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On level subsets of intuitionistic L-fuzzy graphs

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Abstract

In this paper, we introduce the concept of intuitionistic L-fuzzy graphs and partial intuitionistic L-fuzzy subgraphs. Next, we define (s,t) level subsets and t cut sets of Intuitionistic L-Fuzzy Graphs and then move on to a systematic study of their structural properties especially with regard to the associated crisp graphs.

Keywords

Intuitionistic L-fuzzy graph, Partial intuitionistic L-fuzzy subgraph, (s,t) level subsets of Intuitionistic L-fuzzy graph, t cut sets of Intuitionistic L-fuzzy graph.

AMS Subject Classification

05C72.

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Contents

1. Introduction

In 1736, Euler introduced Graph theory, the most inter disciplinary branches in mathematics with a great variety of applications. To describe the phenomena of uncertainity, in 1965 Lotfi. A. Zadeh introduced a new mathematical frame work in his seminal paper entitled "Fuzzy Sets" [9]. The fuzzy sets give the degree of membership of an element in a given set. To describe the uncertainity in objects and in their relationships, Rosenfeld[6] introduced fuzzy graph theory in 1975. Yeh and Bang[8] also introduced fuzzy graphs independently. As a generalization of fuzzy sets, Atanassov[2] introduced the concept of intuitionistic fuzzy sets in 1986 by adding a new component which determines the degree of non-membership in the definition of fuzzy set. i.e., Intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership(which are more or less independent from each other) such that the only requirement is that the sum of these two degrees should be less than or equal to 1. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry etc. Atanassov[3] introduced the concept of intuitionistic fuzzy relations and Intuitionistic Fuzzy Graphs(IFG). Akram M and Davvaz B introduced the concept of intuitionistic fuzzy graph elaborately and analysed its components[1]. Pramada Ramachandran and K. V. Thomas introduced the concept of L-Fuzzy graph[5] as another extension of fuzzy graph.

In this paper, we introduce Intuitionistic L-fuzzy graph as a generalization of L-fuzzy graph. We discuss about its (s,t) level sets and t-cuts. We also try to study the properties of (s,t) level subset of an Intuitionistic L-fuzzy graph.

2. Preliminaries

In this section, we review some basic definitions that are necessary to understand the new concepts introduced in this paper.

Definition 2.1. *[4]*

A fuzzy graph $G = (V, \sigma, \mu)$ with the underlying set V is a nonempty, finite set V together with a pair of functions (σ, μ) where $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y), \quad \forall\, (x,y) \in V \times V$.

Definition 2.2. *[1]*

An intuitionistic fuzzy graph with underlying set V is defined to be a pair $G = (\sigma, \mu)$ where

- *the functions $M_\sigma : V \to [0,1]$ and $N_\sigma : V \to [0,1]$ denote the degree of membership and non membership respectively, such that* $0 \le M_\sigma(x) + N_\sigma(x) \le 1, \forall\ x \in V$.
- *the functions $M_\mu : E \subseteq V \times V \to [0,1]$ and $N_\mu : E \subseteq V \times V \to [0,1]$ are defined by* $M_\mu(x,y) \le \min(M_\sigma(x), M_\sigma(y))$ *and* $N_\mu(x,y) \ge \max(N_\sigma(x), N_\sigma(y))$ *such that* $0 \le M_\mu(x,y) + N_\mu(x,y) \le 1, \forall (x,y) \in E$.

  We call σ the intuitionistic fuzzy vertex set of G, μ the intuitionistic fuzzy edge set of G, respectively.

Definition 2.3. *[7] A Lattice is a partially ordered set $(L, \le)$ in which every pair of elements $a, b \in L$ has a greatest lower bound $a \wedge b$ (called a meet b) and a least upper bound $a \vee b$ (called a join b).*

Definition 2.4. *[7] A Lattice is an algebraic system $(L, \wedge, \vee)$ with two binary operations $\wedge$ and $\vee$ which are both commutative, associative and satisfy the absorption laws.*

Definition 2.5. *[7] A lattice is called a complete lattice if each of its nonempty subsets has a least upper bound and a greatest lower bound. Every complete lattice must have a least element* 0 *and a greatest element* 1.

