Unit regularity of semigroup of normal cones

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Abstract

Let S be a unit regular semigroup and $\mathscr{L}(S)$ be the normal category of principal left ideals of S. Then the semigroup $T\mathscr{L}(S)$ of normal cones in $\mathscr{L}(S)$ is a unit regular semigroup. We characterize the category of principal left ideals of a unit regular semigroup as a $\mathscr{U}\mathscr{R}$-category. This is a normal category $\mathscr{C}$ with the additional property that there is a maximum object m in $v\mathscr{C}$ and that every isomorphism in $\mathscr{C}$ is the restriction of an isomorphism of m. We prove that in this case the semigroup $T\mathscr{C}$ is a unit regular semigroup.

Keywords

Unit regular semigroup, normal category, normal cone, semigroup of normal cones.

AMS Subject Classification

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1. Introduction

Unit regular semigroups have been studied by several people including Sreeja and Rajan (cf.[8] and [9]). These semigroups can be built from the set of idempotents and the group of units of the semigroup. K.S.S. Nambooripad in his theory of cross connections introduced the concept of normal categories. Normal categories are essentially categories of principal left[right] ideals of regular semigroups. In turn a normal category $\mathscr{C}$ gives rise to a regular semigroup $T\mathscr{C}$ which is the semigroup of all normal cones in $\mathscr{C}$. Capturing special properties in $T\mathscr{C}$ in terms of properties of $\mathscr{C}$ has been an object of study. For example see [10]. Here we introduce the concept of $\mathscr{U}\mathscr{R}$-categories which has properties associated with the normal category of principal left ideals of unit regular semigroups. We show that in this case the semigroup $\mathscr{C}$ of normal cones is a unit regular semigroup.

2. Preliminaries

For a semigroup S we denote by $E(S)$ the set of all idempotents of S. An element $1 \in S$ is said to be the identity of S if $1 \cdot x = x = x \cdot 1$ for all $x \in S$. In a semigroup S with identity, an element u is said to be a unit if there exists $v \in S$ such that $uv = 1 = vu$. A semigroup S with identity is said to be unit regular if for each $x \in S$ there exists a unit $u \in S$ such that

$$x = xux.$$

A unit regular semigroup S is said to be strongly unit regular if for any two $\mathscr{D}$-related idempotents e and f there is a unit $u \in S$ such that

$$f = u^{-1}eu.$$

We follow [1] and [2] for general notations and terminology on semigroups. The basic concepts and terminology relating to unit regular semigroups are as in [3], [8] and [9].

2.1 Normal Categories

We provide some properties of normal categories and the normal categories associated with regular semigroups which are needed in the sequel. The notations and terminology on normal categories as in [5] and [6] are followed here.

All categories considered here are small categories (cf.[4]). For a category $\mathscr{C}$ we denote by $v\mathscr{C}$ the object set of $\mathscr{C}$ and write $f \in \mathscr{C}$ for morphisms f in $\mathscr{C}$.

Definition 2.1. *A normal cone in a normal category is a map $\gamma : v\mathscr{C} \to \mathscr{C}$ satisfying the following.*

(i) *There is an object $c = c_\gamma$ in $v\mathscr{C}$ called the vertex of γ such that for every $a \in v\mathscr{C}$*

$$\gamma(a) : a \to c.$$

(ii) *Whenever $a \leq b$, $\gamma(a) = j(a,b)\gamma(b)$ where $j(a,b)$ is the inclusion morphism from a to b.*

(iii) *There is an object $d \in v\mathscr{C}$ such that $\gamma(d)$ is an isomorphism.*

The following results on morphisms of normal categories will be used in later proofs.

Proposition 2.2 ([5])**.** *Let $\mathscr{C}$ be a normal category and $f : a \to b$ be a morphism in $\mathscr{C}$. Let $f = quj$ be a normal factorization of f. Then*

(i) *If $f = q_1 u_1 j_1$ is another normal factorization of f then $qu = q_1 u_1$ and $j = j_1$.*

(ii) *If f is a monomorphism then $q = 1$.*

(iii) *If f is an isomorphism and an inclusion then $f = 1$.*

Here 1 stands for identity morphism on any object.

Since every inclusion is a monomorphism it follows from (i) above that in any normal factorization $f = quj$, qu and j are uniquely determined. We denote qu by f^o and call it the epimorphic part of f. The semigroup of all normal cones in a normal category is a regular semigroup as given below.

