Malaya Journal of Matematik, Vol. 8, No. 4, 1859-1866, 2020

https://doi.org/10.26637/MJM0804/0087

A study on bipolar valued multi I-fuzzy subhemirings of a hemiring

K. Meenatchi[1*] and M. Kaliraja[2]

Abstract

In this paper, bipolar valued multi I-fuzzy subhemiring of a hemiring is introduced and some properties are discussed. Bipolar valued multi I-fuzzy subhemiring of a hemiring is a generalized form of bipolar valued multi fuzzy subhemiring of a hemiring. The paper will be useful to further research.

Keywords

Interval valued fuzzy subset, bipolar valued fuzzy subset, bipolar valued multi fuzzy subset, bipolar valued I-fuzzy subset, bipolar valued multi I-fuzzy subset, bipolar valued multi I-fuzzy subhemiring, union, intersection, product, strongest.

AMS Subject Classification

03E72.

[1,2] *PG and Research Department of Mathematics, H.H.The Rajah's College, Pudukkottai-622001, Tamil Nadu, India.*

[1,2] *Affiliated to Bharathidasan University, Tiruchirappalli-620024, Tamil Nadu, India.*

*__Corresponding author__: [1]meenatchianandh@gmail.com; [2]mkr.maths009@gmail.com

Article History: Received **12** August **2020**; Accepted **09** October **2020**

Contents

1. Introduction

In 1965, Zadeh [13] introduced the notion of a fuzzy subset of a crisp set, fuzzy subsets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Multi fuzzy set was introduced by Sabu Sebastian, T.V. Ramakrishnan [9]. Lee [5] introduced the notion of bipolar valued fuzzy sets. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree $(0,1]$ indicates that elements somewhat satisfy the property and the membership degree $[-1,0)$ indicates that elements somewhat satisfy the implicit counter property. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [5, 6]. Fuzzy subgroup was introduced by Azriel Rosenfeld [2]. K. Murugalingam and K. Arjunan [7] have discussed about interval valued fuzzy subsemiring of a semiring. A study on interval valued fuzzy, anti fuzzy, intuitionistic fuzzy subrings of a ring by Somasundra Moorthy [11], the thesis was useful to write the paper. Anitha et.al. [1] defined as bipolar valued fuzzy subgroups of a group and and Balasubramanian et.al. [3] introduced about bipolar interval valued fuzzy subgroups of a group. The papers [4] and [10] was useful to work this field. After that bipolar valued multi fuzzy subsemirings of a semiring have been introduced by Yasodara and KE. Sathappan [12]. Muthukumaran & Anandh [8] defined the bipolar valued multi fuzzy subnearring of a nearing. Here, the concept of bipolar valued multi I-fuzzy subhemiring of a hemiring is introduced and estiablished some results.

2. Preliminaries

Definition 2.1. *Let X be any nonempty set. A mapping $[M]: X \to D[0,1]$ is called an interval valued fuzzy subset (I-fuzzy subset) of X, where $D[0,1]$ denotes the family of all closed subintervals of $[0,1]$ and $[M](x) = [M^-(x), M^+(x)]$, for all x in X, where M^- and M^+ are fuzzy subsets of X such that $M^-(x) \leq M^+(x)$, for all x in X. Thus $M^-(x)$ is an interval (a closed subset of $[0,1]$) and not a number from the interval $[0,1]$ as in the case of fuzzy subset. Note that $[0] = [0,0]$ and $[1] = [1,1]$.*

Definition 2.2 ([5])**.** *A bipolar valued fuzzy set (BVFS) A in*

X is defined as an object of the form

$$A = \left\{ < x, A^+(x), A^-(x) > / x \in X \right\},$$

where $A^+ : X \to [0,1]$ *and* $A^- : X \to [-1,0]$. *The positive membership degree* $A^+(x)$ *denotes the satisfaction degree of an element x to the*

property corresponding to a bipolar valued fuzzy set A and the negative membership degree $A^-(x)$ *denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued fuzzy set A.*

Example 2.3. $A = \{< a, 0.7, -0.3 >, < b, 0.6, -0.5 >, < c, 0.2, -0.8 >\}$ *is a bipolar valued fuzzy subset of* $X = \{a,b,c\}$.

Definition 2.4 ([12])**.** *A bipolar valued multi fuzzy set (BVMFS) A in X is defined as an object of the form*

$$A = \left\{ < x, A_1^+(x), A_2^+(x), \ldots, A_n^+(x), A_1^-(x), A_2^-(x), \ldots, A_n^-(x) > / x \in X \right\},$$

where $A_i^+ : X \to [0,1]$ *and* $A_i^- : X \to [-1,0]$ *for all* $i = 1,2,\ldots, n$. *The positive membership degrees* $A_i^+(x)$ *denote the satisfaction degrees of an element x to the property corresponding to a bipolar valued multi fuzzy set A and the negative membership degrees* $A_i^-(x)$ *denote the satisfaction degrees of an element x to some implicit counter-property corresponding to a bipolar valued multi fuzzy set A.*

Example 2.5. $A = \{< a, 0.7, 0,3, 0.7, -0.4, -0.5, -0.9 >, < b, 0.6, 0.7, 0.2, -0.8, -0.1, -0.4 > < c, 0.5, 0.9, 0.8, -0.2, -0.6, -0.9 >\}$ *is a bipolar valued multi fuzzy subset of* $X = \{a,b,c\}$.

Definition 2.6. *A bipolar interval valued fuzzy subset (bipolar valued I-fuzzy subset)* $[A]$ *in X is defined as an object of the form* $[A] = \{< x, [A]^+(x), [A]^-(x) > / x \in X\}$, *where* $[A]^+ : X \to D[0,1]$ *and* $[A]^- : X \to D[-1,0]$, *where* $D[0,1]$ *denotes the family of all closed subintervals of* $[0,1]$ *and* $D[-1,0]$ *denotes the family of all closed subintervals of* $[-1,0]$ *The positive interval membership degree* $[A]^+(x)$ *denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued I-fuzzy subset* $[A]$ *and the negative interval membership degree* $[A]^-(x)$ *denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued I-fuzzy subset* $[A]$.

Example 2.7. $[A] = \{< a, [0.3, 0,9], [-0.6, -0.2] >, < b, [0.4, 0.8], [-0.8, -0.3] > < c, [0.3, 0.6], [-0.8, -0.2] >\}$ *is a bipolar valued I-fuzzy subset of* $X = \{a,b,c\}$.

