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On $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence in linear 2-normed spaces

AMAR KUMAR BANERJEE*[1] AND NESAR HOSSAIN[2]

[1,2] *Department of Mathematics, The University of Burdwan, Golapbag, Burdwan - 713104, West Bengal, India.*

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Abstract. In this paper we study the notion of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence in linear 2-normed spaces and some of their properties. We also establish the relationship between them.

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Contents

1. Introduction

The idea of usual convergence of a real sequence was extended to statistical convergence independently by Fast [11] and Steinhaus [21] in the year 1951. Lot of developments were made on this notion of convergence after the pioneering works of Šalát [22] and Fridy [12]. After long fifty years, the concept of statistical convergence was extended to the idea of $\mathcal{I}$-convergence depending on the structure of ideals $\mathcal{I}$ of $\mathbb{N}$, the set of natural numbers, by Kostyrko et al. [17]. Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers respectively. $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $B \in \mathcal{I}$ whenever $B \subset A \in \mathcal{I}$. $\mathcal{I}$ is called an admissible ideal of $\mathbb{N}$ if $\{x\} \in \mathcal{I}$ for each $x \in \mathbb{N}$. $\mathcal{I} \subset 2^{\mathbb{N}}$ is called non-trivial ideal if $\mathcal{I} \neq \{\phi\}$ and $\mathbb{N} \notin \mathcal{I}$. If $\mathcal{I}$ is a non-trivial proper ideal of $\mathbb{N}$ then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter on $\mathbb{N}$, called the filter associated with the ideal $\mathcal{I}$. Indeed, the concept of $\mathcal{I}$- convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of $\mathbb{N}$. $\mathcal{I}$-convergence of real sequences coincides with the ordinary convergence if $\mathcal{I}$ is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if $\mathcal{I}$ is the ideal of $\mathbb{N}$ of natural density zero. In [17] the concept of $\mathcal{I}^*$-convergence was also introduced. Last few years several works on $\mathcal{I}$-convergence and its related areas were carried out in different directions in different spaces viz. metric spaces, normed linear spaces, probabilistic metric spaces, S-metric spaces, linear 2-normed spaces, cone metric spaces, topological spaces etc. (see [3, 4, 6, 18] and many more references therein). Ordinary convergence always implies statistical convergence and when $\mathcal{I}$ is admissible ideal, $\mathcal{I}^*$-convergence implies $\mathcal{I}$-convergence. But the reverse implication

*Corresponding author. Email address: **akbanerjee@math.buruniv.ac.in; akbanerjee1971@gmail.com** (Amar Kumar Banerjee)
nesarhossain24@gmail.com (Nesar Hossain)

does not hold in general. But when $\mathcal{I}$ satisfies the condition (AP), $\mathcal{I}$-convergence implies $\mathcal{I}^*$-convergence. A remarkable observation is that a statistically convergent sequence and $\mathcal{I}$ and $\mathcal{I}^*$-convergent sequence need not even be bounded.

Recently some significant investigations have been done on sequences of real valued functions by using the idea of statistical and $\mathcal{I}$-convergence [8, 10, 15, 19]. The interesting notion of equal convergence was introduced by Császár and Laczkovich [7] for sequences of real valued functions (also known as quasinormal convergence [2]). It is known that equal convergence is weaker than uniform convergence and stronger than pointwise converegence for the sequences of real valued functions. A detailed investigation was carried out by Császár and Laczkovich in [7] on such type of convergence. In [9, 10, 13] the concept of equal convergence of sequences of real functions was generalized to the ideas of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence using ideals of $\mathbb{N}$ and the relationship between them were investigated. $\mathcal{I}$-equal convergence is weaker than $\mathcal{I}$-uniform convergence and stronger than $\mathcal{I}$-pointwise convergence [10].

The notion of linear 2-normed spaces was initially introduced by Gähler [14] and since then the concept has been studied by many authors. In [24] some significant investigations on $\mathcal{I}$-uniform and $\mathcal{I}$-pointwise convergence have been studied in this space.