Theorem 2.3 (cf.[5])**.** *Let $\mathscr{C}$ be a normal category and $T\mathscr{C}$ be the set of all normal cones in $\mathscr{C}$. For γ, $\delta \in T\mathscr{C}$ define $\gamma * \delta$ by*

$$(\gamma * \delta)(a) = \gamma(a)(\delta(c_\gamma))^o,$$

for all $a \in v\mathscr{C}$ where $(\delta(c_\gamma))^o$ is the epimorphic part of $\delta(c_\gamma)$. Then $\gamma * \delta$ is a normal cone and $(T\mathscr{C}, *)$ is a regular semigroup.

The normal category $\mathscr{L}(S)$ of principal left ideals of a regular semigroup S is described as follows. The set of objects is the set of all principal left ideals of S. That is

$$v\mathscr{L}(S) = \{Se : e \in E(S)\},$$

where $E(S)$ is the set of all idempotents of S. A morphism $\rho : Se \to Sf$ is a right translation $x \mapsto xu$ for some $u \in eSf$. We denote this morphism as

$$\rho(e,u,f) : Se \to Sf.$$

The following theorem on $\mathscr{L}(S)$ will be used later.

Theorem 2.4 ([5])**.** *Let $\rho(e,u,f) : Se \to Sf$ be a morphism in $\mathscr{L}(S)$. Then $\rho(e,u,f)$ is an isomorphism if and only if $a\mathscr{R}u\mathscr{L}b$ where $\mathscr{R}$ and $\mathscr{L}$ are the Greens relations.*

Proposition 2.5. *Let $\mathscr{L}(S)$ be the normal category associated with a regular semigroup S. Then for each $a \in S, \rho^a$ is a normal cone in $\mathscr{L}(S)$ where*

$$\rho^a(Se) = \rho(e,ea,f),$$

for all $Se \in v\mathscr{L}(S)$ where f is such that $Sf = Sa$. These normal cones are called principal cones.

3. $\mathscr{U}\mathscr{R}$- Categories

We begin by observing the special properties of the normal category $\mathscr{L}(S)$ of principal left ideals of a unit regular semigroup S. Various properties of this category has been described in [7].

Proposition 3.1. *Let S be a unit regular semigroup. Then every isomorphism in $\mathscr{L}(S)$ from S to S is of the form*

$$f = \rho(1,u,1),$$

where u is a unit.

Proof. Suppose $\rho(1,u,1) : S = S1 \to S$ is an isomorphism where $u \in S$. By Theorem 2.4 we have $1\mathscr{R}u\mathscr{L}1$ where $\mathscr{R}$ and $\mathscr{L}$ are the Greens relations. It follows that u is a unit. $\square$

Theorem 3.2. *Let S be a strongly unit regular semigroup. Then every isomorphism $\rho(e,x,f) : Se \to Sf$ in $\mathscr{L}(S)$ is the restriction of an isomorphism $\rho(1,u,1)$ from S to S where u is a unit.*

Proof. Let $\rho(e,x,f) : Se \to Sf$ be an isomorphism. Then $e\mathscr{R}x$ by Theorem 2.4. Since S is strongly unit regular $x = eu$ for some unit u. Then $\rho(1,u,1)$ is an isomorphism from S to S. Now for any $y \in Se$,

$$y\rho(e,x,f) = yx = yeu = yu = y\rho(1,u,1).$$

So $\rho(e,x,f)$ is the restriction of the isomorphism $\rho(1,u,1)$. $\square$

We now define $\mathscr{U}\mathscr{R}$-category as an abstraction of the normal category $\mathscr{L}(S)$ of a strongly unit regular semigroup S.

Definition 3.3. *A normal category $\mathscr{C}$ is said to be a $\mathscr{U}\mathscr{R}$-category if the following hold.*

UR1 *There is an object m such that $a \leq m$ for all $a \in v\mathscr{C}$. In this case m is called the maximum object in $\mathscr{C}$.*

UR2 *Every isomorphism $u : a \to b$ in $\mathscr{C}$ is the restriction of an isomorphism of $\alpha : m \to m$. That is,*

$$j(a,m)\alpha = uj(b,m).$$

We now show that the semigroup $T\mathscr{C}$ of normal cones in a $\mathscr{UR}$-category $\mathscr{C}$ is a unit regular semigroup.