Definition 2.8. *A bipolar interval valued multi fuzzy subset (bipolar valued multi I-fuzzy subset)* $[A]$ *in X is defined as an object of the form*

$$[A] = \left\{ < x, [A]_1^+(x), [A]_2^+(x), \ldots, [A]_n^+(x), [A]_1^-(x), [A]_2^-(x), \ldots, [A]_n^-(x) > / x \in X \right\},$$

where for each i, $[A]_i^+ : X \to D[0,1]$ *and* $[A]_i^- : X \to D[-1,0]$, *where* $D[0,1]$ *denotes the family of all closed subintervals of* $[0,1]$ *and* $D[-1,0]$ *denotes the family of all closed subintervals of* $[-1,0]$. *The positive interval membership degrees* $[A]i^+(x)$ *denotes the satisfaction degrees of an element x to the property corresponding to a bipolar valued multi I-fuzzy subset* $[A]$ *and the negative interval membership degrees* $[A]i^-(x)$ *denotes the satisfaction degrees of an element x to some implicit counter-property corresponding to a bipolar valued multi I-fuzzy subset* $[A]$. *Note that*

$$[0] = ([0,0],[0,0],\ldots,[0,0]), [1] = ([1,1],[1,1],\ldots,[1,1])$$

and

$$[-1] = ([-1,-1][-1,-1],\ldots,[-1,-1]).$$

It is denoted as

$$[A] = \left\langle [A]_1^+, [A]_2^+, \ldots, [A]_n^+, [A]_1^-, [A]_2^-, \ldots, [A]_n^- \right\rangle.$$

Example 2.9. $[A] = \{< a, [0.3, 0, 6], [0.2, 0, 5], [0.6, 0, 8], [-0.7, -0.2], [-0.9, -0.1]\ [-0.5, -0.2] >, < b, [0.2, 0.4], [0.2, 0.6], [0.3, 0.6], [-0.6, -0.3], [-0.7, -0.22[-0.5, -0.2] > < c, [0.2, 0.6], [0.5, 0.7], [0.6, 0.8], [-0.4, -0.2], [-0.6, -0.3], [-0.4, -0.2] >\}$ *is a bipolar valued multi I-fuzzy subset of* $X = \{a,b,c\}$

Definition 2.10. *Let* $[A] = \left\langle [A]_1^+, [A]_2^+, \ldots, [A]_n^+, [A]_1^-, [A]_2^-, \ldots, [A]_n^- \right\rangle$ *and* $[B] = \left\langle [B]_1^+, [B]_2^+, \ldots, [B]_n^+, [B]_1^-, [B]_2^-, \ldots, [B]_n^- \right\rangle$ *be two bipolar valued multi I-fuzzy subsets of a set X. We define the following relations and operations:*

(i) $[A] \subset [B]$ *if and only if for each i,* $[A]_i^+(u) \le [B]_i^+(u)$ *and* $[A]_i^-(u) \ge [B]_i^-(u), \forall u \in X$

(ii) $[A] = [B]$ *if and only if for each i,* $[A]_i^+(u) = [B]_i^+(u)$ *and* $[A]_i^-(u) = [B]_i^-(u), \forall u \in X$

(iii) $[A] \cap [B] = \left\{ \left\langle u, \text{rmin}\left([A]_1^+(u), [B]_1^+(u)\right), \text{rmin}\left([A]_2^+(u), [B]_2^+(u)\right), \ldots, rmin\left([A]_n^+(u), [B]_n^+(u)\right), \text{rmax}\left([A]_1^-(u), [B]_1^-(u)\right), \text{rmax}\left([A]_2^-(u), [B]_2^-(u)\right), \ldots, \text{rmax}\left([A]_n^-(u), [B]_n^-(u)\right) \right\rangle / u \in X \right\}$

(iv) $[A] \cup [B] = \left\{ \left\langle u, \text{rmax}\left([A]_1^+(u), [B]_1^+(u)\right), \text{rmax}\left([A]_2^+(u), [B]_2^+(u)\right), \ldots, \text{rmax}\left([A]_n^+(u)\ \ [B]_n^+(u)\right), \text{rmin}\left([A]_1^-(u), [B]_1^-(u)\right), \text{rmin}\left([A]_2^-(u), [B]_2^-(u)\right), \ldots, \text{rmin}\left([A]_n^-(u), [B]_n^-(u)\right) \right\rangle u \in X \right\}$.

Definition 2.11. *Let R be a hemiring. A bipolar valued multi I-fuzzy subset* $[A] = \left\langle [A]_1^+, [A]_2^+, \ldots, [A]_n^+, [A]_1^-, [A]_2^-, \ldots, [A]_n^- \right\rangle$ *of R is said to be a bipolar valued multi I-fuzzy subhemiring of R (BVMIFSHR) if the following conditions are satisfied for each i,*

(i) $[A]_i^+(x+y) \ge \text{rmin}\left\{[A]_i^+(x), [A]_i^+(y)\right\}$

(ii) $[A]_i^+(xy) \ge \text{rmin}\left\{[A]_i^+(x), [A]_i^+(y)\right\}$

(iii) $[A]_i^-(x+y) \le \text{rmax}\left\{[A]_i^-(x), [A]_i^-(y)\right\}$

(iv) $[A]_i^-(xy) \le \text{rmax}\{[A]_i^-(x),[A]_i^-(y)\}$ *for all x and y in R*.

Example 2.12. *Let $R = Z_3 = \{0,1,2\}$ be a hemiring with respect to the ordinary addition and multiplication. Then*

$$\begin{aligned}[A] =\{&< 0,[0.55,0.6],[0.65,0.7],[0.75,0.8],[-0.6,-0.5]\\ &[-0.7,-0.6],[-0.8,-0.75]>,< 1,[0.41,0.5],[0.51,0.6],\\ &[0.61,0.7],[-0.51,-0.4][-0.61,-0.5],[-0.71,-0.6]>,\\ &< 2,[0.41,0.5],[0.51,0.6],[0.61,0.7],[-0.51,-0.4]\\ &[-0.61,-0.5],[-0.71,-0.6]>\}\end{aligned}$$

is a bipolar valued multi I-fuzzy subhemiring of R.

Definition 2.13. *Let $[A] = \langle[A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ and $[B]=\langle[B]_1^+,[B]_2^+,\dots,[B]_n^+,[B]_1^-,[B]_2^-,\dots,[B]_n^-\rangle$ be any two bipolar valued multi I-fuzzy subsets of sets G and H, respectively. The product of $[A]$ and $[B]$, denoted by $[A]\times[B]$, is defined as*

$$\begin{aligned}[A]\times[B] = \{&\langle(x,y),([A]_1\times[B]_1)^+(x,y),([A]_2\times[B]_2)^+(x,y),\\ &\dots([A]_n\times[B]_n)^+(x,y),([A]_1\times[B]_1)^-(x,y),\\ &([A]_2\times[B]_2)^-(x,y),\dots,([A]_n\times[B]_n)^-(x,y)\rangle\\ &\text{for all } x \text{ in } G \text{ and } y \text{ in } H\},\end{aligned}$$

where for each i, $([A]_i\times[B]_i)^+(x,y) = \text{rmin}\{[A]_i^+(x),[B]_i^+(y)\}$ and $([A]_i\times[B]_i)^-(x,y) = \text{rmax}\{[A]_i^-(x),[B]_i^-(y)\}$ for all x in G and y in H.

