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Solution of non-linear partial differential equations by Shehu transform and its applications

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Abstract

The aim of this paper is to combine the homotopy perturbation method and the new integral transform, namely, Shehu transform, which generalize both the Sumudu and Laplace integral transforms, in order to deduce an analytical solution of the non-linear partial differential equations. Several non-linear partial differential equations arising in various models are treated to prove the reliability of this method. It has been shown that the method provides a powerful tool for solving these equations.

Keywords

Homotopy perturbation method, Shehu transform method, non-linear partial differential equations, He's polynomials.

AMS Subject Classification

44A05, 26A33, 44A20, 34K37.

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1. Introduction

Many problems in physics (quantum physics, electromagnetism), chemistry (reaction-diffusion mechanisms) and biology (population dynamics) can be described by nonlinear partial differential equations NLPDEs. However, this type of equation is difficult to study. There are no general techniques that work for all such equations. In fact, finding exact solutions of NLPDEs is a challenge in certain cases. For this reason, a great improvement has been made in this field of research and different methods and techniques are developed in recent years. Among these we can mention the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), the variational iteration method (VIM), and others.

On the other hand, a new option has recently emerged, including the formation of the Laplace, the Sumudu, the Natural, the Elzaki, the Aboodh and the Z Transform and other. These important transformations are frequently used to solve linear differential equations or integral differential equations.

By taking advantage of these transformations to solve nonlinear differential equations, researchers in the field of mathematics were guided to the idea of their composition with some methods such as: Adomian decomposition method ([2]-[4], [15],[16], [17] and [26]), variational iteration method([1], [11]-[12], [5], [6]) and homotopy perturbation method (HPM) ([18], [19]-[20], [13]-[14], [23], [24], [10], [9]-[22]).

In this context, the objective of the present study is to combine the homotopy perturbation method and Shehu transform method ([21], [7] and [8]) to get an appropriate method to solve non-linear partial differential equations. The resulting method is called the Shehu Homotopy Perturbation Method (SHPM). Next, this modified method is applied to solve some examples related to nonlinear partial differential equations.

2. Shehu transform homotopy perturbation method

To illustrate the basic idea of this method, we consider a general non-linear non-homogeneous partial differential equation

$$\frac{\partial^m U(x,t)}{\partial t^m} + RU(x,t) + NU(x,t) = g(x,t), \tag{2.1}$$

where $\frac{\partial^m U(x,t)}{\partial t^m}$ is the partial derivative of the function $U(x,t)$ of order m ($m = 1,2,3$), R is the linear differential operator, N represents the general non-linear differential operator, and $g(x,t)$ is the source term. Applying the Shehu transform on both sides of Equ.(2.1), we get:

$$\mathbb{H}\left[\frac{\partial^m U(x,t)}{\partial t^m}\right] + \mathbb{H}\left[RU(x,t) + NU(x,t)\right] = \mathbb{H}\left[g(x,t)\right]. \tag{2.2}$$

Using the properties of Shehu transform, we obtain:

$$\begin{aligned}\frac{s^m}{u^m}\mathbb{H}\left[U(x,t)\right] = \sum_{k=0}^{m-1}\left(\frac{s}{u}\right)^{m-(k+1)}\frac{\partial^k U(x,0)}{\partial t^k} \\ +\mathbb{H}\left[g(x,t)\right] - \mathbb{H}\left[RU(x,t) + NU(x,t)\right],\end{aligned} \tag{2.3}$$

where $m = 1,2,3$.

And thus, we have:

$$\begin{aligned}\mathbb{H}\left[U(x,t)\right] = \sum_{k=0}^{m-1}\left(\frac{u}{s}\right)^{k+1}\frac{\partial^k U(x,0)}{\partial t^k} \\ +\frac{u^m}{s^m}\mathbb{H}\left[g(x,t)\right] - \frac{u^m}{s^m}\mathbb{H}\left[RU(x,t) + NU(x,t)\right].\end{aligned} \tag{2.4}$$

Applying the inverse transform on both sides of Equ.(2.4), we get:

$$U(x,t) = G(x,t) - \mathbb{H}^{-1}\left(\frac{u^m}{s^m}\mathbb{H}\left[RU(x,t) + NU(x,t)\right]\right), \tag{2.5}$$

where $G(x,t)$, represents the term arising from the source term and the prescribed initial conditions.

