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# Some salient feature of topological simple ring

U. Jerseena[1]*, S. Syed Ali Fathima[2] and K. Alli[3]

**Abstract**
In this paper, we manifest the distinct feature of topological simple ring. A topological simple ring has the algebraic structure of ring and topological structure of a topological space. Further we provide a view of some basic results and theorem related to topological simple.

**Keywords**
Topological space, Continuous function, Topological ring, topological simple ring, ideals.

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54Hxx, 54C05.

[1] *Research Scholar, Department of Mathematics, Sadakathullah Appa College, Tirunelveli-627003, Tamil Nadu, India.[ Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.]*
[2] *Department of Mathematics, Sadakathullah Appa College, Tirunelveli-627011, Tamil Nadu, India.*
[3] *Department of Mathematics, The M.D.T Hindu College, Tirunelveli-627010, Tamil Nadu, India.*
***Corresponding author**: [1] jerseena.riyaz@gmail.com; [2]syedalifathima2014@sadakath.ac.in; [3]allimdt@gmail.com
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## Contents



## 1. Introduction

This paper attributes the concept of topological simple ring. Also here we elucidate some examples and basic results related to topological simple ring.

The concept of topological ring was introduced by D.Van Dantzig and N.Jacobson introduced the totally disconnected locally compact ring and Kalpanasy introduced the compact ring . Later the concept of topological ring was developed and studied by S. Warner[6]. Koteswara Rao introduced the topological 3-ring.

## 2. Preliminaries

In this section, we recall some definitions and basic results of Topology and algebra which will be used throughout the paper.

**Definition 2.1.** *[3] A topology on a set X is a collection T of subsets of X having the following properties*
*(i) $\phi$ and X are in T*
*(ii)The union of the elements of any subcollection of T is in T*
*(iii)The intersection of the elements of any finite subcollection of T is in T .*
*A set X for which a topology T has been specified is called a topological space.*

**Definition 2.2.** *[3] A subset U of X is an open set of X if U belongs to the collection T .The complement of open set is called closed set.*

**Definition 2.3.** *[3] Let X and Y be topological space. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

**Definition 2.4.** *[3] Let X and Y be the topological space; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a homeomorphism.*

**Definition 2.5.** *[6] A topology T on a ring A is a ring topology and A furnished with T is a topological ring if the following conditions hold*
*(i)$(x,y) \to x+y$ is continuous from $A \times A \to A$*
*(ii) $x \to -x$ is continuous from $A \to A$*
*(iii)$(x,y) \to xy$ is continuous from $A \times A \to A$ .*

## 3. Topological Simple Ring

**Definition 3.1.** *[2] A non-zero ring S whose only (two sided) ideals are S itself and zero is called simple ring.*

**Definition 3.2.** *A topological simple ring S is a simple ring which is also a topological space if the following conditions are satisfied*
*(i)The addition mapping $a : S \times S \to S$ defined by $a(s,t) = s+t$ $\forall s,t \in S$ is continuous*
*(ii) The additive inverse mapping $i : S \to S$ defined by $i(s) = -s$ $\forall s \in S$ is continuous*
*(iii)the multiplication mapping $m : S \times S \to S$ defined by $m(s,t) = st$ $\forall s,t \in S$ is continuous*
*(iv)the multiplicative inverse mapping $i_1 : S \to S$ defined by $i_1(s) = s^{-1}$ $\forall s \in S$ is continuous.*

**Example 3.3.** *Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be a simple ring ring under addition and multiplication we define a topology on S by $T = \left\{\phi, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.*
*Now $S \times S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ and $T \times T = \left\{\phi, \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\right\}$ Clearly (i),(ii),(iii) and (iv) are continuous .Therefore (S,+,. ,T) is a topological simple ring.*

**Example 3.4.** *[5] The ring of real number in the interval topology is a topological simple ring.*

**Example 3.5.** *The ring of complex number in the topology of the plane is a topological simple ring.*

**Example 3.6.** *[4] $Z_P$ (where p is prime) is a topological simple ring with discrete or indiscrete topology.*

**Remark 3.7.** *Topological simple ring $\Rightarrow$ Topological ring. Converse is not true.*

**Example 3.8.** *The set of integer Z is a topological ring with integer topology but it is not topological simple ring.*

**Theorem 3.9.** *Let $(S,T)$ be a topological simple ring and let $s \in S$. Then (i)the map $R_s : S \to S, x \to x+s$ and $L_s : S \to S, x \to s+x$ are homeomorphism. (ii) The additive inverse map $x \to -x$ is homeomorphism.*