2. Preliminaries

Throughout the paper $\mathcal{I} \subset 2^{\mathbb{N}}$ will stand for an admissible ideal. Now we recall some basic definitions and notations.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that x is the limit of the subsequence $\{x_{m_k}\}_{k\in\mathbb{N}}$ [17].

Let f, f_n be real valued functions defined on a non empty set X. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be equally convergent ([7]) to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_{n\to\infty} \varepsilon_n = 0$ such that for every $x \in X$ there is $m = m(x) \in \mathbb{N}$ with $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geq m$. In this case we write $f_n \xrightarrow{e} f$.

Now we see the key ideas of $\mathcal{I}$-uniform convergent [5] and $\mathcal{I}$ and $\mathcal{I}^*$-equal convergent [10] sequences of real valued functions which will be needed for generalizations into linear 2-normed spaces.

A sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for each $\varepsilon > 0$ there exists a set $B \in \mathcal{I}$ such that for all $n \in B^c$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. In this case we write $f_n \xrightarrow{\mathcal{I}-u} f$. f is called $\mathcal{I}$-equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ such that for any $x \in X$, the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. In this case we write $f_n \xrightarrow{\mathcal{I}-e} f$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-equal convergent to f if there exists a set $M = \{m_1 < m_2 < \cdots < m_k \cdots\} \in \mathcal{F}(\mathcal{I})$ such that f is the equal limit of the subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-e} f$.

Now we recall the following two important notions which are basically equivalent to each other (due to Lemma 3.9. and Definition 3.10. in [20]). Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. $\mathcal{I}$ is called P-ideal if for every sequence of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a sequence $\{B_1, B_2, \cdots\}$ of sets belonging to $\mathcal{I}$ such that $A_j \triangle B_j$ is finite for $j \in \mathbb{N}$ and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$. This notion is also called condition (AP) while in [20] it is denoted as $AP(\mathcal{I}, Fin)$. An ideal $\mathcal{I}$ is a P-ideal if for any sets $A_1, A_2, \cdots$ belonging to $\mathcal{I}$ there exists a set $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for $n \in \mathbb{N}$.

Now we state some results from [16] for the sequences of real numbers.

Theorem 2.1. *Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers and $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. If $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = \xi$ then $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$.*

Theorem 2.2. *$\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$ implies $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = \xi$ if and only if $\mathcal{I}$ satisfies the condition (AP).*

We will now recall the definition of linear 2-normed spaces which will play very important role throughout the paper.

Definition 2.3. *([14]) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies the following conditions:*
(C1) $\|x, y\| = 0$ if and only if x and y are linearly dependent in X;
(C2) $\|x, y\| = \|y, x\|$ for all x, y in X;
(C3) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for all α in $\mathbb{R}$ and for all x, y in X;
(C4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in X.

The pair $(X, \|.,.\|)$ is called a linear 2-normed space. A simple example ([24]) of a linear 2-normed space is $(\mathbb{R}^2, \|.,.\|)$ where the equipped 2-norm is given by $\|x, y\| = |x_1y_2 - x_2y_1|,\ x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Let X be a 2-normed space of dimension d, $2 \leq d < \infty$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent ([1]) to $\xi \in X$ if $\lim_{n\to\infty} \|x_n - \xi, z\| = 0$, for every $z \in X$. In such a case ξ is called limit of $\{x_n\}_{n\in\mathbb{N}}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent ([23]) to $\xi \in X$ if for each $\varepsilon > 0$ and $z \in X$, the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - \xi, z\| \geq \varepsilon\} \in \mathcal{I}$. The number ξ is called $\mathcal{I}$- limit of $\{x_n\}_{n\in\mathbb{N}}$.

3. Main Results

In this paper we study the concepts of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence of sequences of functions and investigate relationship between them in linear 2-normed spaces. Throughout the paper we propose X as a non empty set and Y as a linear 2-normed space having dimension d with $2 \leq d < \infty$.