The classical homotopy perturbation technique HPM for Eq.(2.5) is constructed as follows:

The solution can be expressed by the infinite series given below

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{2.6}$$

where p is considered as a small parameter ($p \in [0,1]$). The non-linear term can be decomposed as:

$$NU(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{2.7}$$

where H_n are He's polynomials of $U_0, U_1, U_2, ..., U_n$, which can be calculated by the following formula

$$H_n(u_0,...,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i U_i\right)\right]_{p=0}, \quad n = 0,1,2,3,\cdots \tag{2.8}$$

Substituting (2.6) and (2.7) in Eq.(2.5) and using HPM by He ([11]), we get:

$$\begin{aligned}\sum_{n=0}^{\infty} p^n U_n = G(x,t) \\ - p\left(\mathbb{H}^{-1}\left[\left(\frac{u}{s}\right)^m \mathbb{H}\left[R\sum_{n=0}^{\infty} p^n U_n + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right),\end{aligned} \tag{2.9}$$

comparing the coeffcients of powers of p; yields

$$\begin{aligned} p^0 &: \quad U_0(x,t) = G(x,t), \\ p^n &: \quad U_n(x,t) = -\mathbb{H}^{-1}\left(\left(\frac{u}{s}\right)^m \mathbb{H}\left[RU_{n-1}(x,t) + H_{n-1}(u)\right]\right),\end{aligned}$$

where $n > 0$, $n \in \mathbb{N}$.

Finally, we approximate the analytical solution, $U(x,t)$, by he above series solutions generally converge very rapidly

$$U(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} U_n(x,t). \tag{2.10}$$

A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault .

3. Application

In this section we examine some examples in order to illustrate the homotopy perturbation transform method:

Example 3.1. *We consider the following nonlinear KdV equation [25]:*

$$U_t + UU_x - U_{xx} = 0, \tag{3.1}$$

with the initial condition:

$$U(x,0) = x. \tag{3.2}$$

Applying the Shehu transform on both sides of Eq.(3.1), we get:

$$\mathbb{H}\left[U_t\right] + \mathbb{H}\left[UU_x - U_{xx}\right] = 0. \tag{3.3}$$

Using the properties of Shehu transform, we get:

$$\mathbb{H}\left[U(x,t)\right] = x - \frac{u}{s}\mathbb{H}\left[UU_x - U_{xx}\right]. \tag{3.4}$$

Taking the inverse Shehu transform on both sides of Eq.(3.4), we obtain:

$$U(x,t) = x - \mathbb{H}^{-1}\left(\frac{u}{s}\mathbb{H}\left[UU_x - U_{xx}\right]\right). \tag{3.5}$$

By applying the HPM, we have:

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x - p\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(\sum_{n=0}^{\infty} p^n H_n(U) - \sum_{n=0}^{\infty} p^n (U_n)_{xx}\right)\right] \tag{3.6}$$

Comparing both sides of Eq.(3.6), we get

$$\begin{aligned} U_0(x,t) &= x, \\ U_1(x,t) &= -\mathbb{H}^{-1}\left[\tfrac{u}{s}\mathbb{H}\left(H_0(U) - U_{0,xx}(x,t)\right)\right], \\ U_2(x,t) &= -\mathbb{H}^{-1}\left[\tfrac{u}{s}\mathbb{H}\left(H_1(U) - U_{1,xx}(x,t)\right)\right], \\ U_3(x,t) &= -\mathbb{H}^{-1}\left[\tfrac{u}{s}\mathbb{H}\left(H_2(U) - U_{2,xx}(x,t)\right)\right], \\ &\vdots \end{aligned} \tag{3.7}$$