Definition 2.14. *Let $[A] = \langle[A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ be a bipolar valued multi I-fuzzy subset in a set S, the strongest bipolar valued multi I-fuzzy relation on S, that is a bipolar valued multi I-fuzzy relation on [A] is*

$$\begin{aligned}[V] = \{&\langle(x,y),[V]_1^+(x,y)\ [V]_2^+(x,y),\dots,[V]_n^+(x,y),\\ &[V]_1^-(x,y),[V]_2^-(x,y),\dots,[V]_n^-(x,y)\rangle/x \text{ and yin } S\}\end{aligned}$$

given by $[V]_i^+(x,y) = \text{rmin}\{[A]_i^+(x),[A]_i^+(y)\}$ and $[V]_i^-(x,y) = \text{rmax}\{[A]_i^-(x),[A]_i^-(y)\}$, for all i. for all x and y in S.

Theorem 2.15. *If $[A] = \langle[A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ be a bipolar valued multi I-fuzzy subhemiring of a hemiring R, then for each i, $[A]_i^+(x) \le [A]_i^+(0)$ and $[A]_i^-(x) \ge [A]_i^-(0)$ for x in R and the zero element 0 in R.*

Proof. For x in R and 0 is zero element of R. For each i,

$$[A]_i^+(x) = [A]_i^+(x+0) \ge rmin\{[A]_i^+(x),[A]_i^+(0)\}$$

and

$$[A]_i^+(0) = [A]_i^+(x.0) \ge \text{rmin}\{[A]_i^+(x),[A]_i^+(0)\}.$$

If $x+y = 0$ then

$$[A]_i^+(0) = [A]_i^+(x+y) \ge \text{rmin}\{[A]_i^+(x),[A]_i^+(y)\}.$$

Hence $[A]_i^+(0) \ge [A]_i^+(x)$ for all x in R and

$$[A]_i^-(x) = [A]_i^-(x+0) \le \text{rmax}\{[A]_i^-(x),[A]_i^-(0)\}$$

and

$$[A]_i^-(0) = [A]_i^-(x.0) \le rmax\{[A]_i^-(x),[A]_i^-(0)\}.$$

If $x+y = 0$ then

$$[A]_i^-(0) = [A]_i^-(x+y) \le \text{rmax}\{[A]_i^-(x),[A]_i^-(y)\}.$$

Hence $[A]_i^-(0) \le [A]_i^-(x)$ for all x in R. □

Theorem 2.16. *Let $[A] = \langle[A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ be a bipolar valued multi I-fuzzy subhemiring of a hemiring R.*

(i) For each i, if $[A]_i^+(x+y) = [0]$ then either $[A]_i^+(x) = [0]$ or $[A]_i^+(y) = [0]$ for x,y in R

(ii) For each i, if $[A]_i^+(xy) = [0]$ then either $[A]_i^+(x) = [0]$ or $[A]_i^+(y) = [0]$ for x,y in R

(iii) For each i, if $[A]\ _i(x+y) = [0]$ then either $[A]_i^-(x) = [0]$ or $[A]_i^-(y) = [0]$ for x,y in R.

(iv) For each i, if $[A]_i^-(xy) = [0]$ then either $[A]_i^-(x) = [0]$ or $[A]_i^-(y) = [0]$ for x,y in R.

Proof. Let x,y in R.

(i) For each i, $[A]\ _i^+(x+y) \ge \text{rmin}\{[A]_i^+(x),[A]_i^+(y)\}$ which implies that $[0] \ge \text{rmin}\{[A]_i^+(x),[A]_i^+(y)\}$. Therefore either $[A]_i^+(x) = [0]$ or $[A]_i^+(y) = [0]$.

(ii) For each $i,[A]_i^+(xy) \ge \text{rmin}\{[A]_i^+(x),[A]_i^+(y)\}$ which implies that $[0] \ge \text{rmin}\{[A]_i^+(x),[A]_i^+(y)\}$ Therefore either $[A]_i^+(x) = [0]$ or $[A]_i^+(y) = [0]$.

(iii) For each $i,[A]_i^-(x+y) \le \text{rmax}\{[A]_i^-(x)\ \ [A]_i^-(y)\}$ which implies that $[0] \le \text{rmax}\{[A]_i^-(x),[A]_i^-(y)\}$. Therefore either $[A]_i^-(x) = [0]$ or $[A]_i^-(y) = [0]$.

(iv) For each i, $[A]_i^-(xy) \le \text{rmax}\{[A]_i^-(x),[A]_i^-(y)\}$ which implies that $[0] \le \text{rmax}\{[A]_i^-(x),[A]_i^-(y)\}$. Therefore either $[A]_i^-(x) = [0]$ or $[A]_i^-(y) = [0]$

□

Theorem 2.17. *If $[A] = \langle[A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ and $[B] = \langle[B]_1^+,\ \ [B]_2^+,\dots,[B]_n^+,[B]_1^-,[B]_2^-,\dots,[B]_n^-\rangle$ are two bipolar valued multi I-fuzzy subhemirings of a hemiring R, then their intersection $[A]\cap[B]$ is also a bipolar valued multi I-fuzzy subhemiring of R.*

Proof. Let $[C] = [A]\cap[B]$ and let x,y in R. For each i,

$$\begin{aligned}&[C]_i^+(x+y)\\ &= \text{rmin}\{[A]_i^+(x+y)[B]_i^+(x+y)\}\\ &\ge \text{rmin}\{\text{rmin}\{[A]_1^+(x),[A]_i^+(y)\},\text{rmin}\{[B]_i^+(x),[B]_i^+(y)\}\}\\ &\ge \text{rmin}\{rmin[A]_1^+(x),[B]_i^+(x)\},\text{rmin}\{[A]_i^+(y),[B]_i^+(y)\}\}\\ &= \text{rmin}\{[C]_i^+(x),[C]_i^+(y)\ .\end{aligned}$$

