Malaya Journal of Matematik, Vol. 8, No. 4, 1906-1911, 2020
https://doi.org/10.26637/MJM0804/0095

Cellularity and representations of walled cyclic G-Brauer algebras

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Abstract
In this paper, we introduce a new class of diagram algebras which are subalgebras of cyclic G-Brauer algebras, called the Walled cyclic G-Brauer algebras denoted by $W_{r,s}(x)$, where $r,s \in \mathbb{N}$ and x is an indeterminate. The cellularity and the necessary and sufficient condition for $W_{r,s}(x)$ to be quasi- hereditary are established.

Keywords
G-Brauer algebra, walled Brauer algebra, cellular.

AMS Subject Classification
16G30.

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Article History: Received **17** July **2020**; Accepted **14** October **2020**

Contents

1. Introduction

Walled Brauer algebras is an algebra consisting of walled Brauer diagrams as basis which are studied in various aspects by [1, 3, 5, 16]. Walled Brauer algebra is a subalgebra of Brauer algebra introduced by [2].

This motivated Kethesan [9], to introduce walled signed Brauer algebras which are subalgebras of signed Brauer algebras introduced by [14]. Walled signed Brauer algebra is an algebra consisting of walled signed Brauer diagrams as basis.

In this paper, we generalize the algebra to walled cyclic G-Brauer algebra and study its generators, relations, cellularity and the necessary and sufficient condition for cyclic G-Brauer algebra to be quasi heriditary. Walled cyclic G-Brauer algebra is an algebra consisting of walled cyclic G-Brauer diagrams which are subalgebra of cyclic G-Brauer algebra introduced by [15].

2. Preliminaries

In this section, we give the fundamental definitions and theorems from [7].

Definition 2.1. *Let $q, Q_1, \dots, Q_r$ be elements of a commutative ring R with unity and q invertible. An associative unital R-algebra with generators $T_0, T_1, \dots, T_{n-1}$ is said to be Ariki-Koike algebra $\mathfrak{H}$ subject to the following relations*

$$\begin{aligned}
(T_0 - Q_1)\cdots(T_0 - Q_r) &= 0 \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\
(T_i + 1)(T_i q) &= 0 \text{ for } i = 1 \text{ to } n-1 \\
T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i \text{ for } i = 1 \text{ to } n-2 \\
T_iT_j &= T_jT_i \text{ for } 0 \le i < j-1 \le n-2
\end{aligned}$$

Definition 2.2. *Suppose that $\boldsymbol{a} = \{a_1, a_2, \dots, a_r\}$ is an $r-$ tuple of integers such that $0 \le a_i \le n$ for all i. Let $u_a^+ = u_{a,1}u_{a,2}\dots u_{a,r}$ where*

$$u_{a,k} = \prod_{m=1}^{a_k} (L_m - Q_k)$$

for $1 \le k \le r$ where $L_m = q^{1-m}T_{m-1}\cdots T_1T_0T_1\cdots T_{m-1}$ for $m = 1, 2, \dots, n$.

A finite sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of non-negative integers is called a composition. Denote $|\alpha|$ = the sum of this sequence. An ordered r-tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of compositions $\lambda^{(k)}$ such that $\sum_{k=1}^{r} |\lambda^{(k)}| = n$ is called a multicomposition of n. We call $\lambda^{(k)}$ the kth component of λ. A composition whose parts are non-increasing is called a partition. A multipartition if all its components are partitions is said to be multicomposition.

To each multicomposition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of n we associate the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \lambda^{(r)}$ of $\mathfrak{S}_n$.

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a composition of n. A Young diagram is an array of boxes having m left justified rows with row j containing λ_j boxes.

A λ-tableau $t = (t^{(1)}, \dots, t^{(r)})$, with $\lambda = (\lambda_1, \dots, \lambda_r)$ a multicomposition of n, is obtained from the Young diagram of λ by filling each boxes $1, 2, \dots, n$ without repetition. We call the tableaux $t^{(k)}$ the components of t.

