



Mathematical analysis of mosquito population global dynamics using delayed-logistic growth

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Abstract

Malaria is a major public health issue in many parts of the world, and the anopheles mosquitoes which drive transmission are key targets for interventions. Consequently, a best understanding of mosquito populations dynamics is necessary in the fight against the disease. Hence, in this paper we propose a delayed mathematical model of the life cycle of anopheles mosquitoes by using delayed-logistic population growth. The model is formulated by inserting the time delay into the logistic population growth rate, that accounts for the period of growth from eggs to the last aquatic stage, which is pupae. Depending on the system parameters, we establish a threshold for survival and extinction of the anopheles mosquitoes population. Moreover, by choosing the time delay as a bifurcation parameter, we prove that the system loses its stability and a Hopf bifurcation occurs when time delay passes through some critical values. Finally, we perform some numerical simulations and the results are well in keeping with the analytical analysis.

Keywords

Mosquitoes population, delayed-logistic growth, malaria transmission, mathematical analysis.

AMS Subject Classification

65L12, 65M20, 65N40.

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1. Introduction

Many diseases such as malaria, dengue, West Nile virus etc, are transmitted by mosquitoes. To achieve a high level of effectiveness in reducing the mosquito population and accordingly the vector-borne diseases, a best understanding of mosquito populations dynamics is necessary [12, 18]. Mathematical models have been used for many years to gain insights into the complex underlying the global dynamic of the mosquito populations. And then, different perspectives of the mosquito growth mechanism such as density dependence, age

structure, seasonal variation have been raised and treated by several authors. Therefore, most of the earlier studies dealing with mosquito population dynamics are ordinary differential equations (ODE). Otherwise, it is well known that introducing the time delays into ODE models is an excellent approach to deal with the latency period in the vector growth dynamic. The last source of delays in vector-borne models comes from the adaptive maturation delays of the vector. Many vectors, which are arthropods, undergo several life stages before they reach adulthood and are able to transmit the disease. For instance, a mosquito's life cycle consists of three successive juvenile phases (egg, larva, pupa) before reaching the adult phase. It usually takes about 1-2 weeks before mosquitoes mature to adulthood, a time frame which is largely relative to the average lifespan of the mosquito. To account for this delay, delay-differential equation models with delay in recruitment are composed [11, 24]. Indeed, the three first stages are all aquatic and the adult stage is aerial. The newly emerged adult rests on the surface of the water for a short time to allow itself to dry and all to harden its body parts. The wings have to spread out and dry properly before the mosquito can fly.

Blood feeding and mating does not occur before a couple of days following the adults emergence.

Studies on the population structure of mosquitoes have important implications for the prediction and the assessment of the effects of many vector control strategies. In the existing literatures there are several papers which deal with mosquito populations growth dynamics. In [12], Koutou et al. have proposed an autonomous mathematical model of mosquito growth dynamics including the immature stages of the vectors. And more recently, by considering the climate effects and applying the theory of uniform persistence and the Floquet theory, Traoré et al. [24] have extended the study proposed by Koutou et al. Due to the complexity of the dynamics of mosquito populations, and since only the adult female mosquitoes are responsible for transmitting diseases, therefore in general, models only focus on describing the dynamics of adult female mosquitoes, [7, 17]. Usually, the total size of the mosquito population is treated as a constant value or the simplest model of population growth due to Malthus is used. This model is fairly unrealistic, especially for a long term population growth. Indeed, for a better growth, each of these aquatic stages needs some available sustaining resources which is called the carrying capacity of environment. All these reasons lead to the introduction of the classical logistic equation by P. F. Verhulst, [9, 18, 27] to describe populations growth in an environment with limited resources. Furthermore, many other factors enter in the development from the immature stages to the adult one. For instance, how long each stage lasts depends on both temperature and species characteristics. Hutchinson [9, 25] pointed out that the classical logistic population growth would be inappropriate for the description of population growth in the case where there is a time lag involved in the processes. He has then proposed in [10] a delay logistic-growth model to study the dynamics of single species.

