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Edge induced V_4- magic labeling of graphs

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Abstract

Let $V_4 = \{0,a,b,c\}$ be the Klein-4-group with identity element 0 and $G = (V(G),E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \to V_4$ denote the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in (V(G)$. Then f^+ again induces an edge labeling $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$. Then a graph $G = (V(G),E(G))$ is said to be an edge induced V_4-Magic graph if f^{++} is constant function. The function f, so obtained is called an edge induced V_4-Magic labeling of G. In this paper we discuss edge induced V_4 magic labeling of some graphs.

Keywords

Klein-4-group, edge induced V_4-magic graphs, edge induced magic labeling

AMS Subject Classification

05C78, 05C25.

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Contents

1. Introduction

In this paper we consider simple, connected, finite and undirected graphs and the Klein 4-group, denoted by $V_4 = \{0,a,b,c\}$, which is a noncyclic Abelian group of order 4 in which every nonidentity element has order 2. We refer to Frank Harary [1] for the standard terminology and notations related to graph theory.

Let $G = (V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex v in G is the number of edges incident at v and it is denoted as $deg(v)$. Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \to V_4$ denote the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in (V(G)$. Then f^+ again induces an edge labeling $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$. Then a graph $G = (V(G),E(G))$ is said to be an edge induced V_4-magic graph or simply edge induced magic graph if $f^{++}(e)$ is a constant for all $e \in E(G)$. If this constant is x, then x is said to be the induced edge sum of the graph G. The function f so obtained is called an edge induced V_4-magic labeling of G or simply edge induced magic labeling of G and it is denoted by EIMV_4L or simply EIML. In this paper we discuss edge induced V_4-Magic labeling of some graphs which belongs to the following categories:

(i) $\sigma_a(V_4) :=$ Set of all edge induced V_4-magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = a$ for all $u \in V$.

(ii) $\sigma_0(V_4) :=$ Set of all edge induced V_4-magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = 0$ for all $u \in V$.

(iii) $\sigma(V_4) := \sigma_a(V_4) \bigcap \sigma_0(V_4)$.

Figure 1 and Figure 2 represent edge induced V_4 magic labeling of graph G_1 and G_2 with induced edge sums 0 and a respectively.

2. Main results

Theorem 2.1. *Let $G = (V,E)$ be a graph with either each vertex is of odd degree or even degree then $G \in \sigma_0(V_4)$.*

Proof. Let G be a graph with $deg(v_i) = r_i$ for $v_i \in V$, $i = 1,2,3,\ldots,n$.

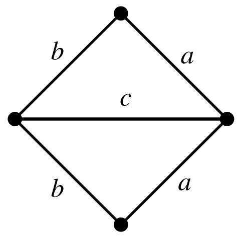

Figure 1. Graph G_1

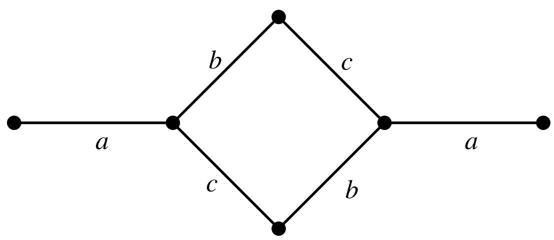

Figure 2. Graph G_2

Case 1: r_i is odd.
In this case, define $f : E \to V_4 \setminus \{0\}$ as $f(e) = a$ for all $e \in E$. Then $f^+(u_i) = deg\ (u_i)a = r_i a = a$. Thus $f^{++}(e) = 0$ for all $e \in E$.

Case 2: r_i is even.
In this case, define $f : E \to V_4 \setminus \{0\}$ as $f(e) = a$ for all $e \in E$. Then $f^+(u_i) = deg\ (u_i)a = r_i a = 0$. Thus $f^{++}(e) = 0$ for all $e \in E$.

