Malaya Journal of Matematik, Vol. 8, No. 4, 1885-1889, 2020
https://doi.org/10.26637/MJM0804/0091

Minimum inclusive degree sum dominating set

K. Saleel Mohammed[1*] and Raji Pilakkat[2]

Abstract
In this paper we introduce the inclusive degree of a vertex and minimum inclusive degree sum dominating set of a graph. Moreover, we determine minimum inclusive degree sum dominating set of some special types of graphs and some operations on graphs.

Keywords
Inclusive degree of a vertex, Minimum inclusive degree sum dominating set, Minimum inclusive degree sum of dominating set.

AMS Subject Classification
05C30.

[1,2] *Department of Mathematics, University of Calicut, Malappuram-673635, Kerala, India.*
*Corresponding author: [1] saleelkmohd@gmail.com; [2]rajipilakkat@gmail.com
Article History: Received **04** August **2020**; Accepted **27** September **2020**

Contents

1. Introduction

In a country if we want to install military check points or corona virus checkpoints such that any township is either having a check point or must be near to a township having a check point. That means a check point takes care of the township it is located in and the neighbouring townships. Suppose the maintenance cost of each check point is directly proportional to the number of townships it is taking care of. Then in order to minimize the total maintenance cost of the check points, we must follow a strategy in deciding which all townships must contain a check point. This when considered as a graph theoretical problem by considering each township as a vertex and the connections between them by an edge, motivates to define a new dominating set, namely minimum inclusive degree sum dominating set.

By a graph $G=(V,E)$ we mean a finite undirected graph with neither loops nor multiple edges. That means we only consider finite simple undirected graphs.

2. Minimum inclusive degree sum dominating set of a graph

In this section we first introduce the definition of inclusive degree of a vertex and then define the minimum inclusive degree sum dominating set of a graph. Moreover, we determine minimum inclusive degree sum dominating set of some special types of graphs and some operations on graphs.

Definition 2.1 (Inclusive degree of a vertex)**.** *Inclusive degree of a vertex in a graph G is defined as degree of the vertex $+1$, the additional $+1$ accounts for the contribution of vertex itself. In other words the inclusive degree of a vertex is the cardinality of closed neighbourhood of that vertex.*

Definition 2.2 (Inclusive degree sum of a set of vertices)**.** *Let $G=(V,E)$ be a graph of order n, then the inclusive degree sum of $X \subseteq V$ is defined as the sum of inclusive degrees of vertices in X.*
Inclusive degree sum of $X = |X| +$ degree sum of vertices in X.

$$= |\{N[v] : v \in X\}|.$$

Theorem 2.3. *Let G be a simple graph of order n and let D be a dominating set [2]. Then the inclusive degree sum of $D \geq n$.*

Proof. In order for a set D to be dominating, each element in D^c should be dominated by at least one element in D. Therefore degree sum of $D \geq |D^c|$. Hence the inclusive degree sum of vertices in $D = |D| +$ degree sum of vertices in $D \geq |D| + |D^c| = n$. □

Definition 2.4 (Minimum inclusive degree sum dominating set)**.** *Let G be a simple graph of order n. Then the minimum inclusive degree sum dominating set of G is defined as the dominating set of G with minimum inclusive degree sum.*

Theorem 2.5. *Let G be a simple graph of order n and let D be a dominating set with inclusive degree sum n. Then it's a minimum inclusive degree sum dominating set.*

Theorem 2.6. *Let G be a simple graph of order n and let D be a minimum inclusive degree sum dominating set. Then it's a minimal dominating set.*

Proof. If $\exists$ a dominating set $D' \subsetneq D$, then that will be a minimum inclusive degree sum dominating set with a lesser inclusive degree sum, which contradicts that D is a minimum inclusive degree sum dominating set. Therefore minimum inclusive degree sum dominating set is always a minimal dominating set. □

Theorem 2.7. *Let G be a simple graph of order n and let X be a set of independent vertices and any vertex in X^c is dominated by exactly one vertex in X. Then X is a minimum inclusive degree sum dominating set with inclusive degree sum n.*

Proof. Clearly X is a dominating set of G. Inclusive degree sum of vertices in $X = |X| +$ degree sum of vertices in $X = |X| + |X^c| = n$ □

Theorem 2.8. *Let G be a simple graph of order n and let D be a minimum inclusive degree sum dominating set with inclusive degree sum n. Then it's a minimum dominating set.*