The first few components of $H_n(U)$ polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_0 U_{0,x}, \\ H_1(U) &= U_0 (U_1)_x + U_1 U_{0,x}, \\ H_2(U) &= U_0 U_{2,x} + U_2 U_{0,x} + U_1 U_{1,x}, \\ &\vdots \end{aligned} \tag{3.8}$$

Using the formulas (3.7) and (3.8), we obtain:

$$\begin{aligned} U_0(x,t) &= x, \\ U_1(x,t) &= -xt, \\ U_2(x,t) &= xt^2, \\ U_3(x,t) &= -xt^3, \\ &\vdots \end{aligned} \tag{3.9}$$

we get:

$$U(x,t) = x - xt + xt^2 - xt^3 + \cdots \tag{3.10}$$

That gives:

$$U(x,t) = \frac{x}{1+t}, \quad |t| < 1, \tag{3.11}$$

which is an exact solution to the KdV equation.

Example 3.2. *Next, we consider the following nonlinear gas dynamics equation:*

$$U_t + UU_x - U(1-U) = 0, t > 0, \tag{3.12}$$

with the initial condition:

$$U(x,0) = e^{-x}. \tag{3.13}$$

Applying the Shehu transform and its inverse on both sides of Eq.(3.12), we get:

$$U(x,t) = e^{-x} - \mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(UU_x + U^2 - U\right)\right]. \tag{3.14}$$

By applying our method, we have:

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = e^{-x} - p\left(\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(\sum_{n=0}^{\infty} p^n H_n(U) - \sum_{n=0}^{\infty} p^n U_n\right)\right]\right). \tag{3.15}$$

On comparing both sides of Eq.(3.15), we obtain:

$$\begin{aligned} U_0(x,t) &= e^{-x}, \\ U_1(x,t) &= -\mathbb{H}^{-1}\left[\tfrac{u}{s}\mathbb{H}\left(H_0(U) - U_0(x,t)\right)\right], \\ U_2(x,t) &= -\mathbb{H}^{-1}\left[\tfrac{u}{s}\mathbb{H}\left(H_1(U) - U_1(x,t)\right)\right],, \\ &\vdots \end{aligned} \tag{3.16}$$

The first few components of $H_n(U)$ polynomials are given as follows:

$$\begin{aligned} H_0(U) &= U_0 U_{0,x} + U_0^2, \\ H_1(U) &= U_1 U_{0,x} + U_0 U_{1,x} + 2U_0 U_1, \\ H_2(U) &= U_2 U_{0,x} + U_1 U_{1,x} + U_0 U_{2,x} + 2U_2, \\ &\vdots \end{aligned} \tag{3.17}$$

we get the first terms of the solution:

$$\begin{aligned} U_0(x,t) &= e^{-\varkappa}, \\ U_1(x,t) &= e^{-\varkappa} t, \\ U_2(x,t) &= e^{-\varkappa}\tfrac{t^2}{2!}, \\ &\vdots \end{aligned} \tag{3.18}$$

So, the approximate series solution of Eq.(3.12), are given as follows:

$$U(x,t) = e^{-x}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right], \tag{3.19}$$

and in the closed form, is given by:

$$U(x,t) = e^{-x}e^{t} = e^{t-x}. \tag{3.20}$$

This result is the same as that obtained in [13] using homotopy analysis method.

Example 3.3. *We consider the following non-linear wave-like equation with variable coefficients*

$$U_{tt} = x^2\frac{\partial}{\partial x}\left(U_x U_{xx}\right) - x^2\left(U_{xx}\right)^2 - U, \; 0 < x < 1, \; t > 0, \tag{3.21}$$

with the initial conditions:

$$U(x,0) = 0, \; U_t(x,0) = x^2. \tag{3.22}$$

Applying the Shehu transform and its inverse on both sides of Equ.(3.21), we get:

$$U(x,t) = x^2 t + \mathbb{H}^{-1}\left[\frac{u^2}{s^2}\mathbb{H}\left(x^2\frac{\partial}{\partial x}\left(U_x U_{xx}\right) - x^2\left(U_{xx}\right)^2 - U\right)\right]. \tag{3.23}$$