Generally, two different techniques are usually used to determine the asymptotic stability of delay differential equations. One is to study the eigenvalues of the linearized equations. The other is the direct Lyapunov’s method, which is most frequently and simply used to establish the global stability [15, 22]. However, the main difficulty is to find suitable Lyapunov functionals for delay differential equations. In this work we propose a delay logistic growth model to investigate how delay affects the mosquitoes growth dynamics. Moreover, the stability of the model has been established by using the eigenvalues of the linearized system. The analysis presented in this paper improves and extends some earlier results on global stability for delay single species population models [2, 3, 20, 26].

The rest of the paper is organized as follows. In Section 2, we first derive the main model and then we give some results for our mathematical analysis in Section 3. Section 4 is devoted to the model analysis and the numerical results illustrations. Finally, we comment ours findings and conclude the paper in Section 5.

2. Mathematical formulation of the model

As described above, the complete metamorphosis of the mosquito entails going through four distinct stages of development, namely egg, larva, pupa that are aquatic states and adult mosquito stages that is terrestrial. Eggs can resist dessication and wait for several months before hatching. When stimulated or under suitable conditions, eggs hatch and give rise to larvae, who feed on small particles in the water, and take from 3 days to several weeks especially in tempered areas to develop fully and reach the pupa stage. Pupae still move in the water but do not feed anymore. This stage lasts 1-3 days and leads to emergence, that is the beginning of the aerial or adult stage. So, we have two life stages for mosquitoes with different nutritional and habitat requirements. However, for the sake of simplicity we consider the same natural death rate d_m for both aquatic state L_m and terrestrial state N_m (that are assumed to be only female mosquitoes).

Several works do exist that are addressing mosquito population dynamics modeling. However, in most of the previous models that have been proposed, the authors have explicitly considered each stage of the vector life cycle when formulating the compartmental model [14, 23, 25]. To study the impact of time on the dynamics, we skip certain steps in the mosquito’s biology and then represent the rest of the steps by a delay parameter. Furthermore, to be more realistic, we consider the bio-transition or mass transfer coefficient from the aquatic stage to the terrestrial state, that is v_m . Perhaps using $v_m \alpha_m L_m$ as a first term in the second equation will account for successful transition from L_m to N_m through the mass transfer coefficient v_m .

So, when we denote by τ , the maturation delay, then the classical logistic coefficient becomes $1 - \frac{L_m(t - \tau)}{K}$.

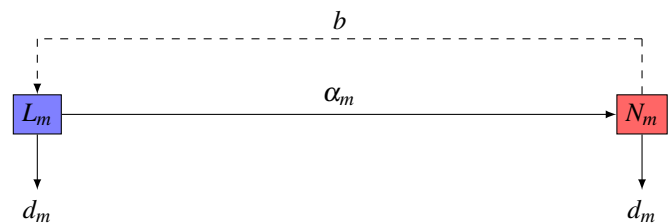


Figure 1. Transfer diagram of mosquito population growth: the dashed arrows indicate the direction of oviposition and the solid arrows represent the transition from one stage to another.

On the one hand, with regards to the different parameters that have been carefully described in Table 1 and on the other hand by making the input-output balance through the



Para.	Description
b	represents the intrinsic laying egg of the females,
α_m	is the transfer rate from the last immature stage to adult mosquito,
d_m	is the natural death rate,
v_m	is the bio-transition coefficient from L_m to N_m ,
τ	is the length of maturation delay from egg to pupa.

Table 1. Description of the model parameters

flowchart in Figure 1 we obtain the following system :

$$\begin{cases} L'_m(t) = bN_m(t) \left(1 - \frac{L_m(t-\tau)}{K} \right) - (d_m + \alpha_m)L_m(t), \\ N'_m(t) = v_m\alpha_m L_m(t) - d_m N_m(t), \end{cases} \quad (2.1)$$

3. Mathematical analysis of the model

Let us introduce the following initial conditions for system (2.1).