Definition 2.6. *A unary operation $c : L \to L$ is said to be an involutive order reversing operation if it is an involutive (i.e., $c(c(a)) = a$, for all $a \in L$) that inverts the ordering (i.e., $a \le b$ implies $c(b) \le c(a)$).*

Definition 2.7. *Let L be a complete lattice with an involutive order reversing operation $c : L \to L$. An intuitionistic L-fuzzy set (ILFS) A in X is defined as an object of the form $A = \langle x, \mu_A(x), \nu_A(x) \rangle$ where $\mu_A : X \to L$ and $\nu_A : X \to L$ define the degree of membership and the degree of non-membership of an element $x \in X$ respectively and for every $x \in X$ satisfying $\mu_A(x) \le c(\nu_A(x))$.*

Definition 2.8. *[5] An L-fuzzy graph (LFG) $G^L = (V, \sigma, \mu)$ with the underlying set V is a nonempty set V together with a pair of functions $\sigma : V \to L$ and $\mu : V \times V \to L$ such that $\mu(x,y) \le \sigma(x) \wedge \sigma(y)$, $\forall x, y \in V$.*

3. Main Results

In this section we have introduced the intuitionistic L-fuzzy graphs and its associated structures and discussed the properties with regard to the associated crisp graphs.

Definition 3.1. *Let L be a complete lattice with an involutive order reversing operation $c : L \to L$. An intuitionistic L-fuzzy graph (ILFG) G^L with underlying set V is defined to be $G = (V, \sigma, \mu)$ where*

1. *the functions $M_\sigma : V \to L$ and $N_\sigma : V \to L$ should satisfy $M_\sigma(x) \le c\big(N_\sigma(x)\big), \forall\ x \in V$. Here $M_\sigma(x)$ and $N_\sigma(x)$ denote the degree of membership and degree of non-membership of the vertex $x \in V$ respectively.*
2. *the functions $M_\mu : E \to L$ and $N_\mu : E \to L$ where $E = V \times V \setminus \{(x,x)/x \in V\}$ should satisfy*
   $M_\mu(x,y) \le M_\sigma(x) \wedge M_\sigma(y)$
   $N_\mu(x,y) \ge N_\sigma(x) \vee N_\sigma(y)$
   $M_\mu(x,y) \le c\big(N_\mu(x,y)\big), \forall\ (x,y) \in E$. *Here $M_\mu(x,y)$ and $N_\mu(x,y)$ denote the degree of membership and degree of non-membership of the edge $(x,y) \in E$ respectively.*

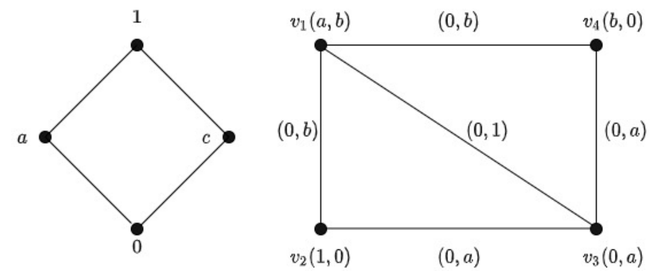

Figure 1. Lattice L and Intuitionistic L-Fuzzy Graph G^L

In figure 1, the order reversing operation c on L is defined by $c : L \to L$ as $c(0) = 1, c(a) = b, c(b) = a$ and $c(1) = 0$

Note 3.2.
1. *$M_\mu(x,y) = N_\mu(x,y) = 0$ for some $x, y \in V$ means there is no edge between the vertices x and y. Otherwise there is always an edge between the vertices x and y*
2. *In an ILFG $G^L = (V, \sigma, \mu)$, the degree of hesitance or hesitation of the vertex $x \in V$ is defined as $\pi_\sigma(x) = c\big(M_\sigma(x) \vee N_\sigma(x)\big)$ and the degree of hesitance or hesitation of the edge $(x,y) \in E$ is defined as $\pi_\mu(x,y) = c\big(M_\mu(x,y) \vee N_\mu(x,y)\big)$*

   In figure 1, $\pi_\sigma(v_1) = 0, \pi_\sigma(v_4) = a$ and $\pi_\mu(v_1, v_2) = a$