A λ-tableau $t = (t^{(1)}, \dots, t^{(r)})$, with $\lambda = (\lambda_1, \dots, \lambda_r)$ is row standard if the entries in each row of each component increase from left to right, let $t \downarrow m$, with $1 \leq m \leq n$, denotes this multicomposition. A λ-tableau $t = (t^{(1)}, \dots, t^{(r)})$, with $\lambda = (\lambda_1, \dots, \lambda_m)$ is standard if the rows are increasing from left to right and columns are increasing from top to bottom in each component.

We say s dominates t $(s \trianglerighteq t)$ if $s \downarrow m \trianglerighteq t \downarrow m$ for all m with $1 \leq m \leq n$. If $s \trianglerighteq t$ and $s \neq t$ then $s \triangleright t$ where s and t are row standard λ-tableau and μ-tableau respectively.

Let t^λ be the λ-tableau where $1, 2, \dots, n$ appear in order along the rows of the first component, second component and so on. The row stabiliser t^λ is the Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$.

For a row standard λ-tableau s, let $s = t^\lambda d(s)$ where $d(s) \in \mathfrak{S}_n$. Then $d(s)$ is a distinguished right coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$.

Notation 2.3. *Let λ be a multicomposition of n and define $\boldsymbol{a} = \{a_1, a_2, \dots, a_r\}$ by $a_k = \sum_{i=1}^{k-1} a_i$. Let $x_\lambda = \sum w \in S_\lambda T_w$, $m_\lambda = u_a^+ x_\lambda$ and $m_{st} = T^*_{d(s)} m_\lambda T_{d(t)}$ where s and t are row standard λ-tableaux.*

Let N^λ be the $R-$ module spanned by m_{st} for all standard μ-tableaux s and t of n with $\mu \trianglerighteq \lambda$ for some multipartition μ of n.

Let $\overline{N^\lambda}$ be the $R-$ module spanned by m_{st} for all standard μ-tableaux s and t of n with $\mu \triangleright \lambda$ for some multipartition μ of n.

Let $z_\lambda = (\overline{N^\lambda} + m_\lambda)/\overline{N^\lambda}$. The Specht module S^λ is the submodule of $\mathfrak{H}/\overline{N^\lambda}$ given by $S^\lambda = z_\lambda \mathfrak{H}$.

Let $D^\lambda = S^\lambda / rad S^\lambda$.

Theorem 2.4. *[7] Suppose that R is a field. Then the non-zero $\mathfrak{H}$ modules in $\{D^\lambda | \lambda$ a multipartition of n$\}$ form a complete set of non-isomorphic irreducible $\mathfrak{H}-$modules. Moreover, each irreducible module D^λ is absolutely irreducible.*

Definition 2.5. *[8, 9]* Let A be an associative algebra over the field K. The associative algebra A is called a *cellular algebra* with cell datum (Λ, M, C, i) if following conditions are satisfied:

1. The finite set Λ is partially ordered. Associated with each $\lambda \in \Lambda$ there is a finite set $M(\lambda)$. The algebra A has an $K-$ basis $C^\lambda_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.
2. The map i is an $K-$ linear anti-automorphism of A with $i^2 = id$ which sends $C^\lambda_{S,T}$ to $C^\lambda_{T,S}$.
3. For each $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ and each $a \in A$, the product $aC^\lambda_{S,T}$ can be written as $\left(\sum_{U \in M(\lambda)} r_a(U, S) C^\lambda_{U,T}\right) + r'$, where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U, S) \in K$ do not depend on T.

An equivalent definition which does not use bases is as follows.

Definition 2.6. *[9, 11]* Let A be an algebra over a Noetherian commutative integral domain R. Assume there is an involution i on A. A two sided ideal J in A is called a cell ideal if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and there is an isomorphism of A-bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative.
The algebra A (with the involution i) is called cellular if and only if there is an R-module decomposition $A = J'_1 \oplus J'_2 \oplus \dots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for some j and such that setting $J_j = \oplus_{i=1}^{j} J'_i$ gives a chain of two-sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \dots \subset J_n = A$ (each of them fixed by i) and for each $j (j = 1, \dots\dots, n)$ the quotient $J'_j = J_j / J_{j-1}$ is a cell ideal(with respect to the involution induced by i on the quotient) of A/J_{j-1}.