*Proof.* (i) Assume that $s \in S$, then the element $x-s$ to $x$. So $R_s$ is surjective . Take $R_s(x) = R_s(y) \Rightarrow x+s = y+s \Rightarrow x = y$. $R_s$ is bijective. Since $R_s : S \to S$ is equal to composition, $S \xrightarrow{f_s} S \times S \xrightarrow{a} S$ where $f_s(x) = (s,x) \forall x \in S$. Let $J \times K$ is a basis open set in $S \times S$. Now $f_s^{-1}(J \times K)$
$= f_s^{-1}(\prod_{s_1}^{-1}(J) \bigcap \prod_{s_2}^{-1}(K)) = (\prod_{s_1} \circ f_s)^{-1}(J) \bigcap (\prod_{s_2} \circ f_s)^{-1}(k)$
Since $\prod_{s_1} \circ f_s$ and $\prod_{s_2} \circ f_s$ are continuous ,$(\prod_{s_1} \circ f_s)^{-1}$ and $(\prod_{s_2} \circ f_s)^{-1}$ are open in $S$. $f_s^{-1}(J \times K)$ is open in S. $f_s$ is continuous. $R_s$ is continuous. Similarly $R_s^{-1}$ is continuous. $R_s$ is homeomorphism. Similarly $L_s$ is homeomorphism.

(ii)Let $i(x) = i(y) \Rightarrow -x = -y \Rightarrow x = y$ . $i$ is one-one. For each $x \in S$, there exist $-x \in S$ such that $i(-x) = -(-x) = x$. $i$ is onto. Since $S$ is topological simple ring, $i$ is continuous. Also $i^{-1}$ is continuous. $i$ is homeomorphism. □

**Theorem 3.10.** *Let $(S,T)$ be a topological simple ring. Then (i)for each $s \in S$, the mapping,$L_s : S \to S, L_s(x) = sx$ ; $R_s : S \to S$, $R_s(x) = xs$ are continuous. If s is invertible, they are homeomorphism*
*(ii)the multiplicative inverse $x \to x^{-1}$ is homeomorphism*
*(iii)the inner automorphism is homeomorphism.*

**Theorem 3.11.** *Let S be a topological simple ring .Then the function $S \times S \to S, (x,y) \to yx$ is continuous.*

*Proof.* Let $m : S \times S \to S$ is defined by $m(x,y) = xy$ .Since S is topological simple ring, then m is continuous. The function $n : S \times S \to S \to S$ is defined by $n(x,y) = (y,x)$ is also continuous.The composition map $m \circ n : S \times S \to S$ is defined by $(x,y) = yx$ is continuous. □

**Corollary 3.12.** *Let S be a topological simple ring and J be an open subset of S, K is closed in S and s be any element of S.Then (i)$s+J, J+s, sJ$ and $Js$ are open in S*
*(ii)$s+K, K+s, Ks$ and $sK$ are closed in S.*

*Proof.* (i) By theorem 3.9, $L_s$ and $R_s$ are open map, $L_s(J) = s+J$ and $R_s(J) = J+s$ are open. By theorem 3.10, $L_s$ and $R_s$ are open map, $L_s(J) = sJ$ and $R_s(J) = Js$ are open. (ii)proof is analog to (i). □

**Corollary 3.13.** *Let S be a topological simple ring and $\Delta_i$ be the collection of all open sets of S at i. Then (i) $\Delta_s = \{J+s/J \in \Delta_i\}$ is also a collection of open set at s. (ii)$\Delta_s = \{Js/J \in \Delta_i\}$ is also a collection of open sets at s.*

**Theorem 3.14.** *Let S be a topological simple ring, J be an open subset of S and L be any subset of S. Then(i) $J+L$ (respectively $L+J$) and $JL$(respectively $LJ$) are open in S.*
*(ii)$-J, J^{-1}$ are open in S.*

*Proof.* (i)Let $s \in S$, By corollary 3.12(i), $s+J, s+J$ are open in $S$. Then $J+L = J+(\bigcup_{s \in L} s)$ is open in $S$. Similarly $JL$ are open in $S$. (ii)By theorem 3.9, the additive inverse map is open, $-J$ is open in $S$. Similarly $J^{-1}$ is open in $S$. □

**Corollary 3.15.** *Let S be a topological simple ring, K be an closed subset of S and M be any subset of S.Then (i)$K+M$ (respectively $M+K$) and $KM$(respectively $MK$) are closed in S.(ii)$-K, K^{-1}$ are closed in S.*

**Definition 3.16.** *Let S be a topological simple ring and $\Delta_i$ be the collection of all open set of S. If for every open neighbourhood J of i in $\Delta_i$ , then there exist a neighbourhood K of i in $\Delta_i$ such that $K \subseteq J$.*

