frac{4(u_i + v_i + w_i)}{3(l_1 + m_1 + n_1) - (l_2 + m_2 + n_2)}$$
$$B(e_i) = \frac{-(u_i + v_i + w_i)}{3(l_1 + m_1 + n_1) - (l_2 + m_2 + n_2)} \tag{7.19}$$

such that

$$3(l_1 + m_1 + n_1) - (l_2 + m_2 + n_2) \neq 0.$$

From (1.3), we have

$$(D_{e_i} R)(X, Y)Z = [A(e_i) + B(e_i)]R(X, Y)Z + B(e_i)[g(Y, Z)X - g(X, Z)Y]. \tag{7.20}$$

By virtue of (7.16), (7.17), (7.18) and (7.19), it can be easily seen that the Riemannian manifold satisfies relation (7.20). Hence the manifold under consideration is a nearly recurrent Riemannian manifold (M^3, g), which is neither recurrent nor symmetric. Thus we have the following theorem:

Theorem 7.2. *There exist a nearly recurrent Riemannian manifold (M^3, g), which is neither recurrent nor symmetric.*

References

[1] H.S. Ruse, On simply harmonic space, *J. London Math..Soc.*, **2**1(1946), https://doi.org/10.1112/jlms/s1-21.4.243.

[2] E.M. Paatterso, Some theorem on Ricci-recurrent space, *J. London. Math. Soc.*, **27**(1952), 287-295, https://doi.org/10.1112/jlms/s1-27.3.287.

[3] A.G.Walker, On ruse spaces of recurrent curvature, *Proc. London Math. Soci.*, **52**, (1951).

[4] H. Singh and Q. Khan, On symmetric Riemannian manifolds, *Novi sad J. Math.*, **29**(3)(1999), 301-308.

[5] H. Singh and Q. Khan, On generalized recurrent Riemannian manifolds, *Publ. Math. Debrecen*, **56**(1-2)(2000), 87-95, https://doi.org/10.5486/PMD.2000.2139.

[6] U.C. De, N. Guha and D. Kamilya, On generalized recurrent manifolds, *National Academy of Math. India*, **9**(1)(1991), 1-4.

[7] Bandyaopadhyay, Mahuya, On generalized ϕ-recurrent -Sasakian manifolds, *Mathematica Pannonica*, **22**(1)(2011), 19-23.

[8] D.G. Prakasha and A. Yildiz, Generalized ϕ-recurrent Lorentzian α-Sasakian manifold, *Common Fac.Sci.Uni. Ank. Series A1*, **59** (1)(2010), 53–62, https://doi.org/10.1501/Commua1_0000000656.

[9] Q. Khan, On generalized recurrent Sasakian manifold with special curvature tensor J(X,Y,Z), *Ganita*, **67**(2)(2017), 189–194.

[10] B. Prasad, On semi-generalized recurrent manifold, *Mathematica Balkanica New Series*, **14**(2000), 1–4.

[11] J.P. Singh, A. Singh and Rajesh Kumar, On a type of Semi generalized recurrent P-Sasakian manifold, *Facta Universitatis (NIS) Ser. Math. Inform*, **31**(1)(2016), 213–225.

[12] J.P. SINGH, A. SINGH AND RAJESH KUMAR, On a type of Semi generalized recurrent Lorentzian α-Sasakian manifold, *Bull. Cal. Math. Soc.,* **107** (5)(2015), 357–370.

[13] J.CHAUDHARY, RAJESH KUMAR, AND J.P. SINGH, Semi generalized ϕ-recurrent trans-Sasakian manifold, *Facta Universitatis (NIS) Ser. Math. Inform*, **31**(4)(2016), 863–871.

[14] H.S. RUSE, Three dimensional spaces of recurrent curvature, *Pro. London Math. Soc.*, **50** (2)(1947), 438–446, https://doi.org/10.1112/plms/s2-50.6.438.

[15] S.KOBAYASHI AND K. NOMIZU, *Foundation of Differential Geometry*, Vol.1, Interscience publishers, New York, (1963).

[16] P. DESAI AND K. AMUR, On symmetric spaces, *Tensor N.S.*, **29**(1975), 119–124.

[17] D.A. FERUS, A remark on Codazzi tensor on constant curvature space, *lecture notes math. 838, Global differential geometry and global analysis, Springer Verlag, New York*, (1981), https://doi.org/10.1007/BFb0088868.

[18] L.P. EISENHERT, Riemannian Geometry, *Princeton University Press*, Princetone, N.J., (1949).

[19] N.J. HICKS, *Notes on Differential Geometry*, D.Van Nostrand company Inc. Princeton, New York, (1965).