Definition 2.7. *Given a k-algebra $\mathscr{C}$, a k-vector space V, and a bilinear form $\phi : V \otimes V \to \mathscr{C}$, Konig and Xi define a (possibly nonunital) algebra structure on $A^\phi_{\mathscr{C},V} = V \otimes V \otimes \mathscr{C}$ by setting the product of two basis elements to be*

$$(a \otimes b \otimes x).(c \otimes d \otimes y) = a \otimes d \otimes x\phi(b, c)y.$$

If i is an involution on $\mathscr{C}$ with $i(\phi(v, w)) = \phi(v, w)$ then there is an involution j on $A^\phi_{\mathscr{C},V}$ given by

$$j(a \otimes b \otimes x) = b \otimes a \otimes i(x).$$

The algebra $A^\phi_{\mathscr{C},V}$ is called the inflation of $\mathscr{C}$ along V.

Proposition 2.8. *[9, 12]* An inflation of a cellular algebra is cellular again. In particular, an iterated inflation of n copies of R is cellular, with a cell chain of length n.

Theorem 2.9. *[9, 12]* Any cellular algebra over R is the iterated inflation of finitely many copies of R. Conversely, any iterated inflation of finitely many copies of R is cellular.

Definition 2.10. *[9, 13]* Let A be a k-algebra. An ideal J in A is called a heredity ideal if J is idempotent, $J(rad(A))J = 0$ and J is a projective left(or, right) A-module. The algebra A is called quasi-hereditary provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \dots \subset J_n = A$ of ideals of A such that $J_j/J_{j-1}$is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a heredity chain of the quasi-hereditary algebra A.

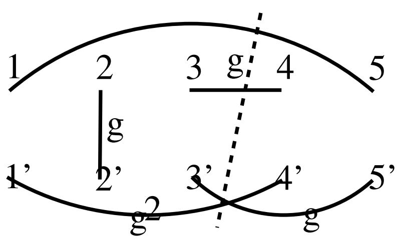
3. Walled cyclic G-Brauer algebra

An edge is said to be a **signed edge** if the edge is labelled by the elements of the cyclic group $\mathbb{Z}_k$. Hereafter, edge means signed edge.

Definition 3.1. *A* ***walled cyclic G-Brauer diagram*** *is a* ***signed diagram*** *with $r+s=n$ signed edges, $2(r+s)$ vertices arranged in two rows of $r+s$ vertices each which is separated by a wall between r and $n-r$ vertices in the top and bottom row such that*

- *degree of each vertex is one*
- *each edge consists of exactly two vertices*
- *the horizontal edge (the edge joining the vertices in the same row) must cross the wall and the vertical edge (the edge joining the vertices in different rows) should not cross the wall.*

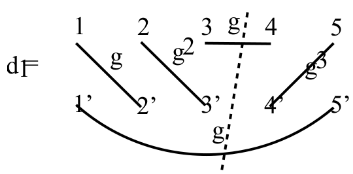
Let $V_{r,s}$ be the set of all walled cyclic G-Brauer diagrams. For example, the diagram in $V_{3,2}$ is

