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Product cordial labeling of hypercube related graphs

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Abstract
In this paper we investigate the product cordial labeling of hypercube graph, path union of the hypercube graphs, $S(t, P_n), P_n^t(t_n, Q_k)$ and graph obtained by joining two copies of hypercube by arbitrary length of path.

Keywords
Product cordial graphs, Hypercube graph, Path union of graphs, Open star of graphs, One point union of path graphs.

AMS Subject Classification
05C78.

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1. Introduction

We begin with a finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. We denote the edge e with end vertices u and v by $e = uv$. For notation and theoretical terminology of any graph, we follow Balakrishnan and Ranganathan [1]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

Motivated through the concept of cordial labeling the product cordial labeling was introduced by Sundaram et al [6] where absolute difference of vertex labels is replaced by product of vertex labels.

Definition 1.1. *Let $f : V \to \{0,1\}$ be a vertex labeling of graph G that induced an edge labeling $f^* : E(G) \to \{0,1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a product cordial labeling if it satisfies the conditions $|vf(0) - vf(1)| \leq 1$ and $|ef(0) - ef(1)| \leq 1$. A graph which admits product cordial labeling is called product cordial graph.*

Definition 1.2. *Let G be a graph and $G_1, G_2, \dots, G_n; n \geq 2$ be n copies of graph G.Then the graph obtained by adding an edge from G_i to $G_{i+1} (i = 1, 2, \dots, n-1)$ is called path union of graph G.*

Definition 1.3. *A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the graph $G_1, G_2, \dots, G_n$ is known as an open star of graphs which is denoted by $S(G_1, G_2, \dots, G_n)$. If we replace each vertex of $K_{1,n}$ except the apex vertex by a graph G. i.e. $G_1 = G_2 = \dots = G_n$. Open star of graphs can be denoted by $S(n \cdot G)$.*

Definition 1.4. *A graph G is obtained by replacing each edge of $K_{1,t}$ by a path P_n of length n on $n+1$ vertices is called one point union for t copies of path P_n which is denoted by P_n^t.*

Definition 1.5. *A graph G is obtained by replacing each vertex of P_n^t except the central vertex by the graph $G_1, G_2, \dots, G_{t_n}$ is known as one point union for path of graphs and is denoted by $P_n^t(G_1, G_2, \dots, G_{t_n})$ If we replace each vertex of P_n^t except the central vertex by graph H. i.e. $G_1 = G_2 = \dots = G_n = H$, then such a one point union of path graph shall be denoted by $P_n^t(t_n \cdot H)$.*

Definition 1.6. *Hypercube is an n-dimensional analogue of a square $(n = 2)$ and a cube $(n = 3)$ which is also known as an n-cube or n-dimensional cube which is denoted by Q_n. In general, it can be defined by $Q_n = Q_{n-1} \times K_2 (n \geq 2)$.*

Hypercubes are part of regular polytopes. Hypercube shapes represent compact, convex and closed figure.

2. Main Results

Theorem 2.1. *The hypercube graph $Q_n (n > 1)$ does not admit a product cordial labeling.*

Proof. We know that the hypercube graph Q_n has 2^n vertices and $n2^{n-1}$ edges. If Q_n admits product cordial labeling then it must satisfy the condition $| vf(0) - vf(1) | \leq 1$, which is only possible when $v_f(0) = v_f(1) = \frac{2^n}{2} = 2^{n-1}$ Now if we assign label zero to one vertex, then we get n corresponding edges with label zero according to the property $f(e = uv) = f(u)f(v)$ of product cordial labeling. Thus in order to minimize the total no of edges with label 0, we need to assign label 0 to the adjacent vertices only in the hypercube graph Q_n.
Case-I: If no vertices with label 0 are adjacent in Q_n If no vertices with label 0 are adjacent in Q_n, then there are $n2^{n-1}$ edges having label 0 and no edges having label 1.
Case-II: If all the vertices with label 0, are adjacent in Q_n If all the vertices with label 0, are adjacent in Q_n, then there are total $(2n-1)2^{n-2}$ edges with label 0 and $n2^{n-2}$ edges with label 1. Thus, in either case we have at least $(2n-1)2^{n-2}$ edges with label 0 and at most $n2^{n-2}$ edges with label 1. Thus

$$|ef(0) - ef(1)| = \left|(2n-1)2^{n-2} - n2^{n-2}\right| = \left|(n-1)2^{n-2}\right| > 1$$

which contradicts with the condition $|ef(0) - ef(1)| \leq 1$. Hence $Q_n(n > 1)$ is not a product cordial graph. □

Theorem 2.2. *The path union of k copies of hypercube graph Q_n is product cordial