**Theorem 3.17.** *Let S be a topological simple ring and $\Delta_i$ be the collection of all open neighbourhood of i. Then*
*(i) for each $J \in \Delta_i$, there is an element $K \in \Delta_i$ such that $-K \subseteq J$*

*and $K^{-1} \subseteq J$.*
*(ii) for each $J \in \Delta_i$ and $s \in J$, there is an element $K \in \Delta_i$ such that $s+K \subseteq J$ and $K+s \subseteq J$.*
*(iii) for each $J \in \Delta_i$ and $s \in J$, there is an $K \in \Delta_i$ such that $sK \subseteq J$ and $Ks \subseteq J$*

*Proof.* (i) Since $S$ is a topological simple ring , for each $J \in \Delta_i$ , there exist $K \in \Delta_i$ such that $i(K) = -K \subseteq J$ because additive inverse mapping is continuous. Similarly we prove that $K^{-1} \subseteq J$. (ii)By theorem 3.9, $L_s$ and $R_s$ are homeomorphism , for each open set $J$ containing $s$, there exist a open set $K$ at $i$ such that $L_s(K) = s+K \subseteq J$. Similarly$R_s(K) = K+s \subseteq J$. (iii)By theorem 3.10, proof is similar to (ii). □

**Theorem 3.18.** *Let $S$ be a topological simple ring. Every neighbourhood $J$ of $i$ containing an open neighbourhood $K$ of $i$ such that $K+K \subseteq J$ and $KK \subseteq J$.*

*Proof.* Let $J \in \Delta_i$ and $J$ is open in $S$. Since addition mapping is continuous, $a^{-1}(J)$ is open in $S \times S$. Then there exist $K_1, K_2 \subseteq J$ such that $(i,i) \in K_1 \times K_2$ and $K_1 + K_2 \subseteq J$. Let $K = K_1 \bigcap K_2$ which is open contain i and which satisfies $K_1 + K_2 \subseteq J$. Similarly $KK \subseteq J$. □

**Corollary 3.19.** *Let $S$ be a topological simple ring. Every neighbourhood $J$ of $i$ contains a neighbourhood $K$ of $i$ such that $K-K \subseteq J$ and $KK^{-1} \subseteq J$.*

**Theorem 3.20.** *Let $S$ be a topological simple ring. If $\{i\}$ is the intersection of the neighbourhood $J$ of $i$. Then $S$ is Hausdorff.*

*Proof.* Let $\{i\}$ be the intersection of neighbourhood Jof $i$. Let $s,t \in S$ and $s \neq t$. Then $s-t \neq J$. Since $s-t \neq J$, then there exist a neighbourhood K of $i$ such that $K+K \subseteq J$. Now $K+s$ is open and contain $s$ and $K+t$ is open and contain $t$. Therefore $(K+s)\bigcap(K+t) = \phi$. Otherwise if $u \in (K+s)\bigcap(K+t)$, then $s-t = -(u-s)+(u-t) \in K+K \subseteq J$. $s-t \in J$ which is contradiction. □

**Theorem 3.21.** *Every topological simple ring is regular.*

*Proof.* Let $J$ be an open set containing $i$. Let $j \in \bar{K}$. Then $K_j \cap K \neq \phi, k_1 J = k_2$ for some $k_1, k_2 \in K$. Therefore $j = k_1^{-1} k_2 \in K_1^{-1} K \subseteq J$. □

**Theorem 3.22.** *Let $\aleph$ be an index set for each $\vartheta \in \aleph$, let $S_\vartheta$ be a topological simple ring Then $S = \prod_{\vartheta \in \aleph} S_\vartheta$ is also topological simple ring.*

*Proof.* Let $L$ be a neighbourhood of $s-t$ in $S$, the there exist an open set $J$ such that $s-t \in J \subseteq L$ where $J = \prod_{\vartheta \in \aleph} J_\vartheta$ with $J_\vartheta$is an open set $J$ such that $s-t \in J \subseteq L$ where $J = \prod_{\vartheta \in \aleph} J_\vartheta$ with $J_\vartheta$ is an open neighbourhood of $s_\vartheta - t_\vartheta$ in $S_\vartheta$. Since $(s_\vartheta, t_\vartheta) \to s_\vartheta - t_\vartheta$ is continuous for each $\vartheta \in \aleph$, there exist neighbourhood $K_{\vartheta_r}, K'_{\vartheta_r}$ of $s_{\vartheta_r}$ and $t_{\vartheta_r}$ respectively such that $K_{\vartheta_r} - K'_{\vartheta_r} \subseteq J_{\vartheta_r}$ for each $1 \leq r \leq n$. Now let $K = \prod_{\vartheta \in \aleph} K_\vartheta$ and $K' = \prod_{\vartheta \in \aleph} K'_r$ then $K$ and $K'$ are neighbourhood of s and t respectively. Therefore $K - K' = \prod_{\vartheta \in \aleph} (K_{\vartheta_r} - K'_{\vartheta_r}) \subseteq \prod_{\vartheta \in \aleph} J_\vartheta \subseteq L$.