Definition 3.1. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be equally convergent to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_{n\to\infty} \varepsilon_n = 0$ such that for every $x \in X$ there is $m = m(x) \in \mathbb{N}$ with $\|f_n(x) - f(x), z\| < \varepsilon_n$ for $n \geq m$ and for every $z \in Y$. In this case we write $f_n \xrightarrow{e} f$.*

Definition 3.2. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for any $\varepsilon > 0$ there exists a set $A \in \mathcal{I}$ such that for all $n \in A^c$ and for all $x \in X, z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$. In this case we write $f_n \xrightarrow{\mathcal{I}-u} f$.*

Definition 3.3. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. Then the the sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-equal convergent to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ such that for any $x \in X$ and for any $z \in Y$, the set $\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon_n\} \in \mathcal{I}$. In this case f is called $\mathcal{I}$-equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ and we write $f_n \xrightarrow{\mathcal{I}-e} f$.*

Example 3.4. *Let $\mathcal{I}$ be a non trivial proper admissible ideal. Let $X = \mathbb{R}^2$ and $Y = \{(a, 0) : a \in \mathbb{R}\}$. Define $f_n(x_1, x_2) = (\frac{1}{n+1}, 0)$ and $f(x_1, x_2) = (0, 0)$ for all $(x_1, x_2) \in \mathbb{R}^2$. Suppose $\varepsilon_n = \frac{1}{n}$. Then $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$. Here we use the 2-norm on $\mathbb{R}^2$ by $\|x, y\| = |x_1y_2 - x_2y_1|,\ x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Now we consider the set $A = \{n \in \mathbb{N} : \|f_n(x_1, x_2) - f(x_1, x_2), z\| \geq \varepsilon_n\}$ for all $z = (y_1, y_2) \in Y$. Then $A = \{n \in \mathbb{N} : \left\|(\frac{1}{n+1}, 0) - (0, 0), (y_1, y_2)\right\| \geq \frac{1}{n}\} = \{n \in \mathbb{N} : \frac{y_2}{n+1} \geq \frac{1}{n}\} = \{n \in \mathbb{N} : 0 \geq \frac{1}{n}\} = \phi \in \mathcal{I}$, since $y_2 = 0$. Therefore $f_n \xrightarrow{\mathcal{I}-e} f$.*

Now we investigate some arithmetical properties of $\mathcal{I}$-equal convergent sequences of functions.

Theorem 3.5. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}-e} f$ then f is unique.*

Proof. If possible let f and g be two distinct $\mathcal{I}$-equal limit of $\{f_n\}_{n\in\mathbb{N}}$. Then there are two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and $\mathcal{I}\text{-}\lim_{n\to\infty} \gamma_n = 0$ and for any $x \in X$ and for any $z \in Y$, the sets $K_1 = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon_n\}$, $K_2 = \{n \in \mathbb{N} : \|f_n(x) - g(x), z\| \geq \gamma_n\} \in \mathcal{I}$. Therefore $K_1^c = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \varepsilon_n\}$, $K_2^c = \{n \in \mathbb{N} : \|f_n(x) - g(x), z\| < \gamma_n\} \in \mathcal{F}(\mathcal{I})$. Let $z \in Y$ be linearly independent with $f(x) - g(x)$. Put $\varepsilon = \frac{1}{2} \|f(x) - g(x), z\| > 0$. As $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and $\mathcal{I}\text{-}\lim_{n\to\infty} \gamma_n = 0$, the sets $K_3^c = \{n \in \mathbb{N} : \varepsilon_n < \varepsilon\}$, $K_4^c = \{n \in \mathbb{N} : \gamma_n <$

$\varepsilon\} \in \mathcal{F}(\mathcal{I})$. As $\phi \notin \mathcal{F}(\mathcal{I})$, $K_1^c \cap K_2^c \cap K_3^c \cap K_4^c \neq \phi$. Then there exists $m \in \mathbb{N}$ such that $m \in K_1^c \cap K_2^c \cap K_3^c \cap K_4^c$. Then $\|f_m(x) - f(x), z\| < \varepsilon_m$, $\|f_m(x) - g(x), z\| < \gamma_m$, $\varepsilon_m < \varepsilon$ and $\gamma_m < \varepsilon$. Now $\|f(x) - g(x), z\| = \|f(x) - f_m(x) + f_m(x) - g(x), z\| \leq \|f_m(x) - f(x), z\| + \|f_m(x) - g(x), z\| < \varepsilon_m + \gamma_m < \varepsilon + \varepsilon = \frac{1}{2}\|f(x) - g(x), z\| + \frac{1}{2}\|f(x) - g(x), z\| = \|f(x) - g(x), z\|$, which is absurd. Hence $\mathcal{I}$-equal limit f of the sequence $\{f_n\}_{n\in\mathbb{N}}$ must be unique if it exists. ∎