For any $\theta \in [-\tau, 0]$,

$$L_m(\theta) = \varphi_1(\theta), \quad N_m(\theta) = \varphi_2(\theta), \quad \varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C} \quad (3.1)$$

such that $\varphi_i(\theta) \geq 0$, ($i = 1, 2$) for all $\theta \in [-\tau, 0]$.

Let \mathcal{C} denotes the Banach space $C([-\tau, 0], \mathbb{R}_+^2)$ of continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R}_+^2 is equipped with the sup norm: $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi_i(\theta)|$, ($i = 1, 2$).

For a biological meaning, [13, 27], we suppose that, $\varphi_1(\theta) > 0$ and $\varphi_2(\theta) > 0$.

Proposition 3.1. *Let $X := C([-\tau, 0], \mathbb{R}_+^2 \setminus \{(0, 0)\})$. Then for any given initial conditions $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta)) \in X$, there is a unique solution of system (2.1) satisfying $(L_m(\theta), N_m(\theta)) = \varphi(\theta)$, $\theta \in [-\tau, 0]$, and the solution remains positive and bounded.*

Proof. We aim at establishing the boundedness of the solution. By method used in [3], one can show that for each $\varphi(\theta) \in X$, there is a unique solution of (2.1) through out $\varphi(\theta)$. Denote this solution by $(L_m(t, \varphi), N_m(t, \varphi))$. If $t_0 > 0$ is the first time that $L_m(t_0)$ and $N_m(t_0)$ become zero respectively, then $L'_m(t) \geq 0$ and $N'_m(t) \geq 0$ by (2.1), which is impossible. Therefore, the solution $(L_m(t), N_m(t))$ remains positive for all $t \geq 0$.

We now show that the solution is bounded. Let $M_1 = \frac{v_m\alpha_m K}{d_m}$ and $M_2 = \max_{-\tau \leq \theta \leq 0} \varphi_2(\theta)$.

If we denote $\bar{M} = \max\{M_1, M_2\}$, then we claim that $N_m(t) \leq \bar{M}$. It is obvious that $N_m(t) \leq M_2 \leq \bar{M}$ for $t \in [-\tau, 0]$. If t_1 is the first time at which $N_m(t)$ reaches \bar{M} , that is $N_m(t) = \bar{M}$. Then,

$$\begin{aligned} N'_m(t) &\leq v_m\alpha_m K - d_m\bar{M} \\ &\leq d_m(M_1 - \bar{M}) \\ &< 0 \end{aligned}$$

This implies that when $N_m(t)$ reaches \bar{M} , it will be decreasing. Therefore, $N_m(t) \leq \bar{M}$ for all $t \geq 0$. As $L_m(t) \leq K$ for all $t \geq 0$, it then follows that the solution is bounded. \square

Theorem 3.2. *The positive orthant*

$$\{(L_m, N_m) \in \mathbb{R}_+^2 : L_m \geq 0, N_m \geq 0\} \quad (3.2)$$

is positively invariant for system (2.1).

Proof. The right hand side of (2.1) is completely continuous and locally Lipschitzian on \mathcal{C} . So, from [8, 15] we deduce that the solution $(L_m(t), N_m(t))$ of (2.1) exists and is unique on $[0, t_{max}]$, where t_{max} represents the maximal time.

Let $x = (L_m, N_m)$. We will prove that the set $\{L_m \geq 0\}$ is positively invariant. Then, let $L(x) = -L_m$. L is differentiable and $\nabla L(x) = (-1, 0) \neq 0_{\mathbb{R}^2}$ for all $x \in L^{-1}(x) = \{x \in \mathbb{R}^2 : L(x) = 0\}$.

The vector field on the set $\{L_m = 0\}$ is given by $A(x) = \begin{pmatrix} bN_m \\ -d_mN_m \end{pmatrix}$. Then, $\langle A(x)/\nabla L(x), -1 \rangle = -bN_m < 0$; this proves that the set $\{L_m \geq 0\}$ is positively invariant. Similarly, we prove that $\{N_m \geq 0\}$ is positively invariant. \square

With regards to the initial conditions given above (3.1), the basic theory of delay differential equations guarantees that the solution of system (2.1) exists and is unique for all $t \geq 0$.