Thus in both cases $f^{++} \equiv 0$. Therefore $G \in \sigma_0(V_4)$.
Hence the proof. □

Theorem 2.2. *Let $G = (V,E)$ be a graph with $uv \in E$ and $f : E \to V_4 \setminus \{0\}$ be an edge label of G then $f^{++}(uv) = \sum\limits_{u\alpha \in E} f(u\alpha) + \sum\limits_{\beta v \in E} f(\beta v)$, where $\alpha \neq v$ and $\beta \neq u$.*

Proof. Let $f : E \to V_4 \setminus \{0\}$ be an edge label of G, then $f^+(u) = \sum\limits_{u\alpha \in E} f(u\alpha)$ for all $u \in V$. Thus we have:

$$\begin{aligned} f^{++}(uv) &= f^+(u) + f^+(v) \\ &= \sum_{u\alpha \in E} f(u\alpha) + \sum_{\beta v \in E} f(\beta v) \\ &= \sum_{\substack{u\alpha \in E \\ v \neq \alpha}} f(u\alpha) + f(uv) + \sum_{\substack{\beta v \in E \\ u \neq \beta}} f(\beta v) + f(uv) \\ &= \sum_{\substack{u\alpha \in E \\ v \neq \alpha}} f(u\alpha) + \sum_{\substack{\beta v \in E \\ u \neq \beta}} f(\beta v) \quad (\text{Since } f(uv) \in V_4). \end{aligned}$$

□

Theorem 2.3. Induced edge sum theorem.
For any graph G, f is an edge induced V_4-Magic labeling of G if and only if the induced edge sum

$$x = f^{++}(uv) = \sum_{\substack{u\alpha \in E \\ v \neq \alpha}} f(u\alpha) + \sum_{\substack{\beta v \in E \\ u \neq \beta}} f(\beta v) \text{ for all } (u,v) \in E. \tag{2.1}$$

The Equation (2.1) corresponding to an edge uv in G, is called induced edge sum equation of the edge uv.

Proof. Proof follows from the definition of edge induced magic labeling and Theorem 2.2. □

3. Edge Induced V_4 Magic Graphs

In this section we discuss the necessary and sufficient condition for the admissibility of edge induced magic labeling of some general graphs like the path graph on n vertices P_n, the cycle graph on n vertices C_n, the star graph $K_{1,n}$, the complete bipartite graph $K_{m,n}$ and the complete graph K_n.

Theorem 3.1. *$P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$.*

Proof. Consider the path $P_2 : v_1 e_1 v_2$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_1) = x$, for some $x \in V_4 \setminus \{0\}$. Then $f^+(v_1) = f^+(v_2) = x$. Therefore $f^{++}(e_1) = 0$. Hence $P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$. □

Corollary 3.2. *$P_2 \notin \sigma(V_4)$.*

Proof. Proof follows from Theorem 3.1. □

Theorem 3.3. *$P_3 \in \sigma_a(V_4)$ and $P_3 \notin \sigma_0(V_4)$.*

Proof. Consider the path $P_3 : v_1 e_1 v_2 e_2 v_3$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_1) = x_1, f(e_2) = x_2$ for some $x_1, x_2 \in V_4 \setminus \{0\}$. Then $f^+(v_1) = x_1, f^+(v_2) = x_1 + x_2, f^+(v_3) = x_2$. Therefore $f^{++}(e_1) = x_2, f^{++}(e_2) = x_1$. Then $P_3 \in \sigma_0(V_4)$ or $P_3 \in \sigma_a(V_4)$ accordingly $x_1 = x_2 = 0$ or $x_1 = x_2 = a$. Since $x_1, x_2 \in V_4 \setminus \{0\}$, $x_1 = x_2 = 0$ is not possible. Therefore $P_2 \notin \sigma_0(V_4)$. Therefore if we take $x_1 = x_2 = a$ then f is an EIML of P_3. Thus $P_3 \in \sigma_a(V_4)$. Hence the proof. □