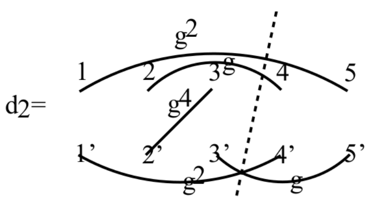


Multiplication in $V_{r,s}$

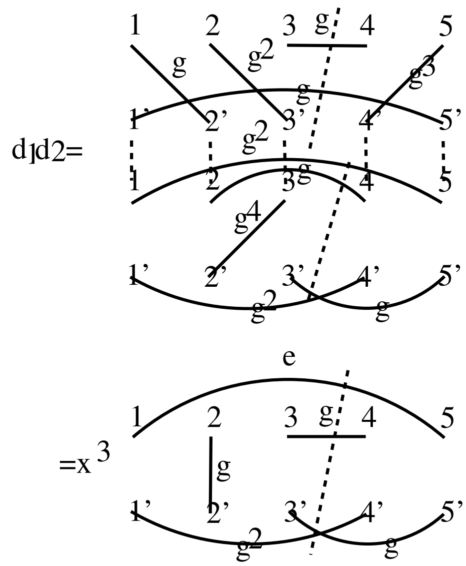
Let $d_1, d_2 \in V_{r,s}$. Draw d_2 below d_1 and join the vertex i of d_2 with vertex (i') of d_1. Then $d_1 d_2 = x^{\sum_i l_i} d$, where l_i is the number of loops labelled by g_i and new edge (new loop) obtained in the product $d_1 d_2$ is labelled by the product of group elements obtained from d_1 and d_2 to form this new edge (new loop).

Let $d_1, d_2 \in V_{3,2}$.





Then $d_1 d_2$ is



Definition 3.2. *The* ***walled cyclic G-Brauer algebra*** *$W_{r,s}(x)$ is a vector space spanned by $V_{r,s}$ over $\mathbb{F}(x)$, where $\mathbb{F}$ is any field of characteristic $p \geq 0$ which is algebraically closed and x is arbitrary.*

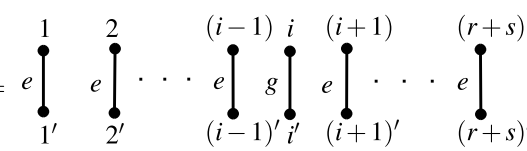
The group algebra $\mathbb{F}(\mathbb{Z}_k \wr S_{r+s})$, where S_{r+s} is a symmetric group of $(r+s)$ symbols and $\mathbb{Z}_k$ is a cyclic group of order k, can be viewed as walled cyclic G-Brauer diagram with no horizontal edge.

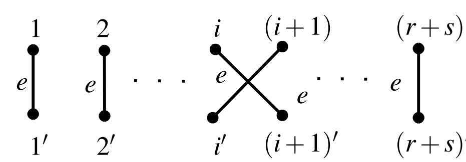
Now we can compare walled cyclic G-Brauer algebra $W_{r,s}(x)$ with $\mathbb{F}(\mathbb{Z}_k \wr S_{r+s})$.

Define a map $f : \mathbb{F}(\mathbb{Z}_k \wr S_{r+s}) \to W_{r,s}(x)$ by making the vertical edges crossing the wall as horizontal edges without changing the labelling. Clearly the map f is an isomorphism.

Hence, $\dim(W_{r,s}(x)) = \dim(\mathbb{F}(\mathbb{Z}_k \wr S_{r+s})) = k^{r+s}(r+s)!$

Let

$h_i =$ 

$s_i =$ 

where g is the generator of $\mathbb{Z}_k$ and e is the identity of $\mathbb{Z}_k$.

Recall, [DJM, AV, etc] $\mathbb{F}(\mathbb{Z}_k \wr S_{r+s})$ is generated by h_1 and s_i for $i=1$ to $(r+s-1)$ subject to the relation

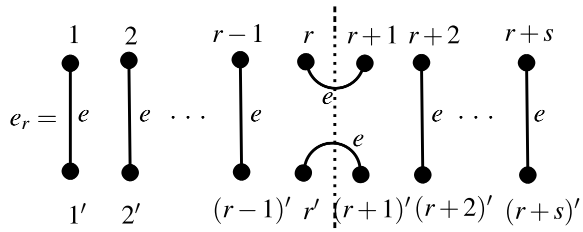
1. $h_1^k = 1$
2. $s_i^2 = 1$ for $i = 1$ to $r+s-1$
3. $s_i s_j = s_j s_i$ for $|i-j| \geq 2$

4. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i = 1$ to $\leq r+s-2$

5. $s_i h_1 = h_1 s_i$ for $2 \leq i \leq r+s-1$

6. $h_1 s_1 h_1 s_1 = s_1 h_1 s_1 h_1$

7. $s_i h_{i+1} = h_i s_i$



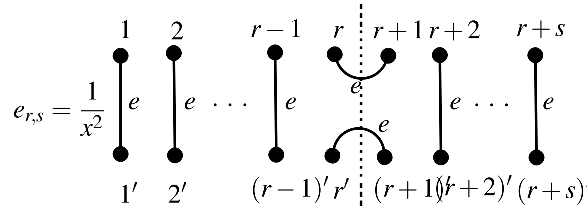
Theorem 3.3. *The walled cyclic G-Brauer algebra $W_{r,s}(x)$ is generated by the elements $h, h_{r+1}, s_1, \ldots, s_{r-1}, e_r, s_{r+1}, \ldots, s_{r+s-1}$ and satisfying the following relation:*

1. $s_i^2 = 1$ *for* $1 \leq i < r$ *and* $r+1 \leq i < r+s$
2. $s_i s_j = s_j s_i$ *for* $|i-j| > 1$
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ *for* $1 \leq i < r-1$ *and* $r+1 \leq i < r+s-1$
4. $h_i^k = 1$ *for* $1 \leq i \leq r+s$
5. $h_1 s_1 h_1 s_1 = s_1 h_1 s_1 h_1$
6. $h_{r+1} s_{r+1} h_{r+1} s_{r+1} = s_{r+1} h_{r+1} s_{r+1} h_{r+1}$
7. $h_1 s_i = s_i h_1$ *for* $i \neq 1$
8. $h_{r+1} s_i = s_i h_{r+1}$ *for* $i \neq r, r+1$
9. $s_i h_{i+1} = h_i s_i$ *for* $i \neq r$
10. $e_r^2 = x e_r$
11. $e_r s_i = s_i e_r$ *for* $1 \leq i < r-1, r+2 \leq i < r+s$
12. $e_r s_i e_r = e_r$ *for* $i = r-1, r+1$
13. $s_{r-1} s_{r+1} e_r s_{r-1} s_{r+1} e_r$ $= e_r s_{r-1} s_{r+1} e_r = e_r s_{r-1} s_{r+1} e_r s_{r-1} s_{r+1}$
14. $e_r h_r e_r = x e_r = e_r h_{r+1} e_r$
15. $e_r h_{r+1} = e_r h_r$
16. $h_{r+1} e_r = h_r e_r$
17. $e_r h_i = h_i e_r$ *for* $i \neq r, r+1$
18. $e_r h_r s_{r+1} e_r = e_r h_{r+2} e_r h_{r+1} s_{r+1} e_r$.

The Proof is similar as in [9] and [15]

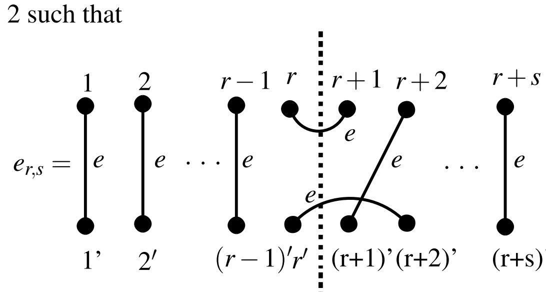
4. Cellularity of Walled cyclic G-Brauer algebra

Consider an arbitrary field F with $r, s > 0$ and $x \neq 0$ and let $e_{r,s} \in W_{r,s}(x)$ be such that



Therefore, $e_{r,s}$ is an idempotent.

If $x = 0$ then we define $\overline{e}_{r,s} \in W_{r,s}(x)$ for $r \geq 2$ or $s \geq 2$ such that



Clearly, $\overline{e}_{r,s}$ is an idempotent in $W_{r,s}(x)$.

Proposition 4.1. *1. For each $r, s > 0$, there is an algebra isomorphism Φ between $W_{r-1,s-1}(x)$ and $e_{r,s} W_{r,s}(x) e_{r,s}$ if $x \neq 0$.*

2. There is an algebra isomorphism between $W_{r-1,s-1}(x)$ and $\overline{e}_{r,s}\ W_{r,s}(x)\ \overline{e}_{r,s}$ for $x = 0$ and $r \geq 2$ or $s \geq 2.x$

The proof is similar to [4].