Theorem 3.4. *Let $\mathscr{C}$ be a $\mathscr{UR}$-category with maximum object m. Then we have the following.*

(i) *For every normal cone γ in $\mathscr{C}$ with vertex m, $\gamma(m)$ is an isomorphism.*

(ii) *$G = \{\gamma \in T\mathscr{C} : c_\gamma = m\}$ is a subgroup of $T\mathscr{C}$.*

Proof. Let γ be a normal cone in $\mathscr{C}$ with vertex m. Let $a \in v\mathscr{C}$ be such that $\gamma(a) : a \to m$ is an isomorphism. Now by condition (UR2) of Definition 3.3 $\gamma(a)$ can be extended to an isomorphism $\alpha : m \to m$. That is

$$j(a,m)\alpha = \gamma(a)j(m,m) = \gamma(a),$$

since $j(m,m) = 1_m$. Since α is an isomorphism we have

$$j(a,m) = \gamma(a)\alpha^{-1}$$

and so $j(a,m)$ is an isomorphism. Now by Proposition 2.2 we have $a = m$. So $\gamma(m) = \gamma(a)$ is an isomorphism. Hence (i).

Towards proving (ii), first we show that the normal cone ε with $\varepsilon(m) = 1_m$ is the identity in the semigroup $T\mathscr{C}$ of normal cones. Let $\gamma \in T\mathscr{C}$. Then for any $a \in v\mathscr{C}$,

$$(\gamma\varepsilon)(a) = \gamma(a)(\varepsilon(m))^o = \gamma(a)1_m = \gamma(a),$$

since $\varepsilon(m) = 1_m = (1_m)^o$. So $\gamma\varepsilon = \gamma$. Similarly $\varepsilon\gamma = \gamma$. So ε is the identity in $T\mathscr{C}$.

Now let $\gamma \in G$. Then by (i) above $\gamma(m)$ is an isomorphism. Consider the normal cone δ with $\delta(m) = (\gamma(m))^{-1}$. Then for any $a \in v\mathscr{C}$

$$(\gamma\delta)(m) = \gamma(m)(\gamma(m))^{-1} = 1_m.$$

So $\gamma\delta = \varepsilon$. Similarly $\delta\gamma = \varepsilon$. It follows that G is a group. □

Theorem 3.5. *Let $\mathscr{C}$ be a $\mathscr{UR}$-category. Then $T\mathscr{C}$ is a unit regular semigroup.*

Proof. Let $\gamma \in T\mathscr{C}$ with $c_\gamma = b$. Let $a \in v\mathscr{C}$ be such that $\gamma(a)$ is an isomorphism. Now by condition (UR2) of Definition 3.3 $(\gamma(a))^{-1}$ can be extended to an isomorphism $\alpha : m \to m$. That is

$$j(b,m)\alpha = (\gamma(a))^{-1}j(a,m).$$

Let σ be the normal cone with $\sigma(m) = \alpha$. Then σ has vertex m and so $\sigma \in G$. Now for any $c \in c\mathscr{C}$,

$$
\begin{aligned}
(\gamma\sigma\gamma)(c) &= (\gamma\sigma)(c)(\gamma(m))^o \\
&= \gamma(c)(\sigma(b)\gamma(m))^o \\
&= \gamma(c)(j(b,m)\sigma(m)\gamma(m))^o \\
&= \gamma(c)(j(b,m)\alpha\gamma(m))^o \\
&= \gamma(c)((\gamma(a))^{-1}j(a,m)\gamma(m))^o \\
&= \gamma(c)((\gamma(a))^{-1}\gamma(a))^o \\
&= \gamma(c)(1_b)^o \\
&= \gamma(c).
\end{aligned}
$$

Therefore $\gamma\sigma\gamma = \gamma$. Thus $T\mathscr{C}$ is unit regular with group of units G.

□

4. Conclusion and Scope

We considered the semigroup $T\mathscr{C}$ of normal cones in a $\mathscr{UR}$-category and proved that it is a unit regular semigroup. Also it has been shown that $\mathscr{UR}$-categories arise from strongly unit regular semigroups. Characterization of the category of principal left ideals of a strongly unit regular semigroup is not complete here as the semigroup $T\mathscr{C}$ has not been shown to be strongly unit regular. Finding such a characterization needs further study of this category.

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