Definition 3.3. *Let L be a complete lattice with an involutive order reversing operation c and $G = (V, E)$ be a crisp graph. Let $M_\sigma, N_\sigma : V \to L$ and $M_\mu, N_\mu : E \to L$ such that*

$$M_\sigma(x) \le c(N_\sigma(x)), \qquad \forall x \in V$$
$$\begin{cases} M_\mu(x,y) \le M_\sigma(x) \wedge M_\sigma(y), \\ N_\mu(x,y) \ge N_\sigma(x) \vee N_\sigma(y), \quad \forall (x,y) \in E \\ M_\mu(x,y) \le c(N_\mu(x,y)) \end{cases}$$

with either $M_\mu(x,y) \neq 0$ or $N_\mu(x,y) \neq 0, \forall (x,y) \in E$

Then the resulting graph is an intuitionistic L-fuzzy graph, called intuitionistic L-fuzzy graph on G

Definition 3.4. *An ILFG $H^L = (V, \nu, \tau)$ is said to be partial intuitionistic L-fuzzy subgraph of the ILFG $G^L = (V, \sigma, \mu)$ if $M_\nu(x) \le M_\sigma(x)$ & $N_\nu(x) \ge N_\sigma(x), \forall x \in V$ and $M_\tau(x,y) \le M_\mu(x,y)$ & $N_\tau(x,y) \ge N_\mu(x,y), \forall\ (x,y) \in E$*

Note 3.5. *In figure 2, the order reversing operation c on L is defined by $c : L \to L$ as $c(0) = 1, c(a) = f, c(b) = d, c(c) = e, c(f) = a, c(d) = b, c(e) = c, c(1) = 0$*

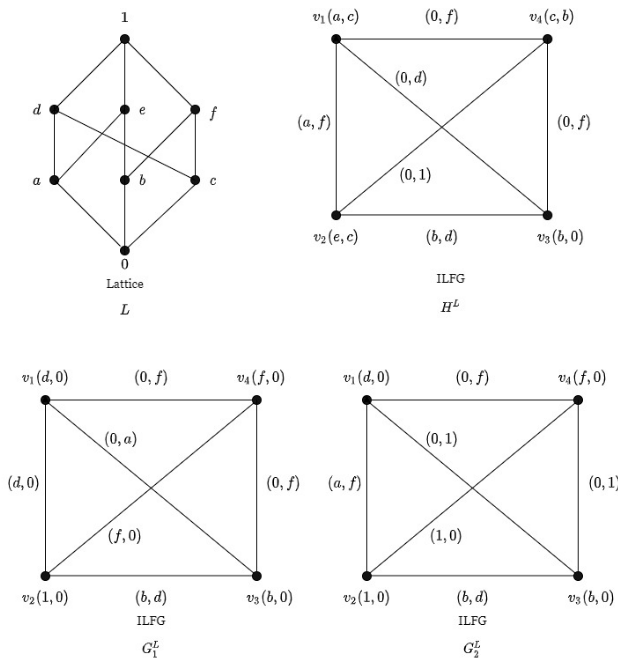

Figure 2. Partial Intuitionistic L-Fuzzy Subgraph

Here H^L is a partial intuitionistic L-fuzzy subgraph of G_1^L, but not a partial intuitionistic L-fuzzy subgraph of G_2^L

Definition 3.6. *An ILFG $G^L = (V, \sigma, \mu)$ is said to be complete ILFG, if $M_\mu(x,y) = M_\sigma(x) \wedge M_\sigma(y)$ and*
$N_\mu(x,y) = N_\sigma(x) \vee N_\sigma(y)$, $\forall x,y \in V$

Definition 3.7. *An ILFG $G^L = (V, \sigma, \mu)$ is said to be semi M-stong ILFG, if*
$M_\mu(x,y) = M_\sigma(x) \wedge M_\sigma(y)$, $\forall (x,y) \in E$

Definition 3.8. *An ILFG $G^L = (V, \sigma, \mu)$ is said to be semi N-stong ILFG, if*
$N_\mu(x,y) = N_\sigma(x) \vee N_\sigma(y)$, $\forall (x,y) \in E$