Proof. We have given

$$|D| + \text{degree sum of vertices in } D = n$$

i.e, each vertex in D^c is dominated by exactly one vertex in D. Thus each vertex in D^c has exactly one private neighbor in D. Suppose D is not a minimum dominating set, then $\exists$ a minimum dominating set D' such that $|D'| < |D|$.
Since D is a minimum degree sum dominating set $D' \not\subset D$.
Let

$$D = \{v_1, v_2, \cdots, v_m\} \cup \{w_1, w_2, \cdots, w_t\}$$

and

$$D' = \{v_1, v_2, \cdots, v_m\} \cup \{u_1, u_2, \cdots, u_k\}$$

where

$$\{v_1, v_2, \cdots, v_m\} = D \cap D', \ \ 0 \leq m < |D'|$$

and $0 < k < t$, and D and D' dominate each other. Which implies,
$\exists\, u_i \in D'$ such that $|N(u_i) \cap D| \geq 2$. Let $\{w_i, w_j\} \subset N(u_i) \cap D$. That is $u_i \in D^c$ is dominated by more than one vertex in D, a contradiction. Therefore D is a minimum dominating set. □

When D contains only one vertex, it dominates all other vertices, hence the inclusive degree sum is n and it's a minimum inclusive degree sum dominating set.

Theorem 2.9. *Let G be a simple graph of order n and let D be a dominating set containing only one vertex. Then it's a minimum inclusive degree sum dominating set with inclusive degree sum n.*

Theorem 2.10. *Let G be a simple graph of order n and let D be a minimum inclusive degree sum dominating set with two vertices. Then it is a minimum dominating set.*

Proof. Suppose there exists another dominating set D' containing only one vertex with same inclusive degree sum. Then by Theorem 2.9, its inclusive degree sum is n. Therefore the inclusive degree sum of D is also n. Then by Theorem 2.8, D is a minimum dominating set, a contradiction. Therefore D is a minimum dominating set. □

Theorem 2.11. *Let G be a simple graph of order n and let D be a minimum inclusive degree sum dominating set with at least* 2 *vertices and inclusive degree sum n. Then the distance between any two vertices in D is at least* 3 *and for any vertex in D there exists another vertex in D at a distance exactly* 3.

Proof. Suppose the distance between a pair of vertices in D is less than 3, then either the pair of vertices in D are adjacent or have a common neighbour in D^c. In the first case the minimum degree sum is greater than n and in the later one vertex in D^c is dominated by more than one vertex in D. In either case we have a contradiction. Therefore the distance between any two vertices in D is at least 3.
For the second part suppose $\exists v \in D$ such that $d(v, D - \{v\}) \geq 4$. Let $u \in D$ be the vertex with minimum distance from v in D, then there exists a minimum path $u, u_1, u_2, u_3, \cdots, u_{m-1}, v$ between u and v of length $m \geq 4$. Clearly each u_i is in D^c and the vertex u_2 is dominated neither by u nor by v. Therefore there exists a vertex $w \in D$ which is adjacent to u_2, so that $d(v, w) = 3$, a contradiction. Hence for any vertex in D there exists another vertex in D which is at a distance exactly 3. □

Theorem 2.12. *Let G be a simple graph of order n. Let D be a dominating set with at least* 2 *vertices and the distance between any two vertices in D be at least* 3*. Then D is a minimum inclusive degree sum dominating set with inclusive degree sum n.*

Proof. Let $v \in D$. Then $N(v) \cap D = \phi$. Therefore $N(v) \subset D^c \ \ \forall v \in D$. We claim that $N(u) \cap N(v) = \phi \ \ \forall u, v \in D$. Suppose that $N(u) \cap N(v) \neq \phi$. Let $w \in N(u) \cap N(v) \implies \exists$ a path u, w, v of length 2, a contradiction.
Therefore $N(u) \cap N(v) = \phi \ \ \forall u, v \in D$ and hence

$$\bigcup_{v \in D} N(v) = D^c.$$

Therefore the inclusive degree sum of

$$D = \sum_{v \in D} degN(v) + |D| = |D^c| + |D| = n.$$

□

Corollary 2.13. *Let G be a simple graph of order n. Let D be a dominating set with at least* 2 *vertices and for any vertex in D the nearest vertex in D is at a distance exactly* 3*. Then D is a minimum inclusive degree sum dominating set with inclusive degree sum n.*

Theorem 2.14. *For a k-regular graph G, minimum inclusive degree sum dominating set is same as the minimum dominating set.*

Proof. Since G is k-regular all vertices have same degree. Therefore the inclusive degree sum of dominating set is minimum when the number of vertices in the dominating set is minimum.