Theorem 3.6. *Let $f, f_n : X \to Y$ and $g, g_n : X \to Y$, $n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}-e} f$ and $g_n \xrightarrow{\mathcal{I}-e} g$, $f_n + g_n \xrightarrow{\mathcal{I}-e} f + g$.*

Proof. Since $f_n \xrightarrow{\mathcal{I}-e} f$ and $g_n \xrightarrow{\mathcal{I}-e} g$, there exist sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\rho_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n = 0$ and $\mathcal{I}\text{-}\lim_{n\to\infty} \rho_n = 0$ such that for $x \in X$ and $z \in Y$, we have $A_1 = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \xi_n\}$, $A_2 = \{n \in \mathbb{N} : \|g_n(x) - g(x), z\| \geq \rho_n\} \in \mathcal{I}$. So $A_1^c = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \xi_n\}$, $A_2^c = \{n \in \mathbb{N} : \|g_n(x) - g(x), z\| < \rho_n\} \in \mathcal{F}(\mathcal{I})$. As $\phi \notin \mathcal{F}(\mathcal{I})$, $A_1^c \cap A_2^c \neq \phi$.

Now let $n \in A_1^c \cap A_2^c$ and consider the set $A_3^c = \{n \in \mathbb{N} : \|f_n(x) + g_n(x) - \{f(x) + g(x)\}, z\| < \xi_n + \rho_n\}$. As $\|f_n(x) + g_n(x) - \{f(x) + g(x)\}, z\| \leq \|f_n(x) - f(x), z\| + \|g_n(x) - g(x), z\| < \xi_n + \rho_n$, therefore $n \in A_3^c$ i.e. $A_1^c \cap A_2^c \subset A_3^c$. So $A_3 \subset A_1 \cup A_2$. Since $A_1 \cup A_2 \in \mathcal{I}$, $A_3 \in \mathcal{I}$. i.e. $\{n \in \mathbb{N} : \|f_n(x) + g_n(x) - \{f(x) + g(x)\}, z\| \geq \xi_n + \rho_n\} \in \mathcal{I}$. As $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n + \rho_n = 0$, $f_n + g_n \xrightarrow{\mathcal{I}-e} f + g$. This proves the theorem. ∎

Theorem 3.7. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. Let $a(\neq 0) \in \mathbb{R}$. If $f_n \xrightarrow{\mathcal{I}-e} f$, $af_n \xrightarrow{\mathcal{I}-e} af$.*

Proof. Since $f_n \xrightarrow{\mathcal{I}-e} f$, there is a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \beta_n = 0$ such that for $x \in X, z \in Y$, the set $B_1 = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\beta_n}{|a|}\} \in \mathcal{I}$. Put $B_2 = \{n \in \mathbb{N} : \|af_n(x) - af(x), z\| \geq \beta_n\}$. As, $\|af_n(x) - af(x), z\| \geq \beta_n \Rightarrow \|f_n(x) - f(x), z\| \geq \frac{\beta_n}{|a|}$. Therefore $B_2 \subset B_1$. So $B_2 \in \mathcal{I}$. This proves the result. ∎

In [10] it has been proved for real valued functions that $\mathcal{I}$-uniform convergence implies $\mathcal{I}$-equal convergence. Now we investigate it in linear 2-normed spaces which will be needed in the sequel. First we give an important lemma which has been stated as remark in [24].