Corollary 3.4. *$P_3 \notin \sigma(V_4)$.*

Proof. Clearly the proof follows from Theorem 3.3. □

Theorem 3.5. *$P_4 \in \sigma_a(V_4)$ and $P_4 \notin \sigma_0(V_4)$.*

Proof. Consider the path $P_4 : v_1 e_1 v_2 e_2 v_3 e_3 v_4$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_1) = x_1, f(e_2) = x_2, f(e_3) = x_3$ for some $x_1, x_2, x_3 \in V_4 \setminus \{0\}$. Then $f^+(v_1) = x_1, f^+(v_2) = x_1 + x_2, f^+(v_3) = x_2 + x_3, f^+(v_4) = x_3$. Therefore $f^{++}(e_1) = x_2, f^{++}(e_2) = x_1 + x_3, f^{++}(e_3) = x_2$. Then $P_4 \in \sigma_0$ or $P_4 \in \sigma_a$ accordingly $x_2 = x_1 + x_3 = 0$ or $x_2 = x_1 + x_3 = a$. But $x_2 = 0$ is not possible. Therefore $P_4 \notin \sigma_0(V_4)$. Thus if we take $x_1 = b, x_2 = a, x_3 = c$, then f is an EIML of P_4. Hence the proof. □

Corollary 3.6. *$P_4 \notin \sigma(V_4)$.*

Proof. Proof follows from Theorem 3.5. □

Theorem 3.7. *P_n is not an edge induced magic graph for any $n \geq 5$.*

Proof. Suppose that $n \geq 5$. Consider the path defined by $P_n := v_1 e_1 v_2 e_2 v_3 e_3 \cdots v_{n-1} e_n v_n$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_i) = x_i$ for some $x_i \in V_4 \setminus \{0\}$ for $i = 1,2,3,\ldots, n-1$. Then $f^+(v_1) = x_1$, $f^+(v_2) = x_1 + x_2$, $f^+(v_3) = x_2 + x_3$, $f^+(v_4) = x_3 + x_4$ and so on. Therefore $f^{++}(e_1) = x_2$, $f^{++}(e_2) = x_1 + x_3$, $f^{++}(e_3) = x_2 + x_4$. Now if possible, suppose f is an EIML of P_n. Then we have $f^{++}(e_1) = x_2 = x_2 + x_4 = f^{++}(e_3)$, which implies $x_4 = 0$, which is a contradiction to our assumption. Hence $P_n \notin \sigma_0(V_4)$ and $P_n \notin \sigma_a(V_4)$ for $n \geq 5$.
Hence the proof. □

Corollary 3.8. *$P_n \notin \sigma(V_4)$ for any n.*

Proof. Proof of the corollary follows from Corollary 3.2, Corollary 3.4, Corollary 3.6 and Theorem 3.7. □

Theorem 3.9. *$C_n \in \sigma_0(V_4)$ for all n.*

Proof. We can observe that the proof follows from Theorem 2.1. □

Theorem 3.10. *$C_n \in \sigma_a(V_4)$ if and only if n is a multiple of 4.*

Proof. Consider the cycle defined by $C_n := v_1 e_1 v_2 e_2 v_3 e_3 \cdots v_{n-1} e_{n-1} v_n e_n v_1$. Suppose n is a multiple of 4, say $n = 4k$, for some integer k. Define $f : E(C_n) \to V_4 \setminus \{0\}$ as

$$f(e_j) = \begin{cases} b & \text{for } j = 1,5,9,\ldots,4k-3,\ 2,6,10,\ldots,4k-2 \\ c & \text{for } j = 3,7,11,\ldots,4k-1,\ 4,8,12,\ldots,4k. \end{cases}$$

Then we can prove that $f^{++}(e_j) = a$ for $j = 1,2,3,\ldots,n$. That is f is an EIML of C_n. Therefore in this case, $C_n \in \sigma_a(V_4)$.
Conversely, suppose that n is not a multiple of 4. Then $n = 4k+1$ or $n = 4k+2$ or $n = 4k+3$ for some integer k. If possible, suppose f is an EIML of C_n with $f(e_i) = x_i$ for $i = 1,2,3,\ldots,n$. Then from the induced edge sum equation of each edge, we get

$$x_n + x_2 = x_1 + x_3 = x_2 + x_4 = x_3 + x_5 = \cdots = x_{n-1} + x_1. \quad (3.1)$$

Case 1: $n = 4k+1$.
In this case, Equation (3.1) implies that $x_1 = x_5 = x_9 = \cdots = x_n = x_4 = x_8 = x_{12} = \cdots = x_{n-1} = x_3 = x_7 = x_{11} = \cdots = x_{n-2} = x_2 = x_6 = x_{10} = \cdots = x_{n-3}$. Thus in this case if we let $x_i = f(e_i) = a$ for all i then $f^{++}(e_i) = 0$ for all i. Hence $C_n \notin \sigma_a(V_4)$.

