



# Changes in binding number and binding degree of a graph under different edge operations

Medha Itagi Huilgol<sup>1\*</sup> and S. Kiran<sup>2</sup>

## Abstract

The binding number of a graph  $G$  is defined as  $bind(G) = \min\{\frac{|N(X)|}{|X|} : X \subseteq V(G), X \neq \emptyset \text{ and } N(X) \neq V(G)\}$ . In this paper we consider the effects of contraction, deletion and/or addition of an edge on the binding number of a graph. Also, invariance of binding number is considered under these operations. A new parameter is defined here, named the binding degree. The variations of binding degree under different edge operations is also considered.

## Keywords

Contraction of edge, Deletion of edge, Addition of edge, Binding Number, Binding degree.

## AMS Subject Classification

05C10.

<sup>1,2</sup>Department of Mathematics, Bengaluru City University, Central College Campus, Bengaluru-560001, India.

\*Corresponding author: <sup>1</sup> medha@bub.ernet.in

Article History: Received 16 August 2020; Accepted 29 October 2020

©2020 MJM.

## Contents

1	Introduction .....	1934
2	Basic definitions and results .....	1935
3	Changes in the binding number due to edge contraction .....	1936
4	Changes in the binding number due to edge addition and deletion .....	1937
5	Binding degree of a graph .....	1938
6	Changes in the binding degree due to edge contraction and deletion .....	1940
7	Conclusion .....	1940
	References .....	1940

## 1. Introduction

Throughout this paper, by a “graph” we mean a finite simple graph without loops as treated in F. Harary [4].

In connection with any graph-theoretic parameter, it is commonly observed that addition or deletion of an edge will either alter the parameter value or not. If the value alters by such an operation, we call the graph as maximal or minimal or popularly known as critical with to the given parameter.

On the other hand, the contraction of edges was done, mainly to check the planarity of a given graph. Harary [4]

defined a contractible graph as a graph which can be obtained by a series of elementary edge contractions. A dual from of the famous Kuratowski’s planarity theorem in the sense of matriod theory, was found independently by Wagner [14], F. Harary and Tutte [5]. This theorem characterizers planar graphs with respect to subgraphs contractible, by elementary edge contractions. Recently, the trend has changed and effect of even one edge-contraction on a given parameter has been studied by Walikar [12], [13]. For any particular property the measure of change was defined as a new graph parameter and studied by Walikar et al. [13], Huilgol et al. [6] [7]. This parameter was called the *edge-essential number* and was defined as the number of edges whose contraction changes a property under consideration.

On the contrary, an edge is called *non-essential* if its contraction does not alter the property. Based on this general set-up, the present paper deals with a comparative study of contractions, deletions and addition with respect to the binding number of a graph.

The binding number of a graph gives a measure on the distribution of edges over vertices. It also serves on a vulnerability parameter, along with vertex/edge connectivity, average lower connectivity etc. The lower the binding number of a graph is, the more vulnerable graph is for connectedness. In any real world network model, the vulnerability parameters measure how robust a network is after a link failure. So the

study of binding number has both theoretical as well as practical implications. Formally the binding number was defined by Woodall [15] in his seminal paper way back in 1947. It is considered to be one of the toughest graph parameters and hence find considerably less number of research papers in seven decades of study. Recently, it is extended for digraphs by Xu et al. [16]. For further details one can refer to [1], [8], [10], [11], [15]. Hence checking the variance of a vulnerability parameter like the binding number under different edge operations itself is an interesting at the same time challenging work.

In this paper we undertake the same and present some results. Also, we find conditions such that the binding number is kept invariant under different edge operations. Hence the changes or invariance of binding number under different edge operations gives a double measure on the reliability of a network under consideration. Till 2019 the study of binding number was considered as a single, major parameter for a graph. But the treatment got a new direction by Aslan [2], where it is considered and defined for each vertex of the graph under consideration and then the average was defined over all the vertices. This approach gives a local vulnerability test of the given graph or network. And this motivated us to define another new parameter namely the binding degree of a graph. This works as a triple layered vulnerability parameter that takes care of the degree of a vertex along with its corresponding local binding number. In the last section of this paper, we have not only defined the binding degree, but also considered effects of different edge operations on the binding degree of a graph.

First we give a list of some basic definitions and results that are required to establish our results in further sections.

## 2. Basic definitions and results

Here are some of the basic definitions and results that help in building the new ones.

**Definition 2.1.** [4] The join  $G+H$  of two graphs  $G$  and  $H$  is the graph with  $V(G+H) = V(G) \cup V(H)$  and  $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in E(H)\}$ .

**Definition 2.2.** [4] In a graph  $G$ , an edge  $e = uv$ , is said to be deleted, if only the edge 'e' is removed from  $G$ , keeping all other adjacencies intact. This is denoted as  $G - e$ .

**Definition 2.3.** [4] In a graph  $G$ , an  $e = uv$ , is said to be added, if only the edge ' $e \in E(\bar{G})$ ' is added to  $G$ , keeping all other adjacencies intact. This is denoted as  $G + e$ .

**Definition 2.4.** [4] An edge  $e = uv$ , is said to be contracted, if both the vertices  $u$  and  $v$  are removed along with their adjacencies and a new vertex  $w$  is introduced in such a way that  $w$  is adjacent to the vertices which were adjacent to either  $u$  or  $v$ .

**Definition 2.5.** [15] The binding number of a graph  $G$  is defined as  $bind(G) = \min\{\frac{|N(X)|}{|X|} : X \subseteq V(G), X \neq \emptyset \text{ and } N(X) \neq$

$V(G)\}$ , where  $N(X) = \{y/xy \in E(G), \forall x \in X\}$ , is called the neighbourhood set of  $X$ .