Let us consider the following threshold parameter which is the mosquito reproduction number

$$\mathcal{R}_m = \frac{bv_m\alpha_m}{d_m(d_m + \alpha_m)}. \quad (3.3)$$

The mosquito population growth dynamic is fully determined by the threshold parameter \mathcal{R}_m and commonly called mosquito reproduction number. It has been highlighted in several studies dealing with mosquito growth dynamic and vector-borne diseases. Mathematical studies have proven that this threshold parameter is highly involved in the transmission dynamics of arboviral diseases such as dengue fever, malaria, chikunguya [12, 16, 25].

Let τ_0 the critical delay value for the mosquito-free equilibrium. Since $(0, 0)$ is a single nonnegative equilibrium in the space X defined in Proposition 3.1 for $\tau < \tau_0$, according to the well-known Hayes theorem [3] and Theorem 2.3.1 of [20, 21], we have the following theorem:

Theorem 3.3.

- (i) *If $\mathcal{R}_m \leq 1$ and $\tau \leq \tau_0$ then system (2.1) admits mosquito-free equilibrium $(0, 0)$ which is globally asymptotically stable in $(\mathbb{R}_+^*)^2$.*
- (ii) *If $\mathcal{R}_m \leq 1$ and $\tau > \tau_0$ then the mosquito-free equilibrium is unstable.*



Proof. By linearizing system (2.1) at mosquito-free equilibrium (0, 0) we obtain that

$$\begin{cases} L'_m(t) = -(d_m + \alpha_m)L(t - \tau) + bN_m(t), \\ N'_m(t) = v_m\alpha_m L_m(t) - d_m N_m(t). \end{cases} \quad (3.4)$$

Seeking for solutions of system (2.1) of the form $(L_m(t), N_m(t)) = (L_m(0)e^{\lambda t}, N_m(0)e^{\lambda t})$ we are led to the following system

$$\begin{cases} -(d_m + \alpha_m)L_m(0)e^{\lambda t}e^{-\lambda\tau} + bN_m(0)e^{\lambda t} = \lambda L_m(0)e^{\lambda t} \\ v_m\alpha_m L_m(0)e^{\lambda t} - d_m N_m(0)e^{\lambda t} = \lambda N_m(0)e^{\lambda t}. \end{cases} \quad (3.5)$$

Canceling $e^{\lambda t}$ from each term and rearranging them gives

$$\begin{cases} -(d_m + \alpha_m + \lambda)L_m(0)e^{-\lambda\tau} + bN_m(0) = 0, \\ v_m\alpha_m L_m(0) - (d_m + \lambda)N_m(0) = 0. \end{cases} \quad (3.6)$$

This is equivalent to

$$0 = \Delta(\lambda) \begin{pmatrix} L_m(0) \\ N_m(0) \end{pmatrix} \quad (3.7)$$

where the matrix $\Delta(\lambda)$ is given by

$$\Delta(\lambda) = \begin{pmatrix} -(d_m + \alpha_m + \lambda)e^{-\lambda\tau} & b \\ v_m\alpha_m & -(d_m + \lambda) \end{pmatrix}. \quad (3.8)$$

There is nontrivial solution if and only if $\det(\Delta(\lambda)) = 0$. Thus, the characteristic polynomial equation for the linearized system (3.4) is

$$bv_m\alpha_m - (d_m + \lambda)(d_m + \alpha_m + \lambda)e^{-\lambda\tau} = 0. \quad (3.9)$$

The asymptotic stability of the zero solution is equivalent to the condition that all roots of the characteristic equation (3.9) have negative real parts.