Now we define a sequence of idempotents $e_{r,s,i} \in W_{r,s}(x)$, for $x \neq 0$ and $r = s$.

Set $e_{r,s,0} = e_{r,s}$ and $e_{r,s,i} = \Phi_{r,s}(\vec{e}_{r-1,s-1,i-1})$, for $1 \leq i \leq min(r,s)$, where $\Phi_{r,s}$ is an algebra isomorphism between $W_{r-1,s-1}(x)$ and $e_{r,s} W_{r,s}(x) e_{r,s}$.

For $x = 0$ and $r = s$, we define associate quotients, $W_{r,s,i} = W_{r,s} / W_{r,s}\ e_{r,s,i}\ W_{r,s}$.

Consider a diagram $d \in W_{r,s}$ with a vertical edges to the left of the wall, b vertical edges to the right and the remaining $r-a$ and $s-b$ vertices are joined by horizontal edges. Define the pair (a,b) to be the propagating vector of $d \in W_{r,s}$.

Let $d_1, d_2 \in W_{r,s}$ with propagating vectors (a_1, b_1) and (a_2, b_2) respectively then $d_1 d_2 \in W_{r,s}$ must be a diagram with propagating vector (a,b) where $a \leq min(a_1, a_2)$ and $b \leq min(b_1, b_2)$.

Let $J_i = W_{r,s} e_{r,s,i} W_{r,s}$ then we get filtration of ideals

$$\cdots \subset J_i \subset J_{i-1} \subset \cdots \subset J_1 \subset J_0 = W_{r,s} \tag{4.1}$$

Thus, the ideal J_i has a basis of all diagrams with propagating vector (a,b) for some $a \leq r-i$ and $b \leq s-i$. In particular, the section J_i / J_{i+1} in the filtration of equation (4.1) has a basis of all diagrams with propagating vector $(r-i, s-i)$.

Clearly, we have

$$W_{r,s}/J_1 \cong F(\mathbb{Z}_k \wr S_r \times \mathbb{Z}_k \wr S_s). \tag{4.2}$$

We will denote $\mathbb{Z}_k \wr S_r \times \mathbb{Z}_k \wr S_s$ by $B_{r,s}$.

Definition 4.2. *A multi-partition $\mu \vdash qn$ is p-regular if it does not contain p equal parts in all the residue classes.*

Since $\mathbb{F}$ is algebraically closed, by Theorem 10.33 in [6], we have $\bigwedge_{reg}^{r,s} = \{(\lambda^L, \lambda^R) | \lambda^L, \lambda^R$ are p-regular multipartition of r and $s$respectively$\}$.

If $p = 0$ or $p > max(r,s)$ then the group algebra $\mathbb{F}B_{r,s}$ is semisimple, and $\bigwedge_{reg}^{r,s}$, consists of all pairs of partitions of r and s. Hence $\mathbb{F}$ is $B_{r,s}$-semisimple when $p = 0$ or $p > max(r,s)$.

Let $\bigwedge^{r,s}$ denote an indexing set for the simple $W_{r,s}$-modules. By Proposition 4.1, define

$$F_{r,s} : W_{r,s}\text{-mod} \to W_{r-1,s-1}\text{-mod}$$

by $F_{r,s}(M) = \bar{e}_{r,s}M$ where $M \in W_{r,s}$. Clearly $F_{r,s}$ is a left exact localisation functor and define $G_{r-1,s-1}$ in the opposite direction which takes a $W_{r-1,s-1}$-module N to $W_{r,s}e_{r,s} \otimes_{e_{r,s}B_{r,s}e_{r,s}} N$ to be the right exact globalisation functor. By standard properties of localisation functors and equation 4.2 we have for $r,s > 0$ that $\bigwedge_{r,s} = \bigwedge_{r-1,s-1} \sqcup \bigwedge_{r,s}^{reg}$.