Case 2: $n = 4k+2$.
In this case, Equation (3.1) implies that $x_1 = x_5 = x_9 = \cdots = x_{n-1} = x_3 = x_7 = x_{11} = \cdots = x_{n-3}$ and $x_2 = x_6 = x_{10} = \cdots = x_n = x_4 = x_8 = x_{12} = \cdots = x_{n-2}$. Then if we let $f(e_1) = a$ and $f(e_2) = b$ then $f^+(v_j) = c$ for all j. Thus $f^{++}(e_i) = 0$ for all i. Hence $C_n \notin \sigma_a(V_4)$

Case 3: $n = 4k+3$.
In this case, Equation (3.1) implies that $x_1 = x_5 = x_9 = \cdots = x_{n-2} = x_2 = x_6 = x_{10} = \cdots = x_{n-1} = x_3 = x_7 = x_{11} = \cdots = x_n = x_4 = x_8 = x_{12} = \cdots = x_{n-3}$. Thus in this case, if we let $x_i = f(e_i) = a$ for all i then $f^{++}(e_i) = 0$ for all i. Hence $C_n \notin \sigma_a(V_4)$.

Thus from all the three cases above, we have $C_n \notin \sigma_a(V_4)$. Thus $C_n \in \sigma_a(V_4)$ if and only if n is a multiple of 4.
Hence the proof. □

Corollary 3.11. *$C_n \in \sigma(V_4)$ if and only if n is a multiple of 4.*

Proof. Proof follows from Theorem 3.9 and Theorem 3.10. □

Theorem 3.12. *Consider the star graph $K_{1,n}$, then we have the following.*

(i) *$K_{1,n} \in \sigma_0(V_4)$ if and only if n is odd.*

(ii) *$K_{1,n} \in \sigma_a(V_4)$ if and only if n is even.*

Proof. Consider $K_{1,n}$ with vertex set $\{v, v_i : 1 = 1,2,3,\ldots,n\}$, where $vv_i \in E(K_{1,n})$ for $i = 1,2,3,\ldots,n$. Let f be an edge label of $K_{1,n}$, with $f(vv_i) = x_i$, then from the induced edge sum equation of each edge we have the equation:

$$\begin{aligned} x_2 + x_3 + x_4 + \cdots + x_n &= x_1 + x_3 + x_4 + \cdots + x_n \\ &= x_1 + x_2 + x_3 + \cdots + x_n \\ &\vdots \\ &= x_1 + x_2 + x_3 + \cdots + x_{n-1}. \end{aligned}$$

Thus we have f is an EIML of $K_{1,n}$ if and only if $x_1 = x_2 = x_3 = \cdots = x_n$.

case (1) n is an odd integer.
Let $f(vv_i) = x_i = a$, then $f^+(v) = na = a$ and $f^+(v_i) = a$. Thus $f^{++}(vv_i) = a + a = 0$ for all i. Hence in this case, we can conclude that $K_{1,n} \in \sigma_0(V_4)$ and $K_{1,n} \notin \sigma_a(V_4)$.

case (2) n is an even integer.
Let $f(vv_i) = x_i = a$, then $f^+(v) = na = 0$ and $f^+(v_i) = a$. Thus $f^{++}(vv_i) = 0 + a = a$ for all i. Hence in this case, we can conclude that $K_{1,n} \in \sigma_a(V_4)$ and $K_{1,n} \notin \sigma_0(V_4)$.

Hence the proof. □

From the Theorem 3.12, we have the following corollary.