By applying the method suggested in this paper, we have:

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,t) &= x^2 t \\ &+ p\left(\mathbb{H}^{-1}\left[\frac{u^2}{s^2}\mathbb{H}\left(x^2\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} p^n H_n(U)\right)\right)\right]\right) \\ &- p\left(\mathbb{H}^{-1}\left[x^2\left(\sum_{n=0}^{\infty} p^n H_n'(U)\right) - \sum_{n=0}^{\infty} U_n\right]\right) \end{aligned} \tag{3.24}$$

Comparing both sides of Eq.(3.24), we obtain:

$$\begin{gathered}U_0(x,t)=x^2t,\\ U_1(x,t)=\mathbb{H}^{-1}\left[\frac{u^2}{s^2}\mathbb{H}\left(x^2\frac{\partial}{\partial x}\left(H_0(U)\right)-x^2H_0'(U)-U_0\right)\right],\\ U_2(x,t)=\mathbb{H}^{-1}\left[\frac{u^2}{s^2}\mathbb{H}\left(x^2\frac{\partial}{\partial x}\left(H_1(U)\right)-x^2H_1'(U)-U_1\right)\right],\\ \vdots\end{gathered} \tag{3.25}$$

The first few components of $H_n(U)$ and $H_n'(U)$ are given by:

$$\begin{gathered}H_0(U)=U_{0,x}U_{0,xx},\\ H_1(U)=U_{0,x}U_{1,xx}+U_{1,x}U_{0,xx},\\ H_2(U)=U_{0,x}U_{2,xx}+U_{2,x}U_{0,xx}+U_{1,x}U_{1,xx},\\ \vdots\end{gathered} \tag{3.26}$$

and

$$\begin{gathered}H_0'(U)=\left(U_{0,xx}\right)^2,\\ H_1'(U)=2U_{0,xx}U_{1,xx},\\ H_2'(U)=2U_{0,xx}U_{2,xx}+\left(U_{1,xx}\right)^2,\\ \vdots\end{gathered} \tag{3.27}$$

Using the iteration formulas (3.25) and the adomian polynomials (3.26) and (3.27), we obtain:

$$\begin{gathered}U_0(x,t)=x^2t,\\ U_1(x,t)=-x^2\frac{t^3}{3!},\\ U_2(x,t)=x^2\frac{t^5}{5!},\\ U_3(x,t)=-x^2\frac{t^7}{7!},\\ \vdots\end{gathered} \tag{3.28}$$

The first terms of the approximate solution of Eq.(3.21), is given by:

$$U(x,t)=x^2\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots\right]. \tag{3.29}$$

And in the closed form:

$$U(x,t)=x^2\sin(t). \tag{3.30}$$

This result represents the exact solution of the equation (3.21)
.

Example 3.4. *Finaly, we consider the two component system of homogeneous KdV equation of order three*

$$\begin{cases}U_t=U_{xxx}+UU_x+V_x\\ V_t=-2V_{xxx}-U_xV\end{cases}, \tag{3.31}$$

subject to to the initial condition:

$$U(x,0)=\left(3-\tanh^2\frac{x}{2}\right),V(x,0)=-\left(3t\sqrt{2}\tanh^2\frac{x}{2}\right). \tag{3.32}$$

By applying the aforesaid method subject to the initial condition, we get

$$U(x,t)=\left(3-\tanh^2\frac{x}{2}\right)+\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(U_{xxx}+UU_x+V_x\right)\right], \tag{3.33}$$

and

$$V(x,t)=-\left(3t\sqrt{2}\tanh^2\frac{x}{2}\right)-\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(2V_{xxx}+U_xV\right)\right]. \tag{3.34}$$