Therefore $[C]_i^+(x+y) \geq \text{rmin}\left\{[C]_i^+(x),[C]_i^+(y)\right\}$, for all x,y in R and

$$\begin{aligned}[C]_i^+(xy) &= rmin\left\{[A]_i^+(xy),[B]_i^+(xy)\right\}\\ &\geq \text{rmin}\left\{\text{rmin}\left\{[A]_i^+(x),[A]_i^+(y)\right\},\text{rmin}\left\{[B]_i^+(x),[B]_i^+(y)\right\}\right\}\\ &\geq \text{rmin}\left\{\text{rmin}\left\{[A]_i^+(x),[B]_i^+(x)\right\},\text{rmin}\left\{[A]_i^+(y),[B]_i^+(y)\right\}\right\}\\ &= \text{rmin}\left\{[C]_i^+(x),[C]_i^+(y)\right. .\end{aligned}$$

Therefore $[C]_i^+(xy) \geq \text{rmin}\left\{[C]_i^+(x),[C]_i^+(y)\right\}$, for all x,y in R. Also

$$\begin{aligned}[C]_i^-(x+y) &= \text{rmax}\left\{[A]_i^-(x+y),[B]_i^-(x+y)\right\}\\ &\leq \text{rmax}\left\{\text{rmax}\left\{[A]_i^-(x),[A]_i^-(y)\right\},\text{rmax}\left\{[B]_i^-(x),[B]_i^-(y)\right\}\right\}\\ &\leq \text{rmax}\{\text{rmax}\left\{[A]_i^-(x),[B]_i^-(x)\right\},\text{rmax}\left\{[A]_i^-(y),[B]_i^-(y)\right\}\right\}\\ &= \text{rmax}\left\{[C]_i^-(x),[C]_i^-(y)\right\}.\end{aligned}$$

Therefore $[C]_i^-(x+y) \leq \text{rmax}\left\{[C]_i^-(x),[C]_i^-(y)\right\}$, for all x,y in R and

$$\begin{aligned}[C]_i^-(xy) &= \text{rmax}\left\{[A]_i^-(xy)\ [B]_i^-(xy)\right\}\\ &\leq \text{rmax}\left\{\text{rmax}\left\{[A]_i^-(x),[A]_i^-(y)\right\},\text{rmax}\left\{[B]_i^-(x),[B]_i^-(y)\right\}\right\}\\ &\leq \text{rmax}\{rmax\left\{[A]_i^-(x),[B]_i^-(x)\right\},\text{rmax}\left\{[A]_i^-(y),[B]_i^-(y)\right\}\right\}\\ &= \text{rmax}\left\{[C]_i^-(x),[C]_i^-(y)\right\}.\end{aligned}$$

Therefore $[C]_i^-(xy) \leq \text{rmax}\left\{[C]_i^-(x),[C]_i^-(y)\right\}$, for all x,y in R. Hence $[A]\cap[B]$ is a bipolar valued multi I-fuzzy subhemiring of R. □

Theorem 2.18. *The intersection of a family of bipolar valued multi I-fuzzy subhemirings of a hemiring R is a bipolar valued multi I-fuzzy subhemiring of R.*

Proof. The proof follows from the Theorem 2.17. □

Theorem 2.19. *If* $[A] = \langle [A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ *is a bipolar valued multi I-fuzzy subhemiring of a hemiring R, then*

$$H = \left\{x\in R \mid [A]_i^+(x) = [1],[A]_i^-(x) = [-1]\ for\ all\ i\right\}$$

is either empty or a subhemiring of R.

Proof. If no element satisfies this condition then H is empty. If x and y in H then for all i,

$$[A]_i^+(x+y) \geq \text{rmin}\left\{[A]_i^+(x),[A]_i^+(y)\right\} = \text{rmin}\{[1],[1]\} = [1].$$

Therefore $[A]_i^+(x+y) = [1]$ and

$$[A]_i^+(xy) \geq \text{rmin}\left\{[A]_i^+(x),[A]_i^+(y)\right\} = \text{rmin}\{[1],[1]\} = [1].$$

Therefore $[A]_i^+(xy) = [1]$. Also

$$\begin{aligned}[A]_i^-(x+y) &\leq \text{rmax}\left\{[A]_i^-(x),[A]_i^-(y)\right\}\\ &= \text{rmax}\{[-1],[-1]\} = [-1].\end{aligned}$$

Therefore $[A]_i^-(x+y) = [-1]$. And

$$\begin{aligned}[A]_i^-(xy) &\leq \text{rmax}\left\{[A]_i^-(x),[A]_i^-(y)\right\}\\ &= \text{rmax}\{[-1],[-1]\} = [-1]\end{aligned}$$

Therefore $[A]_i^-(xy) = [-1]$. That is $x+y\in H$ and $xy\in H$. Hence H is a subhemiring of R. Hence H is either empty or a subhemiring of R. □

Theorem 2.20. *If* $[A] = \langle [A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ *is a bipolar valued multi I-fuzzy subhemiring of a hemiring R, then*

$$H = \left\{x\in R \mid [A]_i^+(x) = [A]_i^+(0)\ \&[A]_i^-(x) = [A]_i^-(0),\forall i\right\}$$

is a subhemiring of R

Proof. Here $H = \left\{x \in R \mid [A]_i^+(x) = [A]_i^+(e)\&[A]_i^-(x) = [A]_i^-(e)\right\}$ by Theorem 2.15. For each i,

$$\begin{aligned}[A]_i^+(x+y) &\geq \text{rmin}\left\{[A]_1^+(x),[A]_i^+(y)\right\}\\ &= \text{rmin}\left\{[A]_i^+(0),[A]_i^+(0)\right\} = [A]_i^+(0).\end{aligned}$$

Hence $[A]_i^+(0) = [A]_i^+(x+y)$. And

$$\begin{aligned}[A]_i^+(xy) &\geq \text{rmin}\left\{[A]_i^+(x),[A]_i^+(y)\right\}\\ &= \text{rmin}\left\{[A]_i^+(e),[A]_i^+(e)\right\} = [A]_i^+(0).\end{aligned}$$

Hence $[A]_i^+(0) = [A]_i^+(xy)$. Also

$$\begin{aligned}[A]_i^-(x+y) &\leq \text{rmax}\left\{[A]_i^-(x),[A]_i^-(y)\right\}\\ &= \text{rmax}\left\{[A]_i^-(0),[A]_i^-(0)\right\} = [A]_i^-(0).\end{aligned}$$

Therefore $[A]_i^-(0) = [A]_i^-(x+y)$. And

$$\begin{aligned}[A]_i^-(xy) &\leq rmax\left\{[A]_i^-(x),[A]_i^-(y)\right\}\\ &= \text{rmax}\left\{[A]_i^-(0),[A]_i^-(0)\right\} = [A]_i^-(0).\end{aligned}$$