As $W_{r,0} \cong W_{0,r} \cong \mathbb{F}(\mathbb{Z}_k \wr S_r)$ we deduce

Proposition 4.3. *For $x \neq 0$ or $r \neq s$, $\bigwedge_{r,s} = \bigsqcup_{i=0}^{min(r,s)} \bigwedge_{reg}^{r-i,s-i}$.*

For convenience, we see the walled cyclic G-Brauer diagrams as partial one-row diagrams.

Consider $d \in W_{r,s}$ with t signed horizontal edges. Let v represent the signed horizontal edge in the first row of d, w represent the signed horizontal edge in the second row of d and $\sigma_d = (\sigma, f) \in \mathbb{Z}_k \wr \Sigma_{r-t} \times \mathbb{Z}_k \wr \Sigma_{s-t}$ represent the signed vertical edge such that $\sigma(i) = j$ if the i^{th} vertex in the first row connected to the j^{th} lower vertex in the second row where $\sigma \in S_{r-t} \times S_{s-t}$ and $f : \{1, \ldots, r-t, r-t+1, \ldots, r+s-2t, \ldots, r+s-kt\} \longrightarrow \mathbb{Z}_k$ with $f(i) = j$, if the corresponding edge is labelled by g^j. Therefore $d = X_{v,w,\sigma_d}$ is unique.

Denote $\nu_{r,s,t} = \{v | v \in d = X_{v,w,\sigma_d}\}$, which is the set of partial one-row (r,s,t) diagrams.

Lemma 4.4. *For $l > 0$, the algebra J_l/J_{l+1} is isomorphic to an inflation $V_l \otimes V_l \otimes \mathbb{F}B_{r-l,s-l}$ of $\mathbb{F}B_{r-l,s-l}$ along a free $\mathbb{F}$-module V_l of rank $| \nu_{r,s,l} |$ with respect to some bilinear form, where V_l is the vector space over $\mathbb{F}(x)$ with basis $\nu_{r,s,l}$.*

Proof. The proof is similar to [5, 9, 10], we give it here for the sake of completion.

Consider a product $X_{u,v,\sigma_1}\dot{X}_{w,z,\sigma_2}$ for some $u, x \in \nu_{r,s,l}$ and $\sigma_1, \sigma_2 \in B_{r-l,s-l}$.

Define the map $\phi : V_l \times V_l \to \mathbb{F}B_{r-l,s-l}$ as

$$\phi(v,w) = \begin{cases} 0, & \text{if the product does not have} \\ & \text{propagating vector } (r-l, s-l); \\ x^t\sigma, & \text{otherwise} \end{cases}$$

where t is the number of closed loops in the product and σ is the unique permutation such that $X_{u,v,\sigma_1}\dot{X}_{w,z,\sigma_2} = x^t X_{u,z,\sigma_1\sigma\sigma_2}$.

Define the map $\psi : V_l \otimes V_l \otimes \mathbb{F}B_{r-l,s-l} \to J_l/J_{l+1}$ such that $v \otimes w \otimes \sigma \mapsto X_{v,w,\sigma}$ where V_l have basis $\nu_{r,s,l}$. Clearly ψ is bijective.

By the definition of inflation of $\mathbb{F}B_{r-l,s-l}$ along V_l, we have for $x = (u \otimes v \otimes \sigma_1)$, $y = (w \otimes z \otimes \sigma_2)$,

$$\begin{aligned} \psi(x\dot{y}) &= \psi(u \otimes z \otimes \sigma_1\phi(v,w)\sigma_2) \\ &= \psi(u \otimes z \otimes \sigma_1 x^t \sigma\sigma_2) \\ &= x^t X_{u,z,\sigma_1\sigma\sigma_2} \\ &= X_{u,v,\sigma_1}\dot{X}_{w,z,\sigma_2} \\ &= \psi(u \otimes v \otimes \sigma_1)\dot{\psi}(w \otimes z \otimes \sigma_2) \end{aligned}$$