Corollary 3.13. *$K_{1,n} \notin \sigma(V_4)$ for any n.*

Theorem 3.14. *Consider the bipartite graph $K_{m,n}$, then we have the following.*

(i) *$K_{m,n} \in \sigma_0(V_4)$ for $m+n$ is even.*

(ii) *$K_{m,n} \in \sigma_a(V_4)$ for $m+n$ is odd.*

Proof. Let $V(K_{m,n}) = \{v_1, v_2, v_3, \ldots, v_m,\ u_1, u_2, u_3, \ldots, u_n\}$, where $v_iu_j \in E(K_{m,n})$ for $i = 1,2,3,\ldots m$ and $j = 1,2,3,\ldots,n$. Let $f : E(K_{m,n}) \to V_4 \setminus \{0\}$ be defined by $f(v_iu_j) = a$ for all $v_iu_j \in E(K_{m,n})$.

Case 1: $m+n$ is even.

subcase (i) m and n are odd.
In this case, we get $f^+(v_i) = na = a$ and $f^+(u_j) = ma = a$. Thus $f^{++}(v_iu_j) = 0$ for all i and j.

subcase (ii) m and n are even.
In this case, we get $f^+(v_i) = na = 0$ and $f^+(u_j) = ma = 0$. Thus $f^{++}(v_iu_j) = 0$ for all i and j.

Thus, if $m+n$ is even then $K_{m,n} \in \sigma_0(V_4)$.

Case 2: $m+n$ is odd.

subcase (i) m is even n is odd.
In this case, we get $f^+(v_i) = na = a$ and $f^+(u_j) = ma = 0$. Thus $f^{++}(v_iu_j) = a+0 = a$ for all i and j.

subcase (ii) m is odd n is even.
In this case, we get $f^+(v_i) = na = 0$ and $f^+(u_j) = ma = a$. Thus $f^{++}(v_iu_j) = 0+a = a$ for all i and j.

Thus if $m+n$ is even, then $K_{m,n} \in \sigma_a(V_4)$.

Hence the proof. □

Theorem 3.15. *Let K_n be the complete graph with n vertices, then $K_n \in \sigma_0(V_4)$ for all n.*

Proof. Consider the complete graph K_n. Let $f : E(K_n) \to V_4 \setminus \{0\}$ be an edge label with $f(e) = a$ for all $e \in E(K_n)$. Then the induced vertex label f^+ becomes $f^+(u) = (n-1)a$, for all $u \in V(K_n)$. Using this, we have the induced edge label f^{++} becomes $f^{++}(e) = 2(n-1)a = 0$, for all $e \in E(K_n)$. Thus $K_n \in \sigma_0(V_4)$ for all n. Hence the proof. □

4. Some Special Edge Induced V_4 Magic Graphs

Here we need the following graphs.

Definition 4.1. [2] *The Bistar $B_{m,n}$ is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.*

Definition 4.2. [2] *The Sun graph on $m = 2n$ vertices, denoted by Sun_n, is the graph obtained by attaching a pendant vertex to each vertex of a $n-$cycle.*

Definition 4.3. *The Corona $P_n \odot K_1$ is called the comb graph CB_n.*

Definition 4.4. [2] *The sum of the graphs C_n and K_1 is called a Wheel graph and it is denoted by W_n,that is $W_n = C_n + K_1$.*

Definition 4.5. [2] *Jelly fish graph $J(m,n)$ is obtained from a $4-$cycle $v_1v_2v_3v_4v_1$ by joining v_1 and v_3 with an edge and appending the central vertex of $K_{1,m}$ to v_2 and appending the central vertex of $K_{1,n}$ to v_4.*

Definition 4.6. [2] *A triangular snake graph TS_n is obtained from a path $v_1, v_2, v_3, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1,2,3,\cdots,n-1$.*

Definition 4.7. [6] *An open ladder graph $O(L_n), n \geq 2$ is obtained from two paths of length $n-1$ with $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 2 \leq i \leq n-1\}$.*

Theorem 4.8. *Consider Bistar graph $B_{m,n}$, then we have the following.*

(i) $B_{m,n} \in \sigma_0(V_4)$ if and only if m and n are even.