**Definition 2.6.** [2] For  $v \in V(G)$ , the local binding number of  $v$  is  $bind_v(G) = \min_{S \in F_v(G)} \{\frac{|N(S)|}{|S|}\}$ , where  $F_v(G) = \{S \subseteq V(G) : v \in S, S \neq \emptyset, N(S) \neq V(G)\}$ .

Clearly  $bind(G) = \min_{v \in V(G)} \{bind_v(G)\}$ .

A local binding set of  $v$  in  $G$  is  $S \in F_v(G)$ , such that  $bind_v(G) = \frac{|N(S)|}{|S|}$ .

**Definition 2.7.** [2] The average binding number of  $G$  is defined as

$$bind_{av}(G) = \frac{1}{n} \sum_{v \in V(G)} bind_v(G),$$

where  $n$  is the number of vertices in graph  $G$ .

Here we briefly list some results proved earlier that help us build the next set.

**Proposition 2.8.** [15] If  $n \geq 2$ , then  $bind(K_n) = n - 1$ .

**Proposition 2.9.** [15] If  $m \geq 2, n \geq 1$  then  $bind(K_{m,n}) = \min\{\frac{m}{n}, \frac{n}{m}\}$ .

**Proposition 2.10.** [15] If  $n \geq 2$ , then

$$bind(P_n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ \frac{n-1}{n+1}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proposition 2.11.** [15] If  $n \geq 3$ , then

$$bind(C_n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ \frac{n-1}{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proposition 2.12.** [3] If  $n \geq 4$ , then

$$bind(W_n) = \begin{cases} \frac{n-1}{n-3}, & \text{if } n \text{ is even} \\ \frac{n+1}{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proposition 2.13.** [3] If  $n \geq 2$ , then for a fan  $F_n$  we have,

$$bind(F_n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ \frac{n-1}{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proposition 2.14.** [8] Let  $t \geq 2$ , then  $bind(\cup_{i=1}^t G_i) = \min\{1, bind(G_1), bind(G_2), \dots, bind(G_t)\}$

**Proposition 2.15.** [10] If  $G$  is a bipartite graph, then  $\beta_0 = \frac{n}{2}$  if and only if  $bind(G) = 1$ , where  $\beta_0$  is the vertex independence number of  $G$ .

**Proposition 2.16.** [15] If  $G$  is a bipartite graph, then  $bind(G) \leq 1$ .

**Proposition 2.17.** [15] For any graph  $G$  with minimum degree  $\delta(G)$ ,  $bind(G) \leq \frac{n-1}{n-\delta(G)}$ .

**Proposition 2.18.** [8] If  $G$  has a 1-factor then  $bind(G) \geq 1$ .

**Proposition 2.19.** [8] For any graph  $G$ ,  $bind(G) \leq \frac{n}{\beta_0} - 1$ , where  $\beta_0$  denotes the vertex independence number of  $G$ .

**Proposition 2.20.** [9] Let  $G$  be a graph with  $bind(G) = c$ . If  $x \in E(G)$ , then for any admissible set  $X$  in  $G - x$  such that  $X \cap x = \emptyset$  and  $\frac{|N_{G-x}(X)|}{|X|} \geq c$ .



### 3. Changes in the binding number due to edge contraction

In this section we try to measure the changes in the binding number of a graph under edge contractions. We first introduce some definitions in line with the ones given for diameter, radius, domination-essential and/or non-essential edges done earlier [12], [13], [6], [7]. In case all these parameters an edge contraction would either increase the parameter or decrease and hence noting the changes was unidirectional. But in the case of binding number of a graph, the situation is slightly not that straight forward. On an edge contraction, the binding number of a graph tends to increase or decrease and sometimes, remain unchanged depending on the edge chosen. This variation makes it even more challenging to study, like a true vulnerability parameter. Keeping all these considerations in mind, we define the various types of edges under edge contractions.

**Definition 3.1.** An edge ‘ $e$ ’ in a graph  $G$  is said to be a binding-variant edge with respect to contraction, if  $bind(G/e) \neq bind(G)$ .

**Definition 3.2.** An edge ‘ $e$ ’ in a graph  $G$  is said to be a binding-invariant edge with respect to contraction, if  $bind(G/e) = bind(G)$ .

The set of such edges in a graph  $G$  is denoted by  $E_{ic}(G)$  and the number of such edges is denoted by  $\sigma_{ic}(G) = |E_{ic}(G)|$ .

As discussed earlier, in case of radius/diameter/domination number on contraction of an edge always decreases or remains unchanged. But in case of binding number, an edge contraction sometimes increases and other-times decreases. Hence, the binding-variant edges are divided into two classes.

- (i) An edge is said to be an essential positive-binding edge, if its contraction increases the binding number of the graph.

That is, an edge  $e$  is an essential positive binding edge (EPBE), if  $bind(G/e) > bind(G)$ . The set of such edges in a graph  $G$  is denoted by  $E_{bc}^+(G)$ .

Thus,  $E_{bc}^+(G) = \{e \in E(G) / bind(G/e) > bind(G)\}$ .

- (ii) An edge is said to be an essential negative-binding edge, if its contraction decreases the binding number of the graph.

That is, an edge  $e$  is an essential negative binding edge (ENBE), if  $bind(G/e) < bind(G)$ . The set of such edges in a graph  $G$  is denoted by  $E_{bc}^-(G)$ .

Thus,  $E_{bc}^-(G) = \{e \in E(G) / bind(G/e) < bind(G)\}$ .

And the number such edges is denoted as  $\sigma_{bc}^+(G) = |E_{bc}^+|$ ,  $\sigma_{bc}^-(G) = |E_{bc}^-|$ .