Lemma 3.8. *(cf.[24]) Let $f, f_n : X \to Y$, $n \in \mathbb{N}$. If $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-uniformly convergent to f then $\{\sup_{x\in X} \|f_n(x) - f(x), z\|\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero for all $z \in Y$.*

Proof. First we assume that $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-uniformly convergent to f. Then for any $\varepsilon > 0$ there exists $M \in \mathcal{I}$ such that for all $n \in M^c$ and for $x \in X$, $z \in Y$, $\|f_n(x) - f(x), z\| < \frac{\varepsilon}{2}$. This implies

$$\sup_{x\in X} \|f_n(x) - f(x), z\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

So the set $\{n \in \mathbb{N} : |\sup_{x\in X} \|f_n(x) - f(x), z\| - 0| \geq \varepsilon\} \subset M \in \mathcal{I}$, for all $z \in Y$. Therefore $\{\sup_{x\in X} \|f_n(x) - f(x), z\|\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero for all $z \in Y$. ∎

Theorem 3.9. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. $f_n \xrightarrow{\mathcal{I}-u} f$ implies $f_n \xrightarrow{\mathcal{I}-e} f$.*

Proof. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-uniformly convergent to f in Y, due to the Lemma 3.8 the sequence $\{u_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero where $u_n = \sup_{x\in X} \|f_n(x) - f(x), z\|$, for all $z \in Y$. Let $\varepsilon > 0$ be given. Then the set $B = \{n \in \mathbb{N} : u_n \geq \varepsilon\} \in \mathcal{I}$. Define $\xi_n = \begin{cases} \frac{1}{n}, & \text{if } n \in B \\ u_n + \frac{1}{n}, & \text{if } n \notin B \end{cases}$. We show $\{\xi_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to zero. For, let $\varepsilon_1 > 0$, we have $\{n : \xi_n \geq \varepsilon_1\} = \{n \in B : \xi_n \geq \varepsilon_1\} \cup \{n \in B^c : \xi_n \geq \varepsilon_1\} = \{n : \frac{1}{n} \geq \varepsilon_1\} \cup \{n : u_n + \frac{1}{n} \geq \varepsilon_1\} = M_1 \cup M_2$. Clearly M_1 is finite. If $n \in M_2$ then $n \in B^c$. So $u_n < \varepsilon$. Now $u_n + \frac{1}{n} \geq \varepsilon_1$ if $\frac{1}{n} \geq \varepsilon_1 - u_n$ i.e. if $\frac{1}{n} \geq \varepsilon_1 - \varepsilon$ which is for finite number values of n. Therefore M_2 is finite. As

$\mathcal{I}$ is admissible, $M_1 \cup M_2 \in \mathcal{I}$. Hence $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n = 0$. Now, for all $z \in Y$, we have $\|f_n(x) - f(x), z\| \leq \sup_{x\in X} \|f_n(x) - f(x), z\| < \sup_{x\in X} \|f_n(x) - f(x), z\| + \frac{1}{n} = u_n + \frac{1}{n} = \xi_n$ if $n \in B^c$ where $B \in \mathcal{I}$. Therefore $\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \xi_n\} \in \mathcal{I}$. As $\mathcal{I}\text{-}\lim_{n\to\infty} \xi_n = 0$, $f_n \xrightarrow{I-e} f$. Hence the theorem follows. ■

Now we intend to proceed with the notion of I^*-equal convergence in linear 2-normed spaces.

Definition 3.10. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-equal convergent to f if there exists a set $M = \{m_1 < m_2 < \cdots < m_k \cdots\} \in \mathcal{F}(\mathcal{I})$ and a sequence $\{\varepsilon_k\}_{k\in M}$ of positive reals with $\lim_{k\to\infty} \varepsilon_k = 0$ such that for every $x \in X$, there is a number $p \in \mathbb{N}$ and for every $z \in Y$, $\|f_{m_k}(x) - f(x), z\| < \varepsilon_k$ for all $k \geq p$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-e} f$.*

We proceed to investigate the relationship between $\mathcal{I}$-equal and $\mathcal{I}^*$-equal convergence in linear 2-normed spaces.