If $\tau = 0$, then the characteristic equation (3.9) becomes

$$\lambda^2 + (\alpha_m + 2d_m)\lambda + d_m(d_m + \alpha_m) - bv_m\alpha_m = 0 \quad (3.10)$$

In this case the discriminant of equation (3.10) is given by

$$D = \alpha_m^2(1 - 4bv_m d_m) + 4d_m^2(1 - bv_m\alpha_m) + 4d_m\alpha_m$$

Assuming $d_m \leq \frac{1}{4}\alpha_m$ and $bv_m\alpha_m \leq 1$, we have the following roots

$$\lambda_1 = -(\alpha_m + 2d_m) - \sqrt{D}$$

and $\lambda_2 = -(\alpha_m + 2d_m) + \sqrt{D}$

It is obvious that $\lambda_1 < 0$. Otherwise, $\lambda_2 < 0$ means leads to $bv_m\alpha_m d_m(d_m + \alpha_m) < 0$ which is absurd since all the constant involved are positive. Thus, for $\tau = 0$ the mosquito-free equilibrium is unstable.

Now, suppose that $\tau > 0$. By corollary 2.4 in [19], it follows that if instability occurs for a particular value of the delay, a characteristic root of (3.9) must intersect the imaginary axis. Suppose that (3.9) has a purely imaginary root $i\omega$, with $\omega > 0$. Let

$$p_1 = \alpha_m + 2d_m, \quad q_1 = d_m(d_m + \alpha_m) \quad \text{and} \quad r_1 = bv_m\alpha_m. \quad (3.11)$$

It follows that

$$\lambda^2 e^{-\lambda\tau} + p_1 \lambda e^{-\lambda\tau} + q_1 e^{-\lambda\tau} - r_1 = 0. \quad (3.12)$$

Then, by developing and separating real and imaginary parts in (3.12), we have

$$(q_1 - \omega^2) \cos(\omega\tau) + p_1 \omega \sin(\omega\tau) = -r_1 \quad (3.13)$$

$$(q_1 - \omega^2) \sin(\omega\tau) + p_1 \omega \cos(\omega\tau) = 0. \quad (3.14)$$

Squaring both sides of (3.13) and (3.14) at a first time and adding them at a second time, the following equation is hold

$$\omega^4 + (p_1^2 - 2q_1)\omega^2 + q_1^2 - r_1^2 = 0. \quad (3.15)$$

This means that, there exists at least one root for the characteristic equation (3.12) which crosses the positive part of the imaginary axis of the complex plane. So, an instability occurs.

From the equation (3.14), we obtain a critical delay value for the mosquito-free equilibrium as follows

$$\tau_0 = \frac{1}{\omega} \left(\arctan \left(\frac{p_1 \omega}{\omega^2 - q_1} \right) + l\pi \right), \quad l \in \mathbb{N}, \quad q_1 - \omega^2 \neq 0. \quad (3.16)$$

□

Theorem 3.4. Let τ_1 be the critical time delay for the endemic equilibrium.

- (i) If $\mathcal{R}_m > 1$ and $\tau \leq \tau_1$ holds, the positive equilibrium $E^* = (L_m^*, N_m^*)$ of system (2.1) is globally asymptotically stable in \mathbb{R}_+^2 .
- (ii) If $\mathcal{R}_m > 1$ and $\tau > \tau_1$ holds, the positive equilibrium $E^* = (L_m^*, N_m^*)$ of system (2.1) is unstable.

Proof. We assume that $\mathcal{R}_m > 1$. The problem of Cauchy associated with system (2.1) is described by

$$\begin{cases} \frac{dF}{dt}(t) = F(X_t), \\ X_0 = \varphi, \end{cases} \quad (3.17)$$

where φ represents an initial condition defined in 3.1 and the function F is \mathcal{C}^∞ given by

$$F(\varphi) = \begin{pmatrix} b \left(1 - \frac{L_m(t-\tau)}{K} \right) N_m - (d_m + \alpha_m)L_m \\ v_m\alpha_m L_m - d_m N_m \end{pmatrix} \quad (3.18)$$

Then, the function is locally Lipschitzian. So, we deduce the existence and the uniqueness of the solution for (3.17).