Hence for the size  $m$  of  $G$ , we have

$$m = \sigma_{bc}^+(G) + \sigma_{bc}^-(G) + \sigma_{ic}(G). \tag{3.1}$$

**Example(1):** In a complete graph  $K_n$ , for  $n \geq 3$  every edge on contraction reduces the binding number, since  $bind(K_n) =$

$n - 1$  and  $bind(K_n/e) = bind(K_{n-1}) = n - 2$ . Therefore every edge is essential negative binding edge (ENBE), with respect to contraction.

**Example (2):** In a star  $K_{1,n}$ , for  $n \geq 3$  every edge on contraction increases the binding number, since  $bind(K_{1,n}) = \frac{1}{n-1}$  and  $bind(K_{1,n-1}) = \frac{1}{n-2}$ . Therefore every edge is essential positive binding edge (EPBE), with respect to contraction.

**Remark 3.3.** For any connected graph  $G$ ,

$$0 \leq \sigma_{bc}^- \leq m.$$

Both the bounds are attainable. The result below shows class of graphs realizing the upper bound. The lower bound is attainable by even cycles,  $C_{2n}$ .

**Lemma 3.4.** For  $G \approx K_n, \forall n \geq 3$  or  $P_{2n}, \forall n \geq 2$  or  $C_{2n+1}, \forall n \geq 1$ , then  $\sigma_{bc}^-(G) = m$ .

*Proof.* For  $G \approx K_n, \forall n \geq 3$  or  $P_{2n}, \forall n \geq 2$  or  $C_{2n+1}, \forall n \geq 1$ , contraction of any edge  $e$  in  $G$  decreases binding number, as  $G/e \approx K_{n-1}$  or  $P_{2n-1}$  or  $C_{2n}$ , respectively. Therefore all edges are essential negative binding edges. Hence  $\sigma_{bc}^-(G) = m$ .  $\square$

**Remark 3.5.** For any connected graph  $G$ ,

$$0 \leq \sigma_{bc}^+ \leq m.$$

Both the bounds are attainable. The result below shows class of graphs realizing the upper bound. The lower bound is attainable by complete graphs,  $K_n$ .

**Lemma 3.6.** For  $G \approx C_{2n}, \forall n \geq 2$  or  $P_{2n+1}, \forall n \geq 1$  or  $K_{1,n}, \forall n \geq 1$ , then  $\sigma_{bc}^+(G) = m$ .

*Proof.* In  $G \approx C_{2n}, \forall n \geq 2$  or  $P_{2n+1}, \forall n \geq 1$  or  $K_{1,n}, \forall n \geq 2$ , contraction of any edge  $e$  in  $G$  binding number in  $G/e$  is increasing, because  $G/e \approx C_{2n-1}$  or  $P_{2n}$  or  $K_{1,n-1}$ , respectively. Therefore all edges are essential positive binding edges. Hence  $\sigma_{bc}^+(G) = m$ .  $\square$

In the next results we consider the graphs wherein no edge is binding invariant with respect to contraction.

**Theorem 3.7.** If  $\gamma(G) = 1$ , then there exists no binding invariant edge in  $G$ .

*Proof.* If  $\gamma(G) = 1$ , then there exists a vertex of full degree and hence contraction of any edge  $e$  in  $G$  alters the binding number of  $G$ . Therefore no edge is binding invariant.  $\square$

**Theorem 3.8.** In a regular graph no edge is binding invariant.

*Proof.* If  $G$  is regular graph then contraction of any edge in  $G$  results in either a regular graph or a non-regular graph. These two cases are considered below and shown that in both cases no edge is binding invariant.

**Case (i):**  $G/e$  is regular.

In this case  $G/e$  must be either a complete graph or a cycle. In both cases we get all edges to alter the binding number of



$G$  and hence no edge is binding invariant.

**Case (ii):**  $G/e$  is non-regular.

Let  $G$  be a regular with regularity  $k$ , say. On contraction of any edge if  $G/e$  is not regular, then a vertex at which the edge is contracted will have its degree more than that before the edge contraction. Hence  $|V(G/e)| = |V(G)| - 1$  and  $m(G/e) \leq \frac{nk}{2} - 1$ . Also we know that

$$\text{bind}(G) \leq \frac{n-1}{n-\delta(G)} = \frac{n-1}{n-k}. \tag{3.2}$$

$$\text{bind}(G/e) < \frac{n-2}{n-k}. \tag{3.3}$$

Therefore every edge is binding variant. Hence the result.  $\square$

**Theorem 3.9.** Let  $G$  be a graph with  $\text{bind}(G) = c$ . If  $e \in E(G)$ , then  $\frac{|N_{G/e}(X)|}{|X|} \geq c$ , for any admissible set  $X$  in  $G/e$  such that  $X \cap \{u, v\} = \emptyset$ , and  $e = uv$  is an edge of  $G$ .

*Proof.* For a graph  $G$  with  $\text{bind}(G) = c$ , let  $X$  be an admissible set of  $G/e$  for an edge  $e = uv$  in  $G$  such that  $X \cap \{u, v\} = \emptyset$ . Therefore  $|N_{G/e}(X)| \geq |N_G(X)|$ . This implies that  $\frac{|N_{G/e}(X)|}{|X|} \geq \frac{|N_G(X)|}{|X|} = c$ . Hence  $\frac{|N_{G/e}(X)|}{|X|} \geq c$ .  $\square$

**Theorem 3.10.** If  $G$  is any graph, then  $\beta_0(G) = 1$  if and only if  $\text{bind}(G) = n - 1$ .

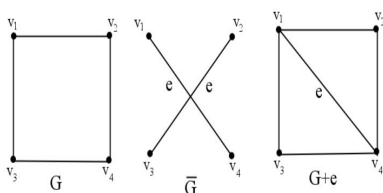
*Proof.* Let  $\beta_0(G) = 1$  where  $\beta_0$  is the vertex independence number. This implies that  $G$  is  $(n - 1)$  regular graph. Therefore  $G$  is a complete graph. Hence  $\text{bind}(G) = n - 1$ . Converse is obvious.  $\square$

#### 4. Changes in the binding number due to edge addition and deletion

We consider two more edge operations namely the edge addition and edge deletion. Again for these operations we check the binding number invariance. First a couple of definitions and illustrations.