Therefore $[A]_i^-(0) = [A]_i^-(xy)$. Therefore $x+y$ and xy are in H. Hence H is a subhemiring of R. □

Theorem 2.21. *If* $[A] = \langle [A]_1^+,[A]_2^+,\dots,[A]_n^+,[A]_1^-,[A]_2^-,\dots,[A]_n^-\rangle$ *and* $[B] = \langle [B]_1^+,[B]_2^+,\dots,[B]_n^+,[B]_1^-,[B]_2^-,\dots,[B]_n^-\rangle$ *are any two bipolar valued multi I-fuzzy subhemirings of the hemirings* R_1 *and* R_2 *respectively, then*

$$\begin{aligned}[A]\times[B] = \langle &([A]_1\times[B]_1)^+,([A]_2\times[B]_2)^+,\dots\\ &([A]_n\times[B]_n)^+,([A]_1\times[B]_1)^-,([A]_2\times[B]_2)^-,\\ &\dots,([A]_n\times[B]_n)^-\rangle\end{aligned}$$

is a bipolar valued multi I-fuzzy subhemiring of $R_1\times R_2$.

Proof. Let $[A]$ and $[B]$ be two bipolar valued multi I-fuzzy subhemirings of the hemirings R_1 and R_2 respectively. Let

x_1, x_2 be in R_1, y_1 and y_2 be in R_2. Then (x_1,y_1) and (x_2,y_2) are in $R_1 \times R_2$. For each i,

$$\begin{aligned}([A]_i \times [B]i)^+ [(x_1,y_1)+(x_2,y_2)] &= ([A]_i \times [B]_i)^+ (x_1+x_2, y_1+y_2) \\ &= rmin\left\{[A]_i^+ (x_1+x_2), [B]_i^+ (y_1+y_2)\right\} \\ &\geq \text{rmin}\left\{rmin\left\{[A]_i^+ (x_1), [A]_i^+ (x_2)\right\},\right. \\ &\qquad \left.\text{rmin}\left\{[B]_i^+ (y_1), [B]_i^+ (y_2)\right\}\right\} \\ &= \text{rmin}\left\{\text{rmin}\left\{[A]_i^+ (x_1), [B]_i^+ (y_1)\right\},\right. \\ &\qquad \left.\text{rmin}\left\{[A]_i^+ (x_2), [B]_i^+ (y_2)\right\}\right\} \\ &= \text{rmin}\left\{([A]_i \times [B]_i)^+ (x_1,y_1)\ ([A]_i \times [B]_i)^+ (x_2,y_2)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}([A]_i \times [B]_i)^+ [(x_1,y_1)+(x_2,y_2)] \\ \geq \text{rmin}\left\{([A]_i \times [B]_i)^+ (x_1,y_1)\ ([A]_i \times [B]_i)^+ (x_2,y_2)\right..\end{aligned}$$

And

$$\begin{aligned}([A]_i \times [B]_i)^+ [(x_1,y_1)(x_2,y_2)] &= ([A]_i \times [B]_i)^+ (x_1x_2, y_1y_2) \\ &= rmin\left\{[A]_i^+ (x_1x_2), [B]_i^+ (y_1y_2)\right\} \\ &\geq \text{rmin}\left\{\text{rmin}\left\{[A]_i^+ (x_1), [A]_i^+ (x_2)\right\},\right. \\ &\qquad \left.\text{rmin}\left\{[B]_i^+ (y_1), [B]_i^+ (y_2)\right\}\right. \\ &= \text{rmin}\left\{\text{rmin}\left\{[A]_i^+ (x_1), [B]_i^+ (y_1)\right\},\right. \\ &\qquad \left.\text{rmin}\left\{[A]_i^+ (x_2), [B]_i^+ (y_2)\right\}\right\} \\ &= \text{rmin}\left\{([A]_i \times [B]_i)^+ (x_1,y_1)\ ([A]_i \times [B]_i)^+ (x_2,y_2)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}([A]_i \times [B]_i)^+ [(x_1,y_1)(x_2,y_2)] \\ \geq \text{rmin}\left\{([A]_i \times [B]_i)^+ (x_1,y_1)\ ([A]_i \times [B]_i)^+ (x_2,y_2)\right\}.\end{aligned}$$

Also

$$\begin{aligned}([A]_i \times [B]_i)^- [(x_1,y_1)+(x_2,y_2)] &= ([A]_i \times [B]_i)^- (x_1+x_2, y_1+y_2) \\ &= \text{rmax}\left\{[A]_i^- (x_1+x_2), [B]_i^- (y_1+y_2)\right\} \\ &\leq \text{rmax}\left\{\text{rmax}\left\{[A]_i^- (x_1), [A]_i^- (x_2)\right\},\right. \\ &\qquad \left.\text{rmax}\left\{[B]_i^- (y_1)\ \ [B]_i^- (y_2)\right\}\right\} \\ &= \text{rmax}\left\{\text{rmax}\left\{[A]_1^- (x_1), [B]_i^- (y_1)\right\},\right. \\ &\qquad \left.\text{rmax}\left\{[A]_i^- (x_2), [B]_i^- (y_2)\right\}\right\} \\ &= \text{rmax}\left\{([A]_i \times [B]_i)^- (x_1,y_1), ([A]_i \times [B]_i)^- (x_2,y_2)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}([A]_i \times [B]_i)^- [(x_1,y_1)+(x_2,y_2)] \\ \leq rmax\left\{([A]_i \times [B]_i)^- (x_1,y_1), ([A]_i \times [B]_i)^- (x_2,y_2)\right\}.\end{aligned}$$

And

$$\begin{aligned}([A]_i \times [B]_i)^- [(x_1,y_1)(x_2,y_2)] &= ([A]_i \times [B]_i)^- (x_1x_2, y_1y_2) \\ &= \text{rmax}\left\{[A]_i^- (x_1x_2), [B]_i^- (y_1y_2)\right\} \\ &\leq rmax\left\{\text{rmax}\left\{[A]_i^- (x_1), [A]_i^- (x_2)\right\}\right. \\ &\qquad \left.\text{rmax}\left\{[B]_i^- (y_1), [B]_i^- (y_2)\right\}\right\} \\ &= \text{rmax}\left\{\text{rmax}\left\{[A]_i^- (x_1), [B]_i^- (y_1)\right\},\right. \\ &\qquad \left.\text{rmax}\left\{[A]_i^- (x_2), [B]_i^- (y_2)\right\}\right\} \\ &= \text{rmax}\left\{([A]_i \times [B]_i)^- (x_1,y_1), ([A]_i \times [B]_i)^- (x_2,y_2)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}([A]_i \times [B]_i)^- [(x_1,y_1)(x_2,y_2)] \\ \leq \text{rmax}\left\{([A]_i \times [B]_i)^- (x_1,y_1), ([A]_i \times [B]_i)^- (x_2,y_2)\right\}.\end{aligned}$$