Let us consider system (2.1) with delay and logistic equation. Since the presence of the delay does not change the number of the equilibrium solutions in the system, the existence of the equilibria follows from the same argument as for the ODE systems in [6]. So, $\bar{L}_m(t) = \bar{L}_m(t - \tau)$ and then we seek the equilibria of our system (2.1) by solving

$$b \left(1 - \frac{L_m^*}{K} \right) N_m^* - (d_m + \alpha_m)L_m^* = 0 \quad (3.19)$$

$$\alpha_m\alpha_m L_m^* - d_m N_m^* = 0. \quad (3.20)$$

From (3.20), we have

$$N_m^* = \frac{v_m\alpha_m}{d_m} L_m^*$$



and substituting it in (3.19) we obtain

$$L_m^* = K \left(1 - \frac{1}{\mathcal{R}_m} \right).$$

So, we find the same equilibrium point as in the ODE case.

The linearization of (2.1) at

$$E^* = \left(K \left(1 - \frac{1}{\mathcal{R}_m} \right), \frac{K v_m \alpha_m}{d_m} \left(1 - \frac{1}{\mathcal{R}_m} \right) \right)$$

gives

$$\begin{aligned} L_m'(t) &= -\frac{b v_m \alpha_m}{d_m} L_m(t - \tau) + \frac{b}{\mathcal{R}_m} N_m(t) \\ N_m'(t) &= v_m \alpha_m L_m(t) - d_m N_m(t). \end{aligned} \tag{3.21}$$

Positing

$$L_m(t) = L_m(0)e^{\lambda t} \quad \text{and} \quad N_m(t) = N_m(0)e^{\lambda t},$$

it follows that

$$\begin{cases} -\left(\lambda + \frac{b v_m \alpha_m}{d_m} \right) L_m(0)e^{-\lambda \tau} + \frac{b}{\mathcal{R}_m} N_m(0) = 0 \\ v_m \alpha_m L_m(0) - (\lambda + d_m) N_m(0) = 0. \end{cases} \tag{3.22}$$

System (3.22) can be rewritten as

$$0 = \Delta_1(\lambda) \begin{pmatrix} L_m(0) \\ N_m(0) \end{pmatrix} \tag{3.23}$$

where the matrix $\Delta_1(\lambda)$ is

$$\Delta_1(\lambda) = \begin{pmatrix} -\left(\lambda + \frac{b v_m \alpha_m}{d_m} \right) e^{-\lambda \tau} & \frac{b}{\mathcal{R}_m} \\ v_m \alpha_m & -(d_m + \lambda) \end{pmatrix} \tag{3.24}$$

There is nontrivial solution if and only if $\det(\Delta_1(\lambda)) = 0$. This implies that

$$(\lambda + d_m) \left(\lambda + \frac{b v_m \alpha_m}{d_m} \right) e^{-\lambda \tau} - d_m(d_m + \alpha_m) = 0 \tag{3.25}$$

If $\tau = 0$, then the characteristic equation (3.25) becomes

$$\lambda^2 + \left(d_m + \frac{b v_m \alpha_m}{d_m} \right) \lambda + b v_m \alpha_m - d_m(d_m + \alpha_m) = 0 \tag{3.26}$$

From (3.26), it follows that all the characteristic roots have negative real parts. And then, the endemic equilibrium is locally asymptotically stable.

Now, consider that $\tau > 0$. We let $\lambda = i\xi$, ($\xi > 0$) be a root of the characteristic equation (3.26). Then, positing

$$p = d_m + \frac{b v_m \alpha_m}{d_m}, \quad q = b v_m \alpha_m \quad \text{and} \quad r = d_m(d_m + \alpha_m)$$

we have

$$((i\xi)^2 + ip\xi + q) (\cos(\xi\tau) - i \sin(\xi\tau)) - r = 0. \tag{3.27}$$

Developing and separating equation (3.27) into real part and imaginary part, we have

$$(q - \xi^2) \cos(\xi\tau) + p\xi \sin(\xi\tau) = -r \tag{3.28}$$

$$(q - \xi^2) \sin(\xi\tau) - p\xi \cos(\xi\tau) = 0 \tag{3.29}$$

Squaring both sides of (3.28) and (3.29) at a first time and adding them at a second time, we obtain the following result

$$\xi^4 + (p^2 - 2q)\xi^2 + q^2 - r^2 = 0. \tag{3.30}$$

It implies that there is at least one positive root for the characteristic equation (3.10). In this case, it exists a critical delay value for what the endemic equilibrium is unstable.