**Definition 4.1.** A graph  $G$  is said to be positive binding invariant, if by adding an edge in  $G$ , it retains its binding number, that is,  $\text{bind}(G + e) = \text{bind}(G)$ , for any edge  $e \in E(\bar{G})$ . This edge  $e$  is called a positive binding invariant edge of  $G$ .

**Illustration:**

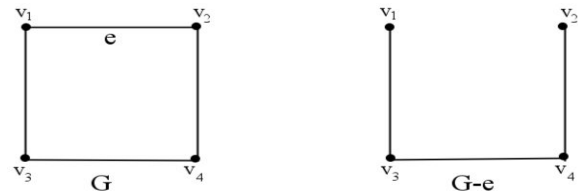


In figure above, we know the  $\text{bind}(G) = 1$ , then  $\text{bind}(G + e) = 1$ . Hence the graph is positive binding invariant.

Next we give the definition of negative binding invariant edge.

**Definition 4.2.** A graph  $G$  is said to be negative binding invariant, if by deleting any edge from  $G$ , it retains its binding number, that is,  $\text{bind}(G - e) = \text{bind}(G)$ , for any  $e \in E(G)$ . This edge  $e$  is called a negative binding invariant edge of  $G$ .

**Illustration:** In the figure above, we know the  $\text{bind}(G) =$



1, then  $\text{bind}(G - e) = 1$ , for all edges  $e$  of  $G$ . Hence the graph is negative binding invariant.

Now, we present some properties of binding invariant, positive binding invariant, negative binding invariant graphs.

**Theorem 4.3.** A graph  $G$  is negative binding invariant graph if and only if for every vertex  $v$ , each vertex  $u \in N_k(v)$  is adjacent to at least two vertices in  $N_{k-1}(v)$ , for  $2 \leq k \leq e(v)$ , where  $N_k(v)$  denotes the  $k^{\text{th}}$  neighbourhood of  $v$  in  $G$  and  $e(v)$  is the eccentricity of vertex  $v$ .

*Proof.* For any arbitrary vertex  $v$  in  $G$  and  $u \in N_k(v)$ , let  $v$  be adjacent to at least two vertices in  $N_{k-1}(v)$ , for  $2 \leq k \leq e(v)$ , where  $N_k(v)$  denotes the  $k^{\text{th}}$  neighbourhood of  $v$  in  $G$  and  $e(v)$  is the eccentricity of vertex  $v$ , then by deleting any edge of the graph, binding number remains unaltered as the realizing set of  $G$  and  $G - e$  remain the same. Hence,  $G$  is a negative binding invariant graph.

Conversely, if  $G$  is a negative binding invariant graph, then  $\text{bind}(G - e) = \text{bind}(G)$ ,  $\forall e \in E(G)$ . Assume that for some vertex  $v$  and some integer  $k$ , a vertex  $u \in N_k(v)$  has at most one neighbour in  $N_{k-1}(v)$ , then removal of an edge incident with the vertex  $u$  will alter the binding number of  $G$ , a contradiction and hence the result.  $\square$

Next result deals with the embedding a graph in a negative binding invariant graph, there by ruling out the characterization using forbidden class of subgraphs.

**Proposition 4.4.** Every graph can be embedded in a negative binding invariant graph.

*Proof.* Let  $G$  be a graph. Partition the vertex set of  $G$  into two sets, one set into an independent set, say  $S$  and another  $V(G) - S$ . Then introduce some more vertices to make  $S$  and  $V(G) - S$  to have the same cardinality. Then introduce a vertex, say  $u$  and join  $u$  to all the vertices  $V(G) - S$ . Make the two sets  $S$  and  $V(G) - S$  to have all edges in between them. Clearly, this graph is negative binding invariant. Hence we can embed any graph into a negative binding invariant graph.  $\square$



**Proposition 4.5.** For a negative binding invariant graph  $G$ , there exists a realizing set  $X$  such that  $X \cap N(X) \neq \emptyset$ .

*Proof.* Let  $G$  be a negative binding invariant graph and  $X$  be its realizing set. Let  $|X \cap N(X)| = k$ . For  $e = uv \in E(G)$  in  $G - e$ , consider  $X' = \{X - (X \cap N(X))\} \cup \{u\}$ . Hence,  $N(X') = N(X) - \{(X \cap N(X)) - \{u\}\}$ .

$$\text{So, } \text{bind}(G - e) \leq \frac{|N(X')|}{|X'|} = \frac{|N(X)| + k - 1}{|X| - k + 1}.$$

Since  $\text{bind}(G - e) = \text{bind}(G)$ , we set

$$\text{bind}(G) \leq \frac{\text{bind}(G) + \frac{k-1}{|X|}}{1 - \frac{k-1}{|X|}} \Rightarrow 1 + \text{bind}(G) \leq K(1 + \text{bind}(G))$$

$$\Rightarrow k > 1.$$

Hence the result.  $\square$

**Proposition 4.6.** For a positive binding invariant graph  $G$ , there exists a realizing set  $X$  with  $X \cap N(X) \neq \emptyset$  then  $\text{bind}(G) > 1$ .