By applying our Method, we have:

$$\begin{aligned}\sum_{n=0}^{\infty}p^nU_n(x,t)&=\left(3-\tanh^2\frac{x}{2}\right)\\ &-p\left(\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(\sum_{n=0}^{\infty}p^nU_{n,xxx}+\sum_{n=0}^{\infty}p^nH_n(U)\right)\right]\right).\end{aligned} \tag{3.35}$$

$$\begin{aligned}\sum_{n=0}^{\infty}p^nV_n(x,t)&=-\left(3t\sqrt{2}\tanh^2\frac{x}{2}\right)\\ &-p\left(\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(2\sum_{n=0}^{\infty}p^nV_{n,xxx}+\sum_{n=0}^{\infty}p^nH_n'(U)\right)\right]\right).\end{aligned} \tag{3.36}$$

where $H_n(U)$ are He's polynomials represent the nonlinear terms. The first few components of He's polynomials, for example, are given by

$$\begin{gathered}H_0(U)=U_0U_{0,x}+V_0V_{0.x},\\ H_1(U)=U_1U_{0,x}+U_0U_{1,x}+V_1V_{0,x}+V_0V_{1,x},\\ H_2(U)=U_2U_{0,x}+U_1U_{1,x}+U_0U_{2,x}+V_2V_{0,x}+V_1V_{1,x}+V_0V_{2,x},\\ \vdots\end{gathered} \tag{3.37}$$

Similarly

$$\begin{gathered}H_0'(U)=U_0V_{0.x},\\ H_1'(U)=U_1V_{0,x}+U_0V_{1,x},\\ H_2'(U)=U_2V_{0,x}+U_1V_{1,x}+U_0U_{2,x},\\ \vdots\end{gathered} \tag{3.38}$$

Comparing the coefficient of like powers of p, we have

$$\begin{gathered}U_0(x,t)=\left(3-\tanh^2\frac{x}{2}\right),\\ U_1(x,t)=\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(U_{1,xxx}+H_0(U)\right)\right],\\ U_2(x,t)=\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(U_{2,xxx}+H_1(U)\right)\right],\\ \vdots\end{gathered} \tag{3.39}$$

$$\begin{gathered}V_0(x,t)=-\left(3t\sqrt{2}\tanh^2\frac{x}{2}\right),\\ V_1(x,t)=\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(V_{1,xxx}+H_0'(U)\right)\right],\\ V_2(x,t)=\mathbb{H}^{-1}\left[\frac{u}{s}\mathbb{H}\left(V_{2,xxx}+H_1'(U)\right)\right],\\ \vdots\end{gathered} \tag{3.40}$$

Finally, we approximate the analytical solution by:

$$U(x,t) = U_0(x,t)+U_1(x,t)+U_2(x,t)+\ldots. \quad (3.41)$$

$$U(x,t)=\left(3-\tanh^2\frac{x}{2}\right)-6t\sec h^2\frac{x}{2}\tanh\frac{x}{2} + \frac{3}{2}t^2\left(2\sec h^2\frac{x}{2}+7\sec h^4\frac{x}{2}-15\sec h^6\frac{x}{2}\right) + \cdots. \quad (3.42)$$

$$V(x,t) = V_0(x,t)+V_1(x,t)+V_2(x,t)+\ldots. \quad (3.43)$$

$$V(x,t) \quad (3.44)$$

$$=-\left(3t\sqrt{2}\tanh^2\frac{x}{2}\right) + 3t\sqrt{2}t\sec h^2\frac{x}{2}\tanh\frac{x}{2} + \frac{3t\sqrt{2}}{4}t^2\left(2\sec h^2\frac{x}{2}+21\sec h^4\frac{x}{2}-24\sec h^6\frac{x}{2}\right) + \cdots. \quad (3.45)$$

4. Conclusion

We have developed an approach to obtain analytically solution of non-linear partial differential equations. The Shehu Homotopy Perturbation Method (SHPM) based on coupling the homotopy perturbation method (HPM) and Shehu transform is applied to some examples in different fields of research. The implementation of this method allows finding the exact solution after calculating only the first terms of the approximate solution**.**

Due to its efficiency and flexibility the (STHPM) can be applied to solve other nonlinear partial differential equations of higher order.

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