(ii) $B_{m,n} \in \sigma_a(V_4)$ if and only if m and n are odd.

Proof. Let $V(B_{m,n}) = \{u, v, v_i, u_j : i = 1,2,3,\ldots,m,\ j = 1,2,3,\ldots,n\}$, where $uv, vv_i, uu_j \in E(B_{m,n})$ for $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n$. Let $f : E(B_{m,n}) \to V_4 \setminus \{0\}$ be an edge label defined as follows:

$$f(e) = \begin{cases} \alpha & \text{if} \quad e = uv \\ x_i & \text{if} \quad e = vv_1, vv_2, vv_3, \ldots, vv_m \\ y_j & \text{if} \quad e = uu_1, uu_2, uu_3, \ldots, uu_n. \end{cases}$$

Then by considering the induced edge sum equation of the edges vv_i we have:

$$\begin{aligned} \alpha + x_2 + x_3 + x_4 + \cdots + x_m &= \alpha + x_1 + x_3 + x_4 + \cdots + x_m \\ &= \alpha + x_1 + x_2 + x_3 + \cdots + x_m \\ &\vdots \\ &= \alpha + x_2 + x_3 + x_4 + \cdots + x_{m-1}. \end{aligned} \tag{4.1}$$

In the light of Equation (4.1), we have $x_1 = x_2 = x_3 = \cdots = x_m = \beta$ (say). Similarly by considering the induced edge sum equation of the edges uu_j one can easily prove that $y_1 = y_2 = y_3 = \cdots = y_n = \gamma$ (say). Thus the induced edge sum equations of the edge vv_i and uu_j are given by

$$\alpha + (m-1)\beta = \alpha + (n-1)\gamma.$$

Also the induced edge sum equation of the edge uv is given by

$$x_1 + x_2 + x_3 + \cdots + x_m + y_1 + y_2 + y_3 + \cdots + y_n = m\beta + n\gamma.$$

Thus f is an edge induced magic label with induced edge sum x if and only if

$$x = \alpha + (m-1)\beta = \alpha + (n-1)\gamma = m\beta + n\gamma. \tag{4.2}$$

Case 1: m and n are even.

In this case, Equation (4.2) becomes $x = \alpha + \beta = \alpha + \gamma = 0$. Thus $\alpha = \beta = \gamma$, and the induced edge sum $x = 0$.

Hence in this case, $B_{m,n} \in \sigma_0(V_4)$.

Case 2: m and n are odd.

In this case, Equation (4.2) becomes $x=\alpha=\beta+\gamma$. Thus in this we can choose $\beta=b$ and $\gamma=c$ then $\alpha=a$ and which implies that the induced edge sum becomes $x=a$.

Hence in this case, $B_{m,n}\in\sigma_a(V_4)$.

Case 3: m is even and n is odd.

In this case, Equation (4.2) becomes $x=\alpha+\beta=\alpha=\gamma$ which implies that $\beta=0$. That is $f(vv_i)=x_i=\beta=0$, which a contradiction to the choice of f. Therefore $B_{m,n}\notin\sigma_0(V_4)$ and $B_{m,n}\notin\sigma_a(V_4)$.

Case 4: m is odd and n is even.

In this case, Equation (4.2) becomes $x=\alpha=\alpha+\gamma=\beta$ which implies $\gamma=0$. That is $f(uu_j)=y_j=\gamma=0$, which a contradiction to the choice of f. Therefore $B_{m,n}\notin\sigma_0(V_4)$ and $B_{m,n}\notin\sigma_a(V_4)$.