Definition 3.9. *An ILFG $G^L = (V, \sigma, \mu)$ is said to be strong ILFG, if*
$M_\mu(x,y) = M_\sigma(x) \wedge M_\sigma(y)$ *and*
$N_\mu(x,y) = N_\sigma(x) \vee N_\sigma(y)$, $\forall (x,y) \in E$

Definition 3.10. *Consider the ILFG $G^L = (V, \sigma, \mu)$ and let s and t are elements of L. Then*
$\sigma_{(s,t)} = \{x \in V : M_\sigma(x) \geq s \ \& \ N_\sigma(x) \leq t\}$ *is called (s,t) level subset of σ and*
$\mu_{(s,t)} = \{(x,y) \in E : M_\mu(x,y) \geq s \ \& \ N_\mu(x,y) \leq t\}$ *is called (s,t) level subset of μ.*
Then (s,t) level subset of G^L is $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ which is always a crisp graph.

Example 3.11. *Let L and H^L are as in figure 2. Then figure 3 shows $(0, f)$ level subset of H^L*

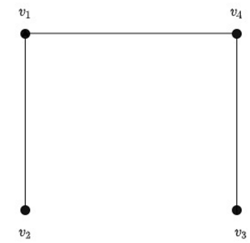

Figure 3. Level Subset

Definition 3.12. *Consider the ILFG $G^L = (V, \sigma, \mu)$. Then for each $t \in L$, there are two types of cut sets corresponding to membership value and non membership value defined by*
$M_{\sigma_t} = \{x \in V : M_\sigma(x) \geq t\}$, $M_{\mu_t} = \{(x,y) \in E : M_\mu(x,y) \geq t\}$ *and*
$N_{\sigma_t} = \{x \in V : N_\sigma(x) \leq t\}$, $N_{\mu_t} = \{(x,y) \in E : N_\mu(x,y) \leq t\}$
Here N_{σ_t}, N_{μ_t} can be called '$\leq t$ cut' of vertices and edges respectively.
Here $(M_{\sigma_t}, M_{\mu_t})$ and $(N_{\sigma_t}, N_{\mu_t})$ are crisp graphs.

Proposition 3.13. *The ILFG $G^L = (V, \sigma, \mu)$ is a complete ILFG if and only if the crisp graph $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ is complete, for all elements s and t of L.*

Proof. Let the ILFG $G^L = (V, \sigma, \mu)$ is a complete ILFG. We have to show if $x, y \in \sigma_{(s,t)}$ then $(x,y) \in \mu_{(s,t)}$.
Now,
$x, y \in \sigma_{(s,t)} \Rightarrow M_\sigma(x) \geq s$, $N_\sigma(x) \leq t$ & $M_\sigma(y) \geq s$, $N_\sigma(y) \leq t$
and since G^L is a complete ILFG,
$M_\mu(x,y) = M_\sigma(x) \wedge M_\sigma(y)$ and
$N_\mu(x,y) = N_\sigma(x) \vee N_\sigma(y)$, $\forall\, x,y \in V$
Hence, $M_\mu(x,y) \geq s$ and $N_\mu(x,y) \leq t$.
That is $(x,y) \in \mu_{(s,t)}$.
So if $x, y \in \sigma_{(s,t)}$, then $(x,y) \in \mu_{(s,t)}$.
i.e., $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ is a complete graph for all elements s and t of L.

Hence, if the ILFG $G^L = (V, \sigma, \mu)$ is complete, then the crisp graph $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ is complete for all elements s and t of L.