□

Theorem 2.15. *Minimum inclusive degree sum of dominating set of a k-regular graph G is* $(k+1)\gamma(G)$*.*

Corollary 2.16. *For a k-regular graph* $\gamma(G) \geq \frac{n}{k+1}$

Corollary 2.17. *For a k-regular graph, minimum inclusive degree sum of dominating set is n iff* $\gamma(G) = \frac{n}{k+1}$*.*

Proof. Let G be a k-regular graph. Then $\gamma(G) = \frac{n}{k+1}$ iff the minimum inclusive degree sum of dominating set is $(k+1)\gamma(G) = n$. □

For a cycle C_n with n vertices minimum inclusive degree sum of dominating set is $3\lceil \frac{n}{3} \rceil$, which is equal to n iff $n \equiv 0$ mod 3, otherwise it's greater than n. Hence we have

Corollary 2.18. *For a cycle* C_n *with n vertices, minimum inclusive degree sum of dominating set is n iff* $n \equiv 0 \mod 3$*.*

Minimum inclusive degree sum dominating set of a graph G may not be unique. That means one graph can have more than one minimum inclusive degree sum dominating sets. One graph can have minimum inclusive degree sum dominating set with different cardinalities as well.

Now we determine minimum inclusive degree sum dominating set and minimum inclusive degree sum of dominating set of some special types of graphs.

3. Minimum inclusive degree sum dominating set of some special types of graphs

Complete graph (K_n)
In complete graphs any one vertex dominates all other vertices, then by Theorem 2.9 it is a minimum inclusive degree sum dominating set with inclusive degree sum n.

Cycle graph (C_n)
Cycle graph C_n $(n \geq 3)$ is a $2-$regular graph and hence by Theorem 2.14 its minimum inclusive degree sum dominating set is same as the minimum dominating set. And by Theorem 2.15, minimum inclusive degree sum of dominating set is $(k+1)\gamma(G) = (2+1)\lceil \frac{n}{3} \rceil = 3\lceil \frac{n}{3} \rceil$.

Path graph (P_n)
Minimum domination number of a path graph P_n is $\lceil \frac{n}{3} \rceil$. We can choose dominating set such that any two vertices are at a distance atleast 3. Label the vertices in a order as $(1,2,3,...,n)$ and choose the vertex set $\{2,5,8,...,n-1\}$ if $3|n$ otherwise the vertex set $\{1,4,7,...,3(\lceil \frac{n}{3} \rceil)-2\}$. Then by Theorem 2.11 the chosen vertex set is a minimum inclusive degree sum dominating set with inclusive degree sum n.

Sunlet graph (S_n)
The sunlet graph S_n is the graph obtained by adjoining a pendant edge at each vertex of the cycle C_n. If we consider all pendant vertices, then that form a dominating set with inclusive degree sum $2n$, which is same as the number of vertices of the graph. Hence it's a minimum degree sum dominating set with inclusive degree sum $2n$.

Wheel graph (W_n)
The wheel graph W_n is the graph obtained by connecting each vertices of a cycle C_n to a single vertex. In wheel graph the center vertex dominates all other vertices, then by Theorem 2.9, it is a minimum inclusive degree sum dominating set with inclusive degree sum $n+1$.

Helm graph (H_n)
The helm graph H_n is the graph obtained from a wheel graph W_n by adjoining a pendant edge at each vertex of the cycle. The minimum inclusive degree sum dominating set is the $n-1$ pendant vertices together with one vertex adjacent to the pendant vertex that is not in the dominating set. Therefore the minimum inclusive degree sum of dominating set is $2(n-1)+5 = 2n+3$

Star graph ($K_{1,n}$)
In star graph $K_{1,n}$ the center vertex dominates all other vertices, then by Theorem 2.9, it is a minimum inclusive degree sum dominating set with inclusive degree sum $n+1$.

Friendship graph (F_n)
In friendship graph F_n the center vertex dominates all other vertices, then by Theorem 2.9, it is a minimum inclusive degree sum dominating set with inclusive degree sum $2n+1$.

Peterson graph
Peterson graph is a $3-$regular graph and hence by Theorem 2.13, its minimum inclusive degree sum dominating set is same as the minimum dominating set. And by Theorem 2.15 minimum inclusive degree sum of dominating set is $(k+1)\gamma(G) = (3+1) \times 3 = 12$.

Complete bipartite graph ($K_{m,n}$)
In order to dominate the complete bipartite graph $K_{m,n}(m,n \geq 2)$, we need at least two vertices, one from each partition of vertices with degrees m and n. And it is a minimum inclusive degree sum dominating set with inclusive degree sum $m+n+2$.