Hence the proof. □

Theorem 4.9. *The Sun graph $Sun_n\in\sigma_0(V_4)$ for all n.*

Proof. Since each vertex Sun_n is of odd degree, by Theorem 2.1 the theorem follows. □

Theorem 4.10. *The Sun graph $Sun_n\in\sigma_a(V_4)$ for n is even.*

Proof. Consider a Sun graph Sun_n with $\{v_1,v_2,v_3,\dots,v_n\}$ as vertex set of the corresponding C_n and w_i be the pendant vertex attached to each v_i, for $i=1,2,3,\dots,n$. Let $f:E(K_n)\to V_4\setminus\{0\}$ be defined by

$$f(e)=\begin{cases} b & \text{if} \quad e=v_1v_2,v_3v_4,v_5v_6,\dots,v_{n-1}v_n\\ c & \text{if} \quad e=v_2v_3,v_4v_5,v_6v_7,\dots,v_nv_1\\ b & \text{if} \quad e=v_1w_1,v_3w_3,v_5w_5,\dots,v_{n-1}w_{n-1}\\ c & \text{if} \quad e=v_2w_2,v_4w_4,v_6w_6,\dots,v_nw_n.\end{cases}$$

Then we can easily prove that $f^{++}(e)=a$ for all $e\in E(Sun_n)$. That is $Sun_n\in\sigma_a(V_4)$. Hence the proof. □

Corollary 4.11. *The Sun graph $Sun_n\in\sigma(V_4)$ if and only if n is even.*

Proof. Proof follows from Theorem 4.9 and Theorem 4.10. □

Theorem 4.12. *The Comb graph CB_n is not an edge induced magic graph, for any n.*

Proof. Let $\{u_i,v_i:1\le i\le n\}$ be the vertex set of CB_n, where $v_i(1\le i\le n)$ are the pendant vertices adjacent to $u_i(1\le i\le n)$. If possible, suppose $f:E(CB_n)\to V_4\setminus\{0\}$ is an induced edge label of CB_n. Then using the induced edge sum equation of the edges u_1v_1 and u_2v_2 we get, $f^{++}(u_1v_1)=f^{++}(u_2v_2)$, which implies $f(u_1u_2)=f(u_1u_2)+f(u_2u_3)$. That is $f(u_2u_3)=0$, which is a contradiction. Thus CB_n is not an edge induced magic graph
Hence the proof. □

Theorem 4.13. *The Wheel graph $W_n\in\sigma_0(V_4)$ for n is odd.*

Proof. Suppose n is odd. Then, since each vertex of W_n is of odd degree, by Theorem 2.1 the proof follows. □

Theorem 4.14. *Let $J(m,n)$ be the Jelly fish graph. Then*

(i) *$J(m,n)\in\sigma_0(V_4)$ if and only m and n are of same parity.*

(ii) *$J(m,n)\notin\sigma_a(V_4)$ for any m and n.*

Proof. Consider the Jelly fish graph with $V(J(m,n))=\{v_k: k=1,2,3,4\}\cup\{u_i:i=1,2,3,\dots,m\}\cup\{w_j:j=1,2,3,\dots,n\}$, where v_k's are the vertices of C_4 and u_i,w_j are the vertices of corresponding $K_{1,m}$ and $K_{1,n}$ respectively. Let $f:E(J(m,n))\to V_4\setminus\{0\}$ be an edge induced magic label with $f(v_1v_2)=x_1$, $f(v_1v_4)=x_2$, $f(v_3v_4)=x_3$, $f(v_2v_3)=x_4$, $f(v_1v_3)=x_5$, $f(v_2u_i)=e_i$, and $f(v_4w_j)=y_j$ for $i=1,2,3,\dots,m$ and $j=1,2,3,\dots,n$.

Using the induced edge sum equation of the edges v_2u_i, we get

$$\begin{aligned}\sum_{\substack{i=1\\ i\neq 1}}^{m}e_i+x_1+x_4 &= \sum_{\substack{i=1\\ i\neq 2}}^{m}e_i+x_1+x_4\\ &= \sum_{\substack{i=1\\ i\neq 3}}^{m}e_i+x_1+x_4\\ &\;\;\vdots\\ &= \sum_{\substack{i=1\\ i\neq m}}^{m}e_i+x_1+x_4.\end{aligned}$$

The above equations imply that $e_1=e_2=e_3=\dots=e_m=\alpha$ (say). Thus the induced edge sum equation of v_2u_i reduces to $(m-1)\alpha+x_1+x_4$.