Hence $[A] \times [B]$ is a bipolar valued multi I-fuzzy subhemiring of $R_1 \times R_2$. □

Theorem 2.22. *Let $[A] = \langle [A]_1^+, [A]_2^+, \ldots, [A]_n^+, [A]_1^-, [A]_2^-, \ldots, [A]_n^- \rangle$ be a bipolar valued multi I-fuzzy subset of a hemiring R and $[V] = \langle [V]_1^+, [V]_2^+, \ldots, [V]_n^+, [V]_1^-, [V]_2^-, \ldots, [V]_n^-$ be the strongest bipolar valued multi I-fuzzy relation of R. Then $[A]$ is a bipolar valued multi I-fuzzy subhemiring of R if and only if $[V]$ is a bipolar valued multi I-fuzzy subhemiring of $R \times R$.*

Proof. Suppose that $[A]$ is a bipolar valued multi I-fuzzy subhemiring of R. Then for any $x = (x_1,x_2), y = (y_1,y_2)$ are in $R \times R$. For each i,

$$\begin{aligned}[V]_i^+(x+y) &= [V]_i^+ [(x_1,x_2)+(y_1,y_2)] \\ &= [V]_i^+ (x_1+y_1, x_2+y_2) \\ &= \text{rmin}\left\{[A]_i^+ (x_1+y_1), [A]_i^+ (x_2+y_2)\right\} \\ &\geq \text{rmin}\left\{\text{rmin}\left\{[A]_i^+ (x_1)\, [A]_i^+ (y_1)\right\},\right. \\ &\qquad \left.\text{rmin}\left\{[A]_i^+ (x_2), [A]_i^+ (y_2)\right\}\right\} \\ &= \text{rmin}\left\{\text{rmin}\left\{[A]_i^+ (x_1), [A]_i^+ (x_2)\right\},\right. \\ &\qquad \left.rmin\left\{[A]_i^+ (y_1), [A]_i^+ (y_2)\right\}\right\} \\ &= \text{rmin}\left\{[V]_i^+ (x_1,x_2), [V]_i^+ (y_1,y_2)\right\} \\ &= \text{rmin}\left\{[V]_i^+(x), [V]_i^+(y)\right\}\end{aligned}$$

Therefore $[V]_i^+(x+y) \geq \text{rmin}\left\{[V]_i^+(x), [V]_i^+(y)\right\}$ for all x,y in $R \times R$. And

$$\begin{aligned}[V]_i^+(xy) &= [V]_i^+ [(x_1,x_2)(y_1,y_2)] \\ &= [V]_i^+ (x_1y_1, x_2y_2) \\ &= \text{rmin}\left\{[A]_i^+ (x_1y_1), [A]_i^+ (x_2y_2)\right\} \\ &\geq \text{rmin}\left\{\text{rmin}\left\{[A]_i^+ (x_1), [A]_i^+ (y_1)\right\},\right. \\ &\qquad \left.\text{rmin}\left\{[A]_i^+ (x_2), [A]_i^+ (y_2)\right\}\right\} \\ &= \text{rmin}\left\{\text{rmin}\left\{[A]_i^+ (x_1), [A]_i^+ (x_2)\right\},\right. \\ &\qquad \left.\text{rmin}\left\{[A]_i^+ (y_1), [A]_i^+ (y_2)\right\}\right\} \\ &= \text{rmin}\left\{[V]_i^+ (x_1,x_2), [V]_i^+ (y_1,y_2)\right\} \\ &= \text{rmin}\left\{[V]_i^+(x), [V]_i^+(y)\right\}\end{aligned}$$

Therefore $[V]_i^+(xy) \geq \text{rmin}\{[V]_i^+(x), [V]_i^+(y)\}$ for all x and y in $R \times R$. Also we have

$$\begin{aligned}
[V]_i^-(x+y) &= [V]_i^-[(x_1,x_2)+(y_1,y_2)] \\
&= [V]_i^-(x_1+y_1, x_2+y_2) \\
&= \text{rmax}\{[A]_i^-(x_1+y_1), [A]_i^-(x_2+y_2)\} \\
&\leq \text{rmax}\{\text{rmax}\{[A]_i^-(x_1), [A]_i^-(y_1)\}, \\
&\qquad \text{rmax}\{[A]_i^-(x_2), [A]_i^-(y_2)\}\} \\
&= \text{rmax}\{\text{rmax}\{[A]_i^-(x_1)[A]_i^-(x_2)\}, \\
&\qquad \text{rmax}\{[A]_i^-(y_1), [A]_i^-(y_2)\}\} \\
&= \text{rmax}\{[V]_i^-(x_1,x_2), [V]_i^-(y_1,y_2)\} \\
&= \text{rmax}\{[V]_i^-(x), [V]_i^-(y)\}.
\end{aligned}$$

Therefore $[V]_i^-(x+y) \leq \text{rmax}\{[V]_i^-(x), [V]_i^-(y)\}$ for all x, y in $R \times R$. And

$$\begin{aligned}
[V]_i^-(xy) &= [V]_i^-[(x_1,x_2)(y_1,y_2)] \\
&= [V]_i^-(x_1y_1, x_2y_2) \\
&= \text{rmax}\{[A]_i^-(x_1y_1), [A]_i^-(x_{2y_2})\} \\
&\leq \text{rmax}\{\text{rmax}\{[A]_i^-(x_1), [A]_i^-(y_1)\}, \\
&\qquad \text{rmax}\{[A]_i^-(x_2), [A]_i^-(y_2)\}\} \\
&= \text{rmax}\{\text{rmax}\{[A]_i^-(x_1)[A]_i^-(x_2)\}, \\
&\qquad \text{rmax}\{[A]_i^-(y_1), [A]_i^-(y_2)\}\} \\
&= \text{rmax}\{[V]_i^-(x_1,x_2), [V]_i^-(y_1,y_2)\} \\
&= \mathit{rmax}\{[V]_i^-(x), [V]_i^-(y)\}.
\end{aligned}$$