Hence ψ is an algebra isomorphism. □

Lemma 4.5. *Let $d_1 \in J_m/J_{m+1}$ and $d_2 \in J_n/J_{n+1}$ be two diagrams in $W_{r,s}$ whose preimage is $u \otimes v \otimes \sigma_{d_1}$ and $w \otimes z \otimes \sigma_{d_2}$ respectively, under the bilinear forms for their respective layers. Then the product d_1d_2 is either an element of J_n/J_{n+1} (J_m/J_{m+1}) or is an element of J_{n+1} (J_{m+1}) if $n \geq m$ $(m \geq n)$.*

The proof follows as in lemma 4.4.

By the definition of involution, it is obvious that the involution on $W_{r,s}$ corresponds to the standard involution on $V_l \otimes V_l \otimes \mathbb{F}B_{r-l,s-l}$ which sends $v \otimes w \otimes \sigma$ to $w \otimes v \otimes \sigma^{-1}$.

Proposition 4.6. *The walled cyclic G-algebra $W_{r,s}(x)$ is an iterated inflation of group algebras of the form $B_{r-l,s-l}$ for $0 \leq l \leq min(r,s)$ along V_l.*

The proof is similar as in [9].

Theorem 4.7. *Let $W_{r,s}(x)$ be the walled cyclic G-Brauer algebra. Then*

1. *$W_{r,s}(x)$ is cellular with a cell module $\Delta_{r,s}(\lambda^L, \lambda^R)$ for each $(\lambda^L, \lambda^R) \in \bigwedge^{r-l,s-l}$ with $0 \leq l \leq min(r,s)$.*

2. *The simple modules of $W_{r,s}(x)$are indexed by all pairs $(l, \lambda^L, \lambda^R)$ where $0 \leq l \leq min(r,s)$, $(\lambda^L, \lambda^R) \in \bigwedge_{reg}^{r-l,s-l}$ if $x \neq 0$ or $r \neq s$.*

3. *The simple modules of $W_{r,s}(x)$are indexed by all pairs $(l, \lambda^L, \lambda^R)$ where $0 \leq l < min(r,s)$, $(\lambda^L, \lambda^R) \in \bigwedge_{reg}^{r-l,s-l}$ if $x = 0$ and $r = s$.*

Proof. (i) A cell basis for $\mathbb{F}B_{r,s}$ can be obtained as a product of cell basis for $\mathbb{F}(\mathbb{Z}_k \wr S_r)$ and $\mathbb{F}(\mathbb{Z}_k \wr S_s)$ from the cellular basis definition in [8, 10].

Hence $\mathbb{F}B_{r,s}$ is cellular with cell modules of the form $M \boxtimes N$, where M, N are cell modules for $\mathbb{F}(\mathbb{Z}_k \wr S_r)$ and $\mathbb{F}(\mathbb{Z}_k \wr S_s)$ respectively. The proof follows from proposition 4.6.

(ii) By Proposition 4.3, for $x \neq 0$ or $r \neq s$, the simple modules of $W_{r,s}$ are indexed by all pair $(l, \lambda^L, \lambda^R)$, where $0 \leq l \leq min(r,s)$ and $(\lambda^L, \lambda^R) \in \Lambda_{reg}^{r-l,s-l}$.

(iii) In the case of $x = 0$, the above assertion is also valid except that the case $l = 0$ (which occurs only for r even) does not contribute a simple module. □

Corollary 4.8. *The walled cyclic G-Brauer algebra $W_{r,s}(x)$ is quasi-hereditary, with heredity chain induced by the idempotent $e_{r,s,i}$ if $\mathbb{F}$ is $B_{r,s}$-semisimple, $r \neq s$ and either $x \neq 0$ or $x = 0$. In all other cases $W_{r,s}(x)$ is not quasi-hereditary.*

Proof. By the theorem we are having the same number of simples as cell modules for $r \neq s$ and either $x \neq 0$ or $x = 0$. Hence the cellular algebra is quasi-hereditary. In all other cases $W_{r,s}$ is not quasi-hereditary. □

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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