Conversly, assume that $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ is complete for all elements s and t of L. i.e., $\forall\, x,y \in \sigma_{(s,t)}$, we have $(x,y) \in \mu_{(s,t)}$. We need to show, $M_\mu(x,y) = M_\sigma(x) \wedge M_\sigma(y)$ and $N_\mu(x,y) = N_\sigma(x) \vee N_\sigma(y)$, $\forall\, x,y \in V$.
Let $M_\sigma(x) = a_1, N_\sigma(x) = b_1$ and $M_\sigma(y) = a_2, N_\sigma(y) = b_2$
So $x \in \sigma_{(a_1,b_1)}$ and $y \in \sigma_{(a_2,b_2)}$
i.e., $x, y \in \sigma_{(a_1 \wedge a_2, b_1 \vee b_2)}$
So $(x,y) \in \mu_{(a_1 \wedge a_2, b_1 \vee b_2)}$, since $G^L_{(s,t)}$ is complete.
i.e., $M_\mu(x,y) \geq a_1 \wedge a_2$ and $N_\mu(x,y) \leq b_1 \vee b_2$
But we have,
$M_\mu(x,y) \leq M_\sigma(x) \wedge M_\sigma(y)$ and $N_\mu(x,y) \geq N_\sigma(x) \vee N_\sigma(y)$
i.e., $M_\mu(x,y) \leq a_1 \wedge a_2$ and $N_\mu(x,y) \geq b_1 \vee b_2$
Hence,
$M_\mu(x,y) = a_1 \wedge a_2$ and $N_\mu(x,y) = b_1 \vee b_2$

Since a_1, a_2, b_1, b_2 are arbitrary elements of L, we have $M_\mu(x,y) = M_\sigma(x) \wedge M_\sigma(y)$ and $N_\mu(x,y) = N_\sigma(x) \vee N_\sigma(y)$, $\forall\, x,y \in V$.
i.e., The ILFG $G^L = (V,\sigma,\mu)$ is a complete ILFG.

Hence if $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ is complete for all elements s and t of L, then the ILFG $G^L = (V,\sigma,\mu)$ is a complete ILFG. □

Remark 3.14. *The ILFG $G^L = (V,\sigma,\mu)$ is a strong ILFG need not imply the crisp graph $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ is complete, for all elements s and t of L.*

Proposition 3.15. *Consider the ILFG $G^L = (V,\sigma,\mu)$ and s and t are elements of L with $s \wedge t = u, s \vee t = v$, then $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$ is a crisp subgraph of $G^L_{(u,v)} = (\sigma_{(u,v)}, \mu_{(u,v)})$*

Proof. Consider $G_{(s,t)}$

$$\begin{aligned} x \in \sigma_{(s,t)} &\Rightarrow M_\sigma(x) \geq s \;\&\; N_\sigma(x) \leq t \\ &\Rightarrow M_\sigma(x) \geq u \;\&\; N_\sigma(x) \leq v \\ &\Rightarrow x \in \sigma_{(u,v)} \end{aligned}$$

Hence,$\sigma_{(s,t)} \subseteq \sigma_{(u,v)}$
Also

$$\begin{aligned} (x,y) \in \mu_{(s,t)} &\Rightarrow M_\mu(x,y) \geq s \;\&\; N_\mu(x,y) \leq t \\ &\Rightarrow M_\mu(x,y) \geq u \;\&\; N_\mu(x,y) \leq v \\ &\Rightarrow (x,y) \in \mu_{(u,v)} \end{aligned}$$

i.e., $\mu_{(s,t)} \subseteq \mu_{(u,v)}$

Hence $G_{(s,t)}$is a crisp subgraph of $G_{(u,v)}$ □

Proposition 3.16. *Consider the ILFG $G^L = (V,\sigma,\mu)$ and $s_1 \leq s_2$ and $t_1 \geq t_2$ are the elements of L, then $G^L_{(s_2,t_2)} = (\sigma_{(s_2,t_2)}, \mu_{(s_2,t_2)})$ is a crisp subgraph of $G^L_{(s_1,t_1)} = (\sigma_{(s_1,t_1)}, \mu_{(s_1,t_1)})$.*