Therefore $[V]_i^-(xy) \leq \text{rmax}\{[V]_i^-(x), [V]_i^-(y)\}$ for all x, y in $R \times R$. This proves that $[V]$ is a bipolar valued multi I-fuzzy subhemiring of $R \times R$.
Conversely assume that $[V]$ is a bipolar valued multi I-fuzzy subhemiring of $R \times R$, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $R \times R$, for all i,

$$\begin{aligned}
&\text{rmin}\{[A]_i^+(x_1+y_1), [A]_i^+(x_2+y_2)\} \\
&= [V]_i^+(x_1+y_1, x_2+y_2) \\
&= [V]_i^+[(x_1,x_2)+(y_1,y_2)] \\
&= [V]_i^+(x+y) \geq \text{rmin}\{[V]_i^+(x), [V]_i^+(y)\} \\
&= \text{rmin}\{[V]_i^+(x_1,x_2)[V]_i^+(y_1,y_2)\} \\
&= \text{rmin}\{\text{rmin}\{[A]_i^+(x_1), [A]_i^+(x_2)\}, \\
&\qquad \text{rmin}\{[A]_i^+(y_1), [A]_i^+(y_2)\}\}.
\end{aligned}$$

If $x_2 = y_2 = 0$, we get,

$$[A]_i^+(x_1+y_1) \geq \text{rmin}\{[A]_i^+(x_1), [A]_i^+(y_1)\}$$

for all x_1 and y_1 in R and

$$\begin{aligned}
&\text{rmin}\{[A]_i^+(x_1y_1), [A]_i^+(x_2y_2)\} \\
&= [V]_i^+(x_1y_1, x_2y_2) \\
&= [V]_i^+[(x_1,x_2)(y_1,y_2)] = [V]_i^+(xy) \\
&\geq \mathit{rmin}\{[V]_i^+(x), [V]_i^+(y)\} \\
&= \text{rmin}\{[V]_i^+(x_1,x_2), [V]_i^+(y_1,y_2)\} \\
&= \text{rmin}\{\mathit{rmin}\{[A]_i^+(x_1), [A]_i^+(x_2)\} \\
&\qquad \text{rmin}\{[A]_i^+(y_1), [A]_i^+(y_2)\}\}.
\end{aligned}$$

If $x_2 = y_2 = 0$, we get

$$[A]_i^+(x_1y_1) \geq \text{rmin}\{[A]_i^+(x_1), [A]_i^+(y_1)\},$$

for all x_1, y_1 in R. For each i,

$$\begin{aligned}
&\text{rmax}\{[A]_i^-(x_1+y_1), [A]_i^-(x_2+y_2)\} \\
&= [V]_i^-(x_1+y_1, x_2+y_2) \\
&= [V]_i^-[(x_1,x_2)+(y_1,y_2)] = [V]_i^-(x+y) \\
&\leq \text{rmax}\{[V]_i^-(x), [V]_i^-(y)\} \\
&= \text{rmax}\{[V]_i^-(x_1,x_2)[V]_i^-(y_1,y_2)\} \\
&= \text{rmax}\{\text{rmax}\{[A]_i^-(x_1), [A]_i^-(x_2)\}, \\
&\qquad \text{rmax}\{[A]_i^-(y_1), [A]_i^-(y_2)\}\}.
\end{aligned}$$

If $x_2 = y_2 = 0$, we get

$$[A]_i^-(x_1+y_1) \leq \text{rmax}\{[A]_i^-(x_1), [A]_i^-(y_1)\}$$

for all x_1 and y_1 in R. And

$$\begin{aligned}
&\text{rmax}\{[A]_1^-(x_1y_1), [A]_i^-(x_2y_2)\} \\
&= [V]_i^-(x_1y_1, x_2y_2) \\
&= [V]_i^-[(x_1,x_2)(y_1,y_2)] = [V]_i^-(xy) \\
&\leq \mathit{rmax}\{[V]_i^-(x), [V]_i^-(y)\} \\
&= \text{rmax}\{[V]_i^-(x_1,x_2), [V]_i^-(y_1,y_2)\} \\
&= \text{rmax}\{\text{rmax}\{[A]_i^-(x_1), [A]_i^-(x_2)\} \\
&\qquad \text{rmax}\{[A]_i^-(y_1), [A]_i^-(y_2)\}\}.
\end{aligned}$$

If $x_2 = y_2 = 0$, we get $[A]_i^-(x_1y_1) \leq \text{rmax}\{[A]_i^-(x_1), [A]_i^-(y_1)\}$ for all x_1 and y_1 in R. Hence $[A]$ is a bipolar valued multi I -fuzzy subhemiring of R. □

Theorem 2.23. *Let $[C]$ be a bipolar valued multi I-fuzzy subset of a hemiring R. Then $[C]$ is a bipolar valued multi I-fuzzy subhemiring of R if and only if $-C$ and ^+C are bipolar valued multi fuzzy subhemiring of R, where $-C$ is the lower limits of the closed intervals and ^+C is the upper limits of the closed intervals.*

Proof. Let a, $b \in R$. Suppose $[C]$ is a bipolar valued multi I-fuzzy subhemiring of R, for each i, $[C]_i^+(a) = [{}^-C_i^+(a), {}^+C_i^+(a)]$ and $[C]_i^-(a) = [{}^-C_i^-(a), {}^+C_i^-(a)]$. So,

$$\begin{aligned}
[C]_i^+(a+b) &\geq \text{rmin}\{[C]_i^+(a)\ [C]_i^+(b)\} \\
&= [\min\{-C_i^+(a), -C_i^+(b)\}, \min\{{}^+C_i^+(a), {}^+C_i^+(b)\}].
\end{aligned}$$

Thus

$$[^{-}C_i^{+}(a+b), {}^{+}C_i^{+}(a+b)]$$
$$\geq [\min\{-C_i^{+}(a), -C_i^{+}(b)\}, \min\{{}^{+}Ci^{+}(a), {}^{+}Ci^{+}(b) - C_i^{+}(b)\}]$$

and ${}^{+}C_i^{+}(a+b) \geq \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}$. And

$$[C]_i^{+}(ab) \geq \text{rmin}\{[C]_i^{+}(a), [C]_i^{+}(b)\}$$
$$= [\min\{-C_i^{+}(a), -C_i^{+}(b)\}, \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}].$$

Thus

$$[^{-}C_i^{+}(ab), {}^{+}C_i^{+}(ab)]$$
$$\geq [\min\{-C_i^{+}(a), -Ci^{+}(b)\}, \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}].$$

Therefore ${}^{-}C_i^{+}(ab) \geq \min\{-C_i^{+}(a), {}^{-}C_i^{+}(b)\}$ and ${}^{+}C_i^{+}(ab) \geq \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}$. Then

$$[C]_i^{-}(a+b) \leq \text{rmax}\{[C]_i^{-}(a), [C]_i^{-}(b)\}$$
$$= [\max\{C_i-(a), {}^{-}C_i-(b)\} \max\{{}^{+}C_i-(a), {}^{+}C_i-(b)\}].$$