Theorem 3.11. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}^*-e} f$ then $f_n \xrightarrow{\mathcal{I}-e} f$.*

Proof. We assume $f_n \xrightarrow{\mathcal{I}^*-e} f$. Then there exist a set $M = \{m_1 < m_2 < \cdots < m_k \cdots\} \in \mathcal{F}(\mathcal{I})$ and a sequence $\{\varepsilon_k\}_{k\in M}$ of positive reals with $\lim_{k\to\infty} \varepsilon_k = 0$ such that for every $x \in X$, there is a number $p \in \mathbb{N}$ and for every $z \in Y$, $\|f_{m_k}(x) - f(x), z\| < \varepsilon_k$ for $k > p$. Then clearly $\|f_n(x) - f(x), z\| \geq \varepsilon_n$ holds for $n \in (\mathbb{N}\backslash M) \cup \{m_1, m_2, \cdots, m_p\}$. This implies $\{n : \|f_n(x) - f(x), z\| \geq \varepsilon_n\} \subset (\mathbb{N}\backslash M) \cup \{m_1, m_2, \cdots, m_p\}$. Since $\mathcal{I}$ is admissible, $\{n : \|f_n(x) - f(x), z\| \geq \varepsilon_n\} \in \mathcal{I}$. Hence $f_n \xrightarrow{\mathcal{I}-e} f$. ■

Remark 3.12. *The converse of the above theorem may not hold in general as shown by the following example.*

Example 3.13. *Consider a decomposition $\mathbb{N} = \bigcup_{i=1}^{\infty} D_i$ such that each D_i is infinite and $D_i \cap D_j = \phi$ for $i \neq j$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ which intersects only a finite number of $D_i^{'}s$. Then $\mathcal{I}$ is a non-trivial admissible ideal. Let $f, f_n : X \to Y, n \in \mathbb{N}$ such that $\{f_n\}_{n\in N}$ is uniformly convergent to f and $f_n \neq f$ for any $n \in \mathbb{N}$. Then for each $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $x \in X, z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$ for all $n > p$. Define a sequence $\{g_n\}_{n\in\mathbb{N}}$ by $g_n = f_j$ if $n \in D_j$. Then for all $x \in X, z \in Y$ the set $\{n \in \mathbb{N} : \|g_n(x) - f(x), z\| \geq \varepsilon\} \subset D_1 \cup D_2 \cup \cdots \cup D_p$. Therefore $\{n \in \mathbb{N} : \|g_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}$. Hence $g_n \xrightarrow{\mathcal{I}-u} f$. By the Theorem 3.9, $g_n \xrightarrow{\mathcal{I}-e} f$.*

Now we shall show that $\{g_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}^*$-equal convergent in Y. If possible let $g_n \xrightarrow{\mathcal{I}^*-e} f$. Now, by definition, if $H \in \mathcal{I}$, then there is a $p \in \mathbb{N}$ such that $H \subset D_1 \cup D_2 \cup \cdots \cup D_p$. Then $D_{P+1} \subset \mathbb{N} \setminus H$ and so we have $g_{m_k} = f_{p+1}$ for infinitely many of $k^{'}s$. Let $z \in Y$ be linearly independent with $f_{p+1} - f(x)$. Now we have $\lim_{n\to\infty} \|g_{m_k}(x) - f(x), z\| = \|f_{p+1}(x) - f(x), z\| \neq 0$. Which shows that $\{g_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}^*$-equal convergent in Y.

Now we see, if X and Y are countable and $\mathcal{I}$ satisfies the condition (AP) then the converse of the Theorem 3.11 also holds. In the next theorem we investigate whether the two concepts $f_n \xrightarrow{\mathcal{I}-e} f$ and $f_n \xrightarrow{\mathcal{I}^*-e} f$ coincide in linear 2-normed spaces when $\mathcal{I}$ is a P-ideal.