Proof. We have

$$\begin{aligned} x \in \sigma_{(s_2,t_2)} &\Rightarrow M_\sigma(x) \geq s_2 \;\&\; N_\sigma(x) \leq t_2 \\ &\Rightarrow M_\sigma(x) \geq s_1 \;\&\; N_\sigma(x) \leq t_1 \\ &\Rightarrow x \in \sigma_{(s_1,t_1)} \end{aligned}$$

i.e., $\sigma_{(s_2,t_2)} \subseteq \sigma_{(s_1,t_1)}$

Also

$$\begin{aligned} (x,y) \in \mu_{(s_2,t_2)} &\Rightarrow M_\mu(x,y) \geq s_2 \;\&\; N_\mu(x,y) \leq t_2 \\ &\Rightarrow M_\mu(x,y) \geq s_1 \;\&\; N_\mu(x,y) \leq t_1 \\ &\Rightarrow (x,y) \in \mu_{(s_1,t_1)} \end{aligned}$$

i.e., $\mu_{(s_2,t_2)} \subseteq \mu_{(s_1,t_1)}$

$G_{(s_2,t_2)}$is a crisp subgraph of $G_{(s_1,t_1)}$. □

Proposition 3.17. *Let the ILFG $H^L = (V,\nu,\tau)$ be a partial intuitionistic L-fuzzy subgraph of the ILFG $G^L = (V,\sigma,\mu)$ and s,t are the elements of L, then $H^L_{(s,t)} = (\nu_{(s,t)}, \tau_{(s,t)})$ is a crisp subgraph of $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$.*

Proof. Let $H^L = (V,\nu,\tau)$ be a partial intuitionistic L-fuzzy subgraph of the ILFG $G^L = (V,\sigma,\mu)$, so that
$M_\nu(x) \leq M_\sigma(x) \;\&\; N_\nu(x) \geq N_\sigma(x), \forall x \in V$ and
$M_\tau(x,y) \leq M_\mu(x,y) \;\&\; N_\tau(x,y) \geq N_\mu(x,y), \forall (x,y) \in E$
Now

$$\begin{aligned} x \in \nu_{(s,t)} &\Rightarrow M_\nu(x) \geq s \;\&\; N_\nu(x) \leq t \\ &\Rightarrow M_\sigma(x) \geq s \;\&\; N_\sigma(x) \leq t \\ &\Rightarrow x \in \sigma_{(s,t)} \end{aligned}$$

i.e., $\nu_{(s,t)} \subseteq \sigma_{(s,t)}$

Also

$$\begin{aligned} (x,y) \in \tau_{(s,t)} &\Rightarrow M_\tau(x,y) \geq s \;\&\; N_\tau(x,y) \leq t \\ &\Rightarrow M_\mu(x,y) \geq s \;\&\; N_\mu(x,y) \leq t \\ &\Rightarrow (x,y) \in \mu_{(s,t)} \end{aligned}$$

i.e.,$\tau_{(s,t)} \subseteq \mu_{(s,t)}$.

Hence $H^L_{(s,t)} = (\nu_{(s,t)}, \tau_{(s,t)})$ is a crisp subgraph of $G^L_{(s,t)} = (\sigma_{(s,t)}, \mu_{(s,t)})$. □

4. Conclusion

In this paper, we have introduced and defined the intuitionistic L-fuzzy graphs and associated structures. We have also proved some interesting properties of the (s,t) level sets and t-cuts in-terms of the crisp graphs derived from them. We hope to extend this work to applications of fuzzy graphs in real life.

References

[1] M. Akram and B. Davvaz, Strong Intuitionistic Fuzzy Graphs, *Filomat*, 26(1) (2012), 177- 196.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1) (1986), 87-96.

[3] K. T. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications: Studies in Fuzziness and Soft Computing*, Heidelberg, New York, Physica-Verl., (1999).

[4] J. N. Mordeson, P. S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica Verlag, (2000).

[5] R. Pramada, K. V. Thomas, On isomorphisms of L-fuzzy graphs, *Annals of Fuzzy Mathematics and Informatics*, 11(2) (2016), 301-313.

[6] A. Rosenfeld, Fuzzy graphs, *Fuzzy Sets and Their Applications*, Academic Press, New York, (1975), 77-95.

[7] J. P. Tremblay and R. Manohar, *Discrete Mathematical Structures with Applications to Computer Science*, New Delhi, Tata McGraw-Hill,(1997).

[8] R. T. Yeh, S. Y. Bang, Fuzzy relations fuzzy graphs and their applications to clustering analysis, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), *Fuzzy Sets and Their Applications*, Academic Press, (1975), 125-149.

[9] L. A. Zadeh, Fuzzy Sets, *Inform. and Control*, 8 (1965), 338-353.

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