From equation (3.29), we have the critical delay value as follows for $\xi\tau \neq k\pi, k \in \mathbb{N}$,

$$\tau_1 = \frac{1}{\xi} \left(\arctan \left(\frac{p\xi}{q - \xi^2} \right) + j\pi \right), \quad j \in \mathbb{N}, \quad q - \xi^2 \neq 0 \tag{3.31}$$

□

4. Some numerical results

In this section, we present a few samples of numerical simulations to illustrate our theoretical results using Matlab solver dde23. Stable, periodic and unstable solutions have been presented according to the different values of the delay τ . For the positive equilibrium, we have first used a value of the delay less than the critical value τ_1 and we have obtained stable solutions. Then, we have taken a delay value greater than the critical delay value τ_1 to show the occurrence of Hopf bifurcation. The following initial conditions have been used: $L_m(0) = 120$ and $N_m(0) = 60$.

Figure 2 shows that mosquito-free equilibrium is globally asymptotically stable whenever the threshold parameter \mathcal{R}_m is less than unity and the time delay is below the critical delay value τ_0 .

Figure 3 illustrates the global stability of the positive equilibrium. It is obtained by choosing parameter values that lead to \mathcal{R}_m greater than one and a delay value that is below to the critical delay τ_1 .

Increasing of the time delay value leads to the loss of monotonicity and to the appearance of oscillations and causes the system to lose stability and exhibit limit cycle behavior (see Figure 4). Then, as the length of the time delay is increasing, the mosquito population size is decreasing.

5. Discussions and conclusion

Time delays are integral parts of ecological systems, and as such, may help in explaining underlying mechanisms of complex population dynamics such as population fluctuations. Moreover, the choice of the delayed underlying ecological process is an important determinant of overall population dynamics. Hence, the idea developed in this study is addressing a very salient issue in mosquito control which in turn will be useful for the vector-borne diseases control problem. The equilibrium points of the system have been determined and the basic reproduction number \mathcal{R}_m of the mosquito population is defined. However, the stability of equilibria is one of the most important issues in the study of any model of single species population [2, 4, 5]. We have mainly investigated the effect of time delays on the stability of the model equilibria by using



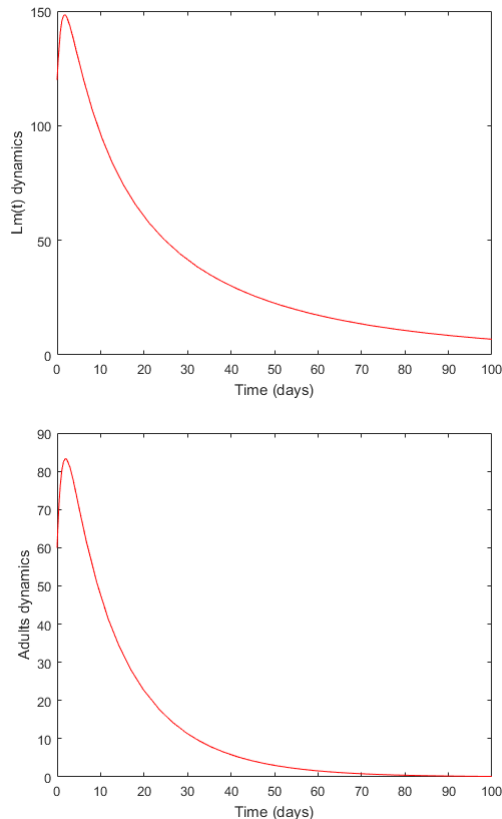


Figure 2. Evolution of eggs, larva, pupa and adults for $b = 2$, $d_m = 0.36$, $\alpha_m = 0.07$, $v_m = 0.27$, $K = 500$, $\tau = 0.64$. We get $\mathcal{R}_m = 0.90$ which is less than unity.