*Proof.* Let  $G$  be a positive binding invariant graph and  $X$  be its realizing set such that  $X \cap N(X) \neq \emptyset$  then  $|X \cap N(X)| = k \geq 1$ . Let  $X' = \{X - (X \cap N(X))\} \Rightarrow N(X') = N(X) - \{(X \cap N(X))\}$ .

$$\text{Hence } \text{bind}(G + e) \leq \frac{|N(X')|}{|X'|} \leq \frac{\text{bind}(G) + \frac{k}{|X|}}{1 - \frac{k}{|X|}}$$

$$\text{But, } \text{bind}(G + e) = \text{bind}(G) \leq \frac{\text{bind}(G) + \frac{k}{|X|}}{1 - \frac{k}{|X|}}.$$

Hence,  $\text{bind}(G) > 1$ .  $\square$

**Proposition 4.7.** Every graph can be embedded in a positive binding invariant graph.

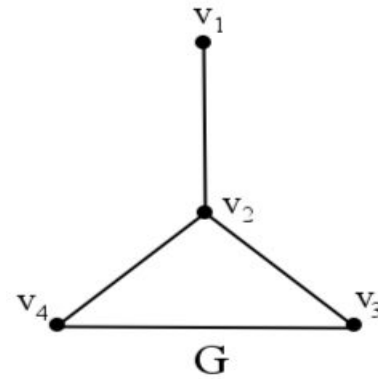
*Proof.* Let  $G$  be a graph of order  $n$ . Let  $M$  be the maximum matching of  $G$ . Add  $n - 2|M| + 2$  new vertices and add edges to have a perfect matching. Clearly by adding any edge does not alter the binding number, hence we get a positive binding invariant graph.  $\square$

### 5. Binding degree of a graph

The binding number and local binding number of a graph give layered measures of the vulnerability of a graph. This motivated us to define a new parameter called the binding degree of a graph. This works as a triple layered parameter combining the degree of a vertex and its local binding number, so that the duality of local connectedness and local vulnerability are addressed. We first define the parameter and then determine it for different class of graphs and then check for the changes occurring in the binding degree of a graph under various edge operations.

**Definition 5.1.** The binding degree of  $G$ , denoted by  $b^d(G)$ , is defined as  $b^d(G) = \sum_{i=1}^n \text{deg}(v_i) \text{bind}_{v_i}(G)$ .

**Illustration:**



In  $G$ , we have  $|V(G)| = 4$  and  $|E(G)| = 4$ . Note that  $\text{bind}_{v_1}(G) = \frac{1}{1}$ ,  $\text{deg}(v_1) = 1$ ,  $\text{bind}_{v_2}(G) = \frac{3}{1}$ ,  $\text{deg}(v_2) = 3$ ,  $\text{bind}_{v_3}(G) = \frac{2}{2} = 1$ ,  $\text{deg}(v_3) = 2$ ,  $\text{bind}_{v_4}(G) = \frac{2}{2} = 1$  and  $\text{deg}(v_4) = 2$ . It follows that,  $b^d(G) = \text{deg}(v_1)\text{bind}_{v_1}(G) + \text{deg}(v_2)\text{bind}_{v_2}(G) + \text{deg}(v_3)\text{bind}_{v_3}(G) + \text{deg}(v_4)\text{bind}_{v_4}(G) = 1 * 1 + 3 * 3 + 2 * 1 + 2 * 1 = 14$ .

**Proposition 5.2.** The binding degree of cycles, paths, complete graphs, complete bipartite graphs, wheels is given as follows:

(i) For any path  $P_n$  with  $n \geq 2$  vertices,

$$b^d(P_n) = \begin{cases} 2n - 2, & \text{if } n \text{ is even} \\ \frac{2n^3 - 5n^2 + 4n - 1}{(n+1)(n-2)}, & \text{if } n \text{ is odd.} \end{cases}$$

(ii) For any cycle  $C_n$  with  $n \geq 3$  vertices,

$$b^d(C_n) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ \frac{2n(n-1)}{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

(iii) For any complete graph  $K_n$  with  $n \geq 3$  vertices,  $b^d(K_n) = n(n-1)^2$ .

(iv) For any complete bipartite graph  $K_{m,n}$ ,  $b^d(K_{m,n}) = m^2 + n^2$ .

(v) For  $n \geq 4$ , the binding degree of a wheel is,

$$b^d(W_n) = \begin{cases} \frac{n(n-1)^2}{n-3}, & \text{if } n \text{ is even} \\ n^2 + n + 4, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* (i)

Let  $P_n$  be a path with  $n \geq 2$  vertices. If the vertices of  $P_n$  are labelled as  $v_1, v_2, \dots, v_n$  then we know that  $\text{deg}(v_1) = \text{deg}(v_n) = 1$  and  $\text{deg}(v_2) = \dots = \text{deg}(v_{n-1}) = 2$ . Here we consider two cases.

Case (a): If  $n$  is even.

From [15] we know that  $\text{bind}(P_n) = 1$ , whenever  $n$  is even. For  $v_i \in V(P_n)$ , the local binding number of  $v_i$  with  $1 \leq i \leq n$



is  $bind_{v_i}(P_n) = 1$ . Therefore  $b^d(G) = 2 + 2(n - 2) = 2n - 2$ .

Case (b): If  $n$  is odd.