Bistar graph ($BS_{m,n}$)
Bistar graph $BS_{m,n}$ is a graph obtained by connecting the centre vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by an edge. When $m,n \geq 2$ the minimum dominating set is the set consisting of the two central vertices with degrees $m+1$ and $n+1$. And it is a minimum inclusive degree sum dominating set with inclusive degree sum $m+n+4$.
Spider graph (Sp_{2n+1})
Spider graph Sp_{2n+1} is the graph obtained by subdividing each edge of a star graph $K_{1,n}$. The minimum inclusive degree sum dominating set is the set consisting of the $n-1$ pendant vertices together with one vertex adjacent to the pendant vertex that is not in the dominating set. And the minimum inclusive degree sum of dominating set is $2n+1$.

Now we determine minimum inclusive degree sum dominating set of Connected graphs of order 1,2,3,4.

There is only one non-isomorphic graph of order 1, which is a single vertex, whose minimum dominating set and minimum inclusive degree sum dominating set is the vertex itself.

There is only one non-isomorphic connected graph of order 2, which is P_2. One of its vertex is a minimum dominating set and minimum inclusive degree sum dominating set.

There are only two non-isomorphic connected graphs of order 3, which are P_3 and C_3. For P_3 the middle vertex is the minimum dominating set and minimum inclusive degree sum dominating set.For C_3 any single vertex is a minimum dominating set and minimum inclusive degree sum dominating set.

There are only six non-isomorphic connected graphs of order 4. Four of them have single vertex as minimum dominating set and minimum inclusive degree sum dominating set. And P_4 has the two pendant vertices as its minimum inclusive degree sum dominating set, which is a minimum dominating set. And C_4 has any two vertices as its minimum dominating set and minimum inclusive degree sum dominating set.

Theorem 3.1. *For a connected graph of order $n \leq 4$, the minimum inclusive degree sum dominating set is a minimum dominating set.*

Consider a connected graph of order 5, the graph obtained from the star graph $K_{1,3}$ by adding a pendant edge to one of its pendant vertices. For this graph, the three pendant vertices is a minimum inclusive degree sum dominating set with inclusive degree sum 6, but it is not a minimum dominating set. But there exist a minimum inclusive degree sum dominating set which is also a minimum dominating set, which is the newly added pendant vertex together with the the vertex of degree 3.

For a connected graph of order $n \geq 5$ every minimum inclusive degree sum dominating set may not be a minimum dominating set.

Consider a simple connected graph of order 6, the graph obtained by adding pendant edges to two adjacent vertices of the cycle graph C_4. The minimum dominating set is the two vertices into which the pendant edges is added , with inclusive degree sum 8. But the minimum inclusive degree sum dominating set is the two pendant vertices together with one vertex of degree 2, with inclusive degree sum 7.

For a simple connected graph of order $n \geq 6$, there may not exist a minimum inclusive degree sum dominating set which is a minimum dominating set.

Now we determine minimum inclusive degree sum dominating set of some operations on graphs.

4. Minimum inclusive degree sum dominating set of operations of graphs

Disjoint union of simple graphs
Let G and H be two disjoint simple graphs, and the union of G and H is denoted by $G \cup H$. Then the minimum inclusive degree sum dominating set of $G \cup H$ is the union of minimum inclusive degree sum dominating sets of G and H.
Join of two graphs
The join of two graphs [1] G and H is a graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H, and it is denoted by $G+H$.
If the minimum dominating set of either G or H is a single vertex, then that vertex itself is the minimum dominating set of $G+H$ and thus the minimum inclusive degree sum dominating set of $G+H$. i.e., if either G or H is a complete graph or star graph or wheel graph etc.
Corona product of two graphs
Let G and H be two simple graphs of order m and n. The corona product [1] $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and m copies of H and joining by an edge each vertex from the i^{th} copy of H with the i^{th} vertex of G.
If the minimum dominating set of H is a single vertex, then m such vetices from each copy of H will be a dominating set of $G \odot H$, and the distance between the vertices in the dominating set is atleast 3. Then by Theorem 2.12, it will be a minimum inclusive degree sum dominating set of $G \odot H$, with inclusive degree sum $mn+m$, which is the number of vertices in $G \odot H$.

Acknowledgment

The first author great fully acknowledge the funding agency, Council of Scientific and Industrial Research(CSIR) of the Government of India, for providing financial support to carry out this research work.

References

[1] F. Harary, *Graph Theory*, Adison-Wesley, 2010.

[2] S. Alikhani, *Dominating Sets and Domination Polynomials of Graphs*, Ph.D. Thesis, 2009.

⋆⋆⋆⋆⋆⋆⋆⋆⋆

ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

⋆⋆⋆⋆⋆⋆⋆⋆⋆