Let $L$ be a neighbourhood of $st^{-1}$ in $S$, then there exists an open set $J$ such that $st^{-1} \in J \subseteq L$, where $J = \prod_{\vartheta \in \aleph} J_\vartheta$ with $J_\vartheta$ is an open neighbourhood of $s_\vartheta t_\vartheta^{-1}$ in $S_\vartheta$. Since $(s_\vartheta, t_\vartheta) \to s_\vartheta t_\vartheta^{-1}$ is continuous, for each $\vartheta \in \aleph$, there exist neighbourhood $K_{\vartheta_r}, K'_{\vartheta_r}$ of $s_{\vartheta_r}$ and $t_{\vartheta_r}$ respectively such that $K_{\vartheta_r} K'_{\vartheta_r} \subseteq J_{\vartheta_r}$ for each $1 \leq r \leq n$. Now Let $K = \prod_{\vartheta \in \aleph} K_\vartheta$ and $K' = \prod_{\vartheta \in \aleph} K'_\vartheta$. Therefore $KK'^{-1} = \prod_{\vartheta \in \aleph} K_{\vartheta_r} K'^{-1}{}_{\vartheta_r} \subseteq \prod_{\vartheta \in \aleph} J_\vartheta = J \subseteq L$. $S = \prod_{\vartheta \in \aleph} S_\vartheta$ is a topological simple ring . □

## 4. Algebraic properties of topological simple ring

**Theorem 4.1.** *Let $S$ be a topological simple ring and $\Delta$ be the collection of all open neighbourhood of $i$ .Then the intersection of all open neighbourhood of $i(\cap J_i)$ is an ideal of $S$.*

*Proof.* (i)Let $j$ and $k$ be two elements of $\cap J_i$. For any open neighbourhood $J$ of $i$, then there exist neighbourhood $K$ of $i$ such that $K-K \subseteq J$. Since $j$ and $k$ are in $K$, $j-k \in A$. Therefore $j-k \in \cap j_i$.
(ii) For any neighbourhood $J$ of $i$ , then there exist a neighbourhood $K$ of $i$ such that $Ks, sK$ is in $J$. Since $j$ is in $K$, $js, sj$ in $J$. Hence $js, sj$ is in $\cap J_i$. Therefore $\cap J_i$ is an ideal of $S$. □

**Theorem 4.2.** *Let $S$ be a topological simple ring and $R$ is an ideal of $S$. Then the closure of $R$ is also an ideal.*

*Proof.* (i) Suppose $R$ is an ideal of $S$. The closure of $R = \{i \in S/$every neighbourhood of $i$ intersect $R\}$. Let $i_1, i_2$ belongs to the closure of $R$. Then every neighbourhood of $i_1, i_2$ intersects $R$. Suppose $J$ is a neighbourhood of $i_1 + i_2$. By theorem 3.18, then there exist a neighbourhood $K$ of $i_1$, neighbourhood $L$ of $i_2$ such that $K+L \subseteq J$. Since $K$ intersects $R$ and $L$ intersects $R$, then $K+L$ intersects $R$ and $J$ intersects $R$. Therefore $i_1 + i_2$ belongs to the closure of $R$. (ii) Let $i$ belongs to the closure of $R$, $s \in S$. Since $i$ belongs to the closure of $R$ , there exist neighbourhood $J$ of $i$ intersect $R$. By theorem 3.19, then there exist neighbourhood $K$ of $i$ such that $Ks \subseteq J$. Since $J$ intersect $R, Ks, sK$ intersect $R$. So $i \in K$ and $is \in J$.Hence $is$ belongs to closure of $R$. □

**Theorem 4.3.** *Every maximal ideal of a topological simple ring is closed.*

*Proof.* Let $M$ be a maximal ideal.By above theorem , the closure of $M$ is an ideal and the closure of $M$ contains $M$. Hence $M$ is closed. □

**Remark 4.4.** $\{0\}$ *is only one maximal ideal of topological simple ring which is closed.*

## 5. Conclusion

In this paper, we developed topological simple ring.The concept is further elaborated with examples counter examples.

Moreover some constancy results and properties of topological simple ring are characterized and explained throughout the paper.

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