In a similar way, by considering the induced edge sum equation of the edges v_4w_j, we get $y_1=y_2=y_3=\dots=y_n=\beta$ (say). Thus the induced edge sum equation of v_4w_j reduces to $(n-1)\beta+x_2+x_3$.

Now consider the induced edge sum equation of the edges v_1v_2 and v_2v_3, then we get $m\alpha+x_2+x_4+x_5=m\alpha+x_1+x_3+x_5$ which implies $x_2+x_4=x_1+x_3$.
Similarly by considering the induced edge sum equation of v_1v_4 and v_3v_4, we get $n\beta+x_1+x_3+x_5=n\beta+x_2+x_4+x_5$. Also from the induced edge sum equation of v_1v_3, we get its induced edge sum equal to $x_1+x_2+x_3+x_4=0$, since $x_2+x_4=x_1+x_3$.
Thus from the above discussion we have the induced edge sum is given by:

$$\begin{aligned}x &= (m-1)\alpha+x_1+x_4=(n-1)\beta+x_2+x_3\\ &= m\alpha+x_2+x_4+x_5=m\alpha+x_1+x_3+x_5 \qquad (4.3)\\ &= n\beta+x_1+x_3+x_5=n\beta+x_2+x_4+x_5\\ &= 0.\end{aligned}$$

Since the induced sum is 0, we have $J(m,n)\notin\sigma_a(V_4)$ for any m,n.
Now consider the following cases

Case 1 : m and n are even.
In this case, equation (4.3) becomes

$$x=\alpha+x_1+x_4=\beta+x_2+x_3=x_2+x_4+x_5=x_1+x_3+x_5=0. \tag{4.4}$$

Choose $\alpha=\beta=x_5=c,\ x_1=x_2=a,\ x_3=x_4=b$, then above Equation (4.4) follows. Thus in this case, $J(m,n)\in\sigma_0(V_4)$

Case 2 : m and n are odd.
In this case, equation (4.3) becomes

$$\begin{aligned} x &= x_1+x_4=x_2+x_3 \\ &= \alpha+x_2+x_4+x_5=\alpha+x_1+x_3+x_5 \quad (4.5) \\ &= \beta+x_1+x_3+x_5=\beta+x_2+x_4+x_5 \\ &= 0. \end{aligned}$$

Choose $\alpha=\beta=x_5=a,\ x_1=x_2=x_3=x_4=b$, then above Equation (4.5) follows. Thus in this case, $J(m,n)\in\sigma_0(V_4)$

Case 3 : m odd and n even.
In this case, Equation (4.3) becomes

$$\begin{aligned} x &= x_1+x_4=\beta+x_2+x_3 \\ &= \alpha+x_2+x_4+x_5=\alpha+x_1+x_3+x_5 \\ &= x_1+x_3+x_5=x_2+x_4+x_5 \\ &= 0. \end{aligned}$$

Note that above equations imply that $\alpha=e_i=f(v_2u_i)=0$, which is not admissible. Thus in this case, $J(m,n)\notin\sigma_0(V_4)$.

Case 4 : m even and n odd.
In this case, Equation (4.3) becomes

$$\begin{aligned} x &= \alpha+x_1+x_4=x_2+x_3 \\ &= x_2+x_4+x_5=x_1+x_3+x_5 \\ &= \beta+x_1+x_3+x_5=\beta+x_2+x_4+x_5 \\ &= 0. \end{aligned}$$

Note that above equations imply that $\beta=y_j=f(v_4w_j)=0$, which is not admissible. Thus in this case, $J(m,n)\notin\sigma_0(V_4)$.

Thus $J(m,n)\in\sigma_0(V_4)$ if and only if m and n are of same parity.
Hence the proof. □

Theorem 4.15. *The triangular snake graph $TS_n\in\sigma_0(V_4)$ for all n.*

Proof. Since each vertex of TS_n is of even degree, by Theorem 2.1 the proof follows. □

Theorem 4.16. *The open ladder graph $O(L_n)\in\sigma_0(V_4)$ for all n.*

Proof. Since each vertex of $O(L_n)$ is of odd degree, by Theorem 2.1 the proof follows. □

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