Theorem 3.14. *Let $f, f_n : X \to Y, n \in \mathbb{N}$ and let X and Y be countable sets. Then $f_n \xrightarrow{\mathcal{I}-e} f$ implies $f_n \xrightarrow{\mathcal{I}^*-e} f$ whenever $\mathcal{I}$ is a P-ideal.*

Proof. From the given condition there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \sigma_n = 0$ and for every $z \in Y$ and for each $x \in X$, there is a set $B = B(x, z) \in \mathcal{F}(\mathcal{I})$, $\|f_n(x) - f(x), z\| < \sigma_n$ for all $n \in B$. Now by Theorem 2.2, $\mathcal{I}^*\text{-}\lim_{n\to\infty} \sigma_n = 0$. So we will get a set $H \in \mathcal{F}(\mathcal{I})$ for which $\{\sigma_n\}_{n\in H}$ is convergent to zero. Since X and Y are countable sets, so $X \times Y$ is countable. So let us enumerate $X \times Y$ by $\{(x_i, z_i) : x_i \in X, z_i \in Y, i = 1, 2, \ldots\}$. So for each element $(x_i, z_i) \in X \times Y$, there is a set $B_i = B(x_i, z_i) \in$

$\mathcal{F}(\mathcal{I})$, we have $\|f_n(x_i) - f(x_i), z_i\| < \sigma_n$ for all $n \in B_i$. $\mathcal{I}$-being a P-ideal, there is a set $A \in \mathcal{F}(\mathcal{I})$ such that $A \setminus B_i$ is finite for all i. So for every $z \in Y$ and for all $n \in A \cap H$ except for finite number of values, we have $\|f_n(x) - f(x), z\| < \sigma_n$. Therefore $f_n \xrightarrow{\mathcal{I}^*-e} f$. Hence the theorem follows. ■

Theorem 3.15. *Let $f, f_n : X \to Y, n \in \mathbb{N}$. Suppose that $f_n \xrightarrow{\mathcal{I}-e} f$ implies $f_n \xrightarrow{\mathcal{I}^*-e} f$. Then $\mathcal{I}$ satisfies the condition (AP).*

Proof. Let $f, f_n : X \to Y, n \in \mathbb{N}$ such that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent to f and $f_n \neq f$ for any $n \in \mathbb{N}$. Then for each $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $x \in X, z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$ for all $n > p$. Suppose $\{M_1, M_2, \cdots\}$ be a class of mutually disjoint non empty sets from $\mathcal{I}$. Define a sequence $\{h_n\}_{n\in\mathbb{N}}$ by $h_n = \begin{cases} f_j, & \text{if } n \in M_j \\ f, & \text{if } n \in \mathbb{N} \setminus \bigcup_j M_j \end{cases}$. First of all we shall show that $h_n \xrightarrow{\mathcal{I}-u} f$. Let $\varepsilon > 0$ be given. Observe that the set $M = M_1 \cup M_2 \cup \ldots \cup M_p \in \mathcal{I}$ and for all $x \in X, z \in Y$, we have $\|h_n(x) - f(x), z\| < \varepsilon$ for all $n \in M^c$. i.e. $\{n \in \mathbb{N} : \|h_n(x) - f(x), z\| \geq \varepsilon\} \subset M_1 \cup M_2 \cup \cdots \cup M_p \in \mathcal{I}$. Therefore $h_n \xrightarrow{\mathcal{I}-u} f$. By the Theorem 3.9 we have $h_n \xrightarrow{\mathcal{I}-e} f$. So by the given condition $h_n \xrightarrow{\mathcal{I}^*-e} f$. Therefore there is a set $B \in \mathcal{I}$ such that

$$H = \mathbb{N} \setminus B = \{a_1 < a_2 < \cdots < a_k < \cdots\} \in \mathcal{F}(\mathcal{I}) \text{ and } h_{a_k} \xrightarrow{e} f. \tag{3.1}$$

Put $B_j = M_j \cap B$ $(j = 1, 2, \cdots)$. So $\{B_1, B_2, \cdots\}$ is a class of sets belonging to $\mathcal{I}$. Now $\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty}(M_j \cap B) = (B \cap \{\bigcup_{j=1}^{\infty} M_j\} \subset B$. Since $B \in \mathcal{I}$ it follows $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. Now from the equation 3.1 we see that the set M_j has a finite number of elements common with the set $\mathbb{N} \setminus B$. So $M_j \triangle B_j \subset M_j \cap (\mathbb{N} \setminus B)$. Therefore $M_j \triangle B_j$ is finite. Therefore $\mathcal{I}$ satisfies the condition AP. ■

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