For  $v_i \in V(P_n)$ , the local binding number of  $\{v_1, v_3, \dots, v_n\}$  is  $bind_{v_i}(P_n) = \frac{n-1}{n+1}$  and the local binding number of  $\{v_2, v_4, \dots, v_{n-1}\}$  is  $bind_{v_i}(P_n) = \frac{n-1}{n-2}$ . Therefore

$$\begin{aligned} b^d(P_n) &= 2 \frac{n-1}{n+1} + 2 \left( \frac{n-3}{2} \right) \left( \frac{n-1}{n+1} \right) + 2 \left( \frac{n-1}{2} \right) \left( \frac{n-1}{n-2} \right) \\ &= (n-1)^2 \left( \frac{2n-1}{(n+1)(n-2)} \right) \\ &= \frac{2n^3 - 5n^2 + 4n - 1}{(n+1)(n-2)}. \end{aligned}$$

Hence the result.

(ii)

Let  $C_n$  be a cycle with  $n \geq 3$  vertices. We know that  $C_n$  is a self-centered, regular graph of regularity 2. Here also we have two cases.

Case (a): If  $n$  is even.

Form [15] we know that  $bind(C_n) = 1$  if  $n$  is even. Therefore  $b^d(C_n) = 2n$ .

Case (b): If  $n$  is odd.

from [15] we know that  $bind(C_n) = \frac{n-1}{n-2}$  for  $n$  is odd. Therefore  $b^d(C_n) = 2n \frac{n-1}{n-2}$ .

Hence the result.

(iii)

Since  $K_n$  for  $n \geq 3$  is a regular graph of regularity  $n - 1$  and all vertices have local binding number is  $n - 1$ , we get  $b^d(K_n) = n(n - 1)^2$ .

(iv)

Let  $K_{m,n}$  be a complete bipartite graph. Let vertices be labelled as  $v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n$ . For  $v_i \in V(K_{m,n})$ , the local binding number of  $v_i$  for  $1 \leq i \leq m$ , is  $bind_{v_i}(K_{m,n}) = \frac{n}{m}$  and the local binding number of  $u_j$  for  $1 \leq j \leq n$ , is  $bind_{u_j}(K_{m,n}) = \frac{m}{n}$ . Therefore  $b^d(G) = mn \frac{n}{m} + mn \frac{m}{n} = m^2 + n^2$ .

(v)

Let  $W_n$  be a wheel with  $n \geq 4$ , we know that  $W_n = C_{n-1} + K_1$ . If  $W_n$  has  $\{v_1, v_2, \dots, v_n\}$  vertices, we know that  $deg(v_n) = n - 1$  and  $deg(v_1) = deg(v_2) = \dots = deg(v_{n-1}) = 3$ . Here we have two cases.

Case (a): If  $n$  is even.

For  $v_i \in V(W_n)$ , from [2], the local binding number of  $v_i$  for  $1 \leq i \leq n - 1$  is  $bind_{v_i}(W_n) = \frac{n-1}{n-3}$  and the local binding number of  $v_n$  is  $bind_{v_n}(W_n) = n - 1$ . Therefore  $b^d(W_n) = 3(n - 1) \left( \frac{n-1}{n-3} \right) + (n - 1)^2 = (n - 1)^2 \left( \frac{n}{n-3} \right)$ .

Case (b): If  $n$  is odd.

For  $v_i \in V(W_n)$ , from [2], the local binding number of  $v_i$  for  $1 \leq i \leq n - 1$  is  $bind_{v_i}(W_n) = \frac{n+1}{n-1}$  and the local binding number of  $v_n$  is  $bind_{v_n}(W_n) = n - 1$ . Therefore  $b^d(W_n) = 3(n - 1) \left( \frac{n+1}{n-1} \right) + (n - 1)^2 = n^2 + n + 4$ .

Hence the result.  $\square$

The next result deals with the binding degree of a fan graph and a double star. For ready reference we first give the

definitions.

**Definition 5.3.** The fan graph  $F_n$  is defined as the graph join  $K_1 + P_{n-1}$ , where  $K_1$  is the trivial one vertex graph and  $P_{n-1}$  is path graph on  $n - 1$  vertices.

**Definition 5.4.** The double star  $S_{m,n}$  is a tree with diameter 3 and central vertices of degree  $m$  and  $n$  respectively and  $m + n$  number of pendant vertices.

**Proposition 5.5.** The binding degree of a double star and a fan graph is given as follows:

(i) For any double star graph  $S_{m,n}$  with  $m, n \geq 2$  vertices,

$$b^d(S_{m,n}) = \begin{cases} \frac{2m^2+6m-2}{2m-1}, & \text{if } m = n \\ \frac{m^3+n^3+m^2n+mn^2+6mn+2m^2-9m-7n+4}{(m+n-1)(m+n-2)}, & \text{if } m < n \\ \frac{m^3+n^3+m^2n+mn^2+6mn+2n^2-7m-9n+4}{(m+n-1)(m+n-2)}, & \text{if } m > n. \end{cases}$$

(ii) For any fan graph  $F_n$  with  $n \geq 4$  vertices,

$$b^d(F_n) = \begin{cases} \frac{n^3-2n^2-4n+6}{n-3}, & \text{if } n \text{ is even} \\ \frac{n^3-n^2-3n+3}{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* (i)

Let  $S_{m,n}$  be a double star with  $m, n \geq 2$ , and hence  $S_{m,n} = K_{m-1} + K_1 + K_1 + K_{n-1}$ . If  $S_{m,n}$  has  $v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n$  vertices with  $deg(v_1) = deg(v_2) = \dots = deg(v_{m-1}) = deg(u_1) = deg(u_2) = \dots = deg(u_{n-1}) = 1$ ,  $deg(v_m) = m$  and  $deg(u_n) = n$ , then we have three cases.

Case (a): If  $m = n$ .