Thus

$$[^{-}C_i-(a+b), {}^{+}C_i^{-}(a+b)]$$
$$\leq [\max\{-C_i^{-}(a), {}^{-}C_i-(b)\}, \max\{{}^{+}C_i^{-}(a){}^{+}C_i^{-}(b)\}].$$

Therefore ${}^{-}C_i-(a+b) \leq \max\{-C_i-(a), -C_i-(b)\}$ and ${}^{+}C_i-(a+b) \leq \max\{{}^{+}C_i-(a) \quad {}^{+}C_i-(b)\}]$. Thus

$$[^{-}C_i-(ab), {}^{+}C_i-(ab)]$$
$$\leq [\max\{-C_i^{-}(a), -C_i-(b)\}, \max\{{}^{+}C_i-(a), {}^{+}C_i^{-}(b)\}]$$

and C^{+} are bipolar valued multi fuzzy subhemiring of R. Conversely, assume that C^{-} and C^{+} $\{-C_i^{+}(b)\}$ and ${}^{+}C_i^{+}(a+b) \geq \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}$ which implies that

$$[^{-}C_i^{+}(a+b), {}^{+}C_i^{+}(a+b)]$$
$$\geq \min\{-C_i^{+}(a), -C_i^{+}(b)\}, \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}]$$

which implies that

$$[C]_i^{+}(a+b) \geq \text{rmin}\{[C]_i^{+}(a), [C]_i^{+}(b)\}$$
$$= [\min\{-C_i^{+}(a), -C_i^{+}(b)\}, \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}].$$

And ${}^{-}C_i^{+}(ab) \geq \min\{-Ci^{+}(a), -Ci^{+}(b)\}$ and ${}^{+}C_i^{+}(ab) \geq \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}$ which implies that

$$[^{-}C_i^{+}(ab), {}^{+}C_i^{+}(ab)]$$
$$\geq [\min\{-C_i^{+}(a), {}^{-}C_i^{+}(b)\}, \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}]$$

which implies that

$$[C]_i^{+}(ab) \geq \text{rmin}\{[C]_i^{+}(a), [C]_i^{+}(b)\}$$
$$= [\min\{-C_i^{+}(a), -C_i^{+}(b)\}, \min\{{}^{+}C_i^{+}(a), {}^{+}C_i^{+}(b)\}].$$

Also for each i, $-C_i-(a+b) \leq \max\{-C_i-(a), -C_i-(b)\}$ and ${}^{+}C_i-(a+b) \leq \max\{{}^{+}C_i^{-}(a), {}^{+}C_i-(b)\}$ which implies that

$$[-C_i-(a+b), {}^{+}C_i-(a+b)]$$
$$\leq [\max\{-C_i^{-}(a), {}^{-}C_i-(b)\}, \max\{{}^{+}C_i-(a), {}^{+}C_i-(b)\}]$$

which implies that

$$^{[}C]_i^{-}(a+b) \leq \text{rmax}\{[C]_i^{-}(a), [C]_i^{-}(b)\} =$$
$$[\max\{\bar{_{Ci}}(a), {}^{-}C_i^{-}(b)\}, \max\{{}^{+}C_i^{-}(a), {}^{+}C_i^{-}(b)\}]$$

And ${}^{-}C_i^{-}(ab) \leq \max\{{}^{-}C_i^{-}(a), {}^{-}C_i^{-}(b)\}$ and ${}^{+}C_i^{-}(ab) \leq \max\{{}^{+}C_i^{-}(a), {}^{+}C_i^{-}(b)\}$ which implies that

$$[^{-}C_i^{-}(ab), {}^{+}C_i^{-}(ab)]$$
$$\leq [\max\{{}^{-}C_i^{-}(a), {}^{-}C_i^{-}(b)\}, \max\{{}^{+}C_i^{-}(a), {}^{+}C_i^{-}(b)\}]$$

which implies that

$$[C]_i^{-}(ab) \leq rmax\{[C]_i^{-}(a), [C]_i^{-}(b)\}$$
$$= [\max\{-C_i^{-}(a), -C_i^{-}(b)\}, \max\{{}^{+}C_i^{-}(a), {}^{+}C_i^{-}(b)\}]$$

Hence $[C]$ is a bipolar valued multi I-fuzzy subhemiring of R. □

References

[1] G. Adomian and G. E. Adomian, Cellular systems and aging models, *Comput. Math. Appl.* 11(1985), 283–291.

[2] M.S. Anitha, Muruganantha Prasad and K. Arjunan, Notes on bipolar valued fuzzy subgroups of a group, *Bulletin of Society for Mathematical Services and Standards*, 2(3), (2013), 52–59.

[3] Azriel Rosenfeld, fuzzy groups, *Journal of mathematical analysis and applications*, 35(1971), 512–517.

[4] A. Balasubramanian, K.L. Muruganantha Prasad and K. Arjunan, Properties of Bipolar interval valued fuzzy subgroups of a group, *International Journal of Scientific Research*, 4(4), (2015), 262–268.

[5] Grattan Guiness, Fuzzy membership mapped onto interval and many valued quantities, *Z. Math. Logik. Grundladen Math*, 22(1975), 149–160.

[6] K.M. Lee, bipolar valued fuzzy sets and their operations, *Proc. Int. Conf. on Intelligent Technologies*, Bangkok, Thailand, (2000), 307–312.

[7] K.M. Lee, Comparison of interval valued fuzzy sets, intuitionistic fuzzy sets and bipolar valued fuzzy sets, *J. fuzzy Logic Intelligent Systems*, 14(2), (2004), 125–129.

[8] K. Murugalingam and K. Arjunan, A study on interval valued fuzzy subsemiring of a semiring, *International Journal of Applied Mathematics Modeling*, 1(5), (2013), 1–6.

[9] S. Muthukumaran and B. Anandh, Some theorems in bipolar valued multi fuzzy subnearring of a nearing, *Infokara*, 8(11), (2019).

[10] Sabu Sebastian, T.V. Ramakrishnan, Multi fuzzy sets, *International Mathematical Forum*, 5(50), (2010), 2471–2476.

[11] V.K Santhi and K. Anbarasi, Bipolar valued multi fuzzy subhemirings of a hemiring, *Advances in Fuzzy Mathematics*, 10(1), (2015), 55–62.

[12] M.G. Somasundra Moorthy, *A study on interval valued fuzzy, anti fuzzy, intuitionistic fuzzy subrings of a ring*, Ph.D Thesis, Bharathidasan University, Trichy, Tamilnadu, India (2014).
[13] B. Yasodara and KE. Sathappan, Bipolar-valued multi fuzzy subsemirings of a semiring, *International Journal of Mathematical Archive*, 6(9), (2015), 75–80.
[14] L.A. Zadeh, fuzzy sets, *Inform. And Control*, 8(1965), 338–353.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