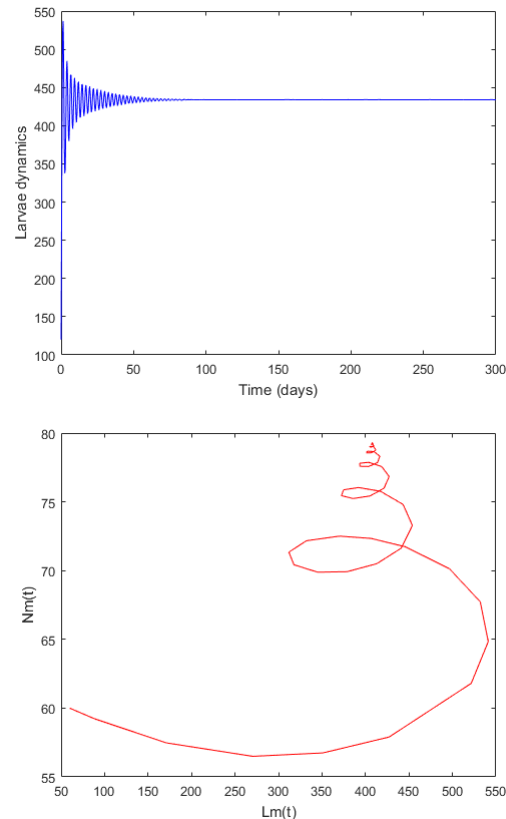


Figure 3. Evolution of larva and adults for $b = 12$, $d_m = 0.3$, $\alpha_m = 0.07$, $v_m = 0.27$, $K = 500$. We get $\mathcal{R}_m = 7.56$ which is greater than unity and $\tau = 0.64$ is less than the critical value τ_0 .

the eigenvalues of the linearized system method. Based on the numerical simulations presented in our paper, we found that large delays would make the steady states unstable because they would lead to oscillations and cyclic behavior that are not observed when delays are very small [1, 2, 11, 22]. However, a delay in negative feedback mechanisms such as intraspecific competition always results in an instability and thus leads to populations cycles or even to populations extinction [11].

Data Availability

Data supporting this publication are from previously published research articles.

Disclosure Policy

The author declare that there is no conflict of interest regarding the publication of this paper.

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Authors' Contributions

All the authors read and approved the final manuscript.

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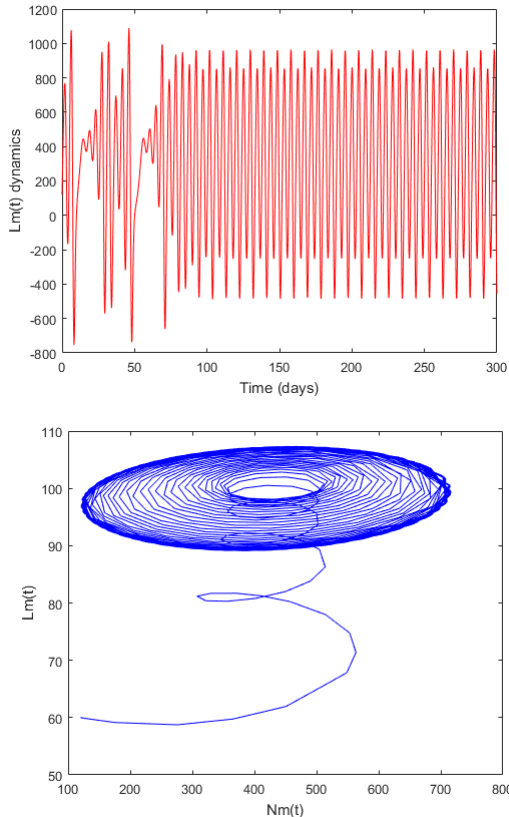


Figure 4. Evolution of larvae and adults for $b = 12$, $d_m = 0.3$, $\alpha_m = 0.07$, $v_m = 0.27$, $K = 500$. We get $\mathcal{R}_m = 7.56$ which is greater than unity and $\tau = 0.76$ is greater than the critical value τ_0 . Then, an instability occurs.

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