For  $v_i \in S_{m,n}$ , the local binding number of each vertex of  $v_1, v_2, \dots, v_{m-1}$  is  $bind_{v_i}(S_{m,m}) = \frac{1}{m-1}$  from [2], and that of each of  $u_1, u_2, \dots, u_{m-1}$  as  $bind_{u_i}(S_{m,m}) = \frac{1}{m-1}$ ,  $bind_{v_m}(S_{m,n}) = \frac{m+1}{2m-1}$  for  $v_m$  and is  $bind_{u_m}(S_{m,n}) = \frac{m+1}{2m-1}$  for  $u_m$ . Therefore, the binding degree of the double star is,

$$b^d(S_{m,m}) = (2m - 2) \left( \frac{1}{m-1} \right) + 2m \left( \frac{m+1}{2m-1} \right) = \frac{2m^2+6m-2}{2m-1}.$$

Case (b): If  $m < n$ .

For  $v_i \in S_{m,n}$ , the local binding number of each vertex  $v_1, v_2, \dots, v_{m-1}$  is  $bind_{v_i}(S_{m,n}) = \frac{2}{m+n-2}$ , and that of  $u_1, u_2, \dots, u_{n-1}$  is  $bind_{u_i}(S_{m,n}) = \frac{1}{n-1}$ ,  $bind_{v_m}(S_{m,n}) = \frac{m+1}{m+n-1}$  for  $v_m$  and  $bind_{u_m}(S_{m,n}) = \frac{n+1}{m+n-1}$  for  $u_m$ . Therefore  $b^d(S_{m,n}) = (n - 1) \frac{1}{n-1} + (m - 1) \frac{2}{m+n-2} + m \left( \frac{m+1}{m+n-1} \right) + n \left( \frac{n+1}{m+n-1} \right) = \frac{m^3+n^3+m^2n+mn^2+6mn+2m^2-9m-7n+4}{(m+n-1)(m+n-2)}$ . Case (c): If  $m > n$ .

Similar to Case(b), we get,

$$b^d(S_{m,n}) = (m - 1) \frac{1}{m-1} + (n - 1) \frac{2}{m+n-2} + m \left( \frac{m+1}{m+n-1} \right) + n \left( \frac{n+1}{m+n-1} \right) = \frac{m^3+n^3+m^2n+mn^2+6mn+2n^2-7m-9n+4}{(m+n-1)(m+n-2)}.$$

Hence the result.

(ii)



If  $F_n$  is a fan graph with  $n \geq 4$ , labelled  $v_1, v_2, \dots, v_n$  vertices. We know that  $deg(v_n) = n - 1$ ,  $deg(v_1) = deg(v_{n-1}) = 2$  and  $deg(v_2) = deg(v_3) = \dots = deg(v_{n-2}) = 3$ . Here we have two cases.

Case (a): If  $n$  is even.

For  $v_i \in V(F_n)$ ,  $i = 2, 4, \dots, n - 1$ , the local binding number is  $bind_{v_i}(F_n) = 1$ , the local binding number of  $v_3, v_5, \dots, v_{n-2}$  is  $bind_{v_i}(F_n) = \frac{n-1}{n-3}$  and the local binding number of  $v_n$  is  $bind_{v_n}(F_n) = n - 1$ .

$$\begin{aligned} \text{Therefore } b^d(F_n) &= 4 + 3\left(\frac{n-4}{2}\right) + 3\left(\frac{n-2}{2}\right)\left(\frac{n-1}{n-3}\right) + (n-1)(n-1) \\ &= \frac{n^3 - 2n^2 - 4n + 6}{n-3}. \end{aligned}$$

Case (b): If  $n$  is odd.

For  $v_i \in V(F_n)$ , the local binding number of  $v_i$  with  $1 \leq i \leq n - 1$  is  $bind_{v_i}(F_n) = \frac{n-1}{n-2}$  and the local binding number of  $v_n$  is  $bind_{v_n}(F_n) = n - 1$ .

$$\text{Therefore } b^d(F_n) = 4\left(\frac{n-1}{n-2}\right) + 3(n-3)\left(\frac{n-1}{n-2}\right) + (n-1)(n-1) = \frac{n^3 - n^2 - 3n + 3}{n-2}.$$

Hence the result.  $\square$

**Remark 5.6.** Since we are concentrating on the changes due to edge operations here, we are not going deep into finding binding degree of other graphs. But we consider the variation in binding degree due to different edge operations in the following section.

## 6. Changes in the binding degree due to edge contraction and deletion

Adhering to the central theme of the paper, in this part we prove results that deal with the variation in binding degree of several graphs due to edge contraction.

**Proposition 6.1.** The change in binding degree on contraction of an edge is as follows:

- (1) On contraction of any edge in complete graph, binding degree defers by  $(n - 1)(3n - 4)$ . That is,  $b^d(K_n) - b^d(K_n/e) = (n - 1)(3n - 4)$ .
- (2) On contraction of any edge in an even cycle, binding degree defers by  $\frac{-4}{n-3}$ . That is,  $b^d(C_n) - b^d(C_n/e) = \frac{-4}{n-3}$ .
- (3) On contraction of any edge in an odd cycle, binding degree defers by  $4\left(\frac{n-1}{n-2}\right)$ . That is,  $b^d(C_n) - b^d(C_n/e) = 4\left(\frac{n-1}{n-2}\right)$ .
- (4) On contraction of any edge in an even ordered path, binding degree defers by  $\frac{3n^2 - 14n + 12}{n(n-3)}$ . That is,  $b^d(P_n) - b^d(P_n/e) = \frac{3n^2 - 14n + 12}{n(n-3)}$ .
- (5) On contraction of any edge in an odd ordered path, binding degree defers by  $\frac{n^2 + 4n - 9}{(n+1)(n-2)}$ . That is,  $b^d(P_n) - b^d(P_n/e) = \frac{n^2 + 4n - 9}{(n+1)(n-2)}$ .

(6) On contraction of any edge in a complete bipartite graph, binding degree defers by  $-(m + n - 2)^2$ . That is,  $b^d(K_{m,n}) - b^d(K_{m,n}/e) = -(m + n - 2)^2$ .

(7) On contraction of any outer cycle edge in a wheel, binding degree defers by  $\frac{2n^2 - 6n + 12}{n-3}$ , whenever  $n$  is even. That is,  $b^d(W_n) - b^d(W_n/e) = \frac{2n^2 - 6n + 12}{n-3}$ .

(8) On contraction of any outer cycle edge in a wheel, binding degree defers by  $\frac{2n^2 - 11n}{n-4}$ , whenever  $n$  is odd. That is,  $b^d(W_n) - b^d(W_n/e) = \frac{2n^2 - 11n}{n-4}$ .

(9) On contraction of any outer path edge in fan graph, binding degree defers by  $\frac{2(n^2 - 3n + 1)}{n-3}$ , whenever  $n$  is even. That is,  $b^d(F_n) - b^d(F_n/e) = \frac{2(n^2 - 3n + 1)}{n-3}$ .

(10) On contraction of any outer path edge in fan graph, binding degree defers by  $\frac{2(n^3 - 6n^2 + 7n + 1)}{(n-2)(n-4)}$ , whenever  $n$  is odd. That is,  $b^d(F_n) - b^d(F_n/e) = \frac{2(n^3 - 6n^2 + 7n + 1)}{(n-2)(n-4)}$ .

*Proof.* The proof follows from Proposition 5.1 and Proposition 5.2.  $\square$

We consider the changes occurring in binding degree on an edge deletion in a graph.

**Proposition 6.2.** The change in binding degree on deletion of an edge is as follows:

- (1) On deletion of an edge in an even cycle, binding degree defers by 2.
- (2) On deletion of an edge in an odd cycle, binding degree defers by  $\frac{(n-1)(5n-1)}{(n+1)(n-2)}$ .
- (3) On deletion of an edge in a complete graph, binding degree defers by  $n^2 - 2$ .

*Proof.* Proof follows from the fact that,  $C_n - e \approx P_n$ , whether  $n$  is odd or even and the difference between respective binding degree can be obtained by applying Proposition 5.1. In case of complete graphs, it is easy to verify that, the local binding numbers are  $\frac{n-2}{2}$  and  $(n - 1)$ . Therefore  $b^d(K_n) - b^d(K_n - e) = n^2 - 2$ .  $\square$

## 7. Conclusion

This paper is mainly divided into two parts, the first one deals with the changes in binding number of a graphs under different edge operations. In the second part we introduce a new graph parameter called the binding degree of a graph and determine its exact value for some class of graphs. This study is being extended to many more graphs in general and its practical applicability is looked into, as the binding degree of a graph has beautiful duality associated with it, in terms of local connectedness and local vulnerability for connectedness. Not to deviate from the theme of the paper, we have checked the changes occurring in binding degree of some graphs due to different edge operations.



## References

- [1] I. Anderson, Binding number of a graph, A survey in graph Theory, *Advances in Graph Theory*, 1991, 1–10.
- [2] E. Aslan, The Average Binding Number of Graphs, *Science Asia*, 45(2019), 85-91.
- [3] R. D. Cordero, R. C. Guerrero, Binding Number of Corona and Join of Graphs, *International Mathematical Forum*, 9(2014), 429-436.
- [4] F. Harary, *Graph Theory*, Addison Wesley/ Narosa Publishing House Reprint, 2001.
- [5] F. Harary, W. T. Tutte, A dual form of Kuratowski's theorem, *Cand. Math. Bull.*, 8(373)(1965), 17–20.
- [6] Medha Itagi Huilgol and S. Kiran, Domination essential edges in a graph: Edge contraction and deletion, (Communicated).
- [7] Medha Itagi Huilgol and S. Kiran, Domination non-essential edges in a graph, *International Journal of Mathematical Analysis* 14(2020), 241–249.
- [8] V. G. Kane, *On binding number of a graph*, Indian Institute of Technology, Doctoral Thesis, Kanpur, India, 1976.
- [9] M. Kwasnik and D. Michalak, The Join of Graphs and the binding minimality, *Casopis Pro Pestovani Matematiky* 114(1989), 262–275.
- [10] H. B. Walikar, B. B. Mulla, On graphs with binding number one, *International Journal of Pure and Applied Mathematics*, 98(2015), 413–418.
- [11] H. B. Walikar, B. B. Mulla, Power of a graph and binding number, *International Journal of Pure and Applied Mathematics*, 101(2015), 505–511.
- [12] H. B. Walikar, F. Buckley, M. K. Itagi, Diameter-essential edges in a graph, *Discrete Mathematics*, 259(2002), 211–225.
- [13] H. B. Walikar, F. Buckley, M. K. Itagi, Radius-edge-invariant and diameter edge-invariant graphs, *Discrete Mathematics*, 272(2003), 119–126.
- [14] K. Wagner, Uber eine Eigenschaft ebenen Komplexe, *Math. Ann.* 114(1937), 570–590.
- [15] D. R. Woodall, The binding number of a graph and its Anderson number, *J. Combinatorial Theory Ser. B*, 15(1973), 225–255.
- [16] G. Xu, X. Li, S. Zhang, The binding number of a digraph, *CJCDGCGT*, 4381(2007), 221–227.

\*\*\*\*\*

ISSN(P):2319 – 3786

Malaya Journal of Matematik

ISSN(O):2321 – 5666

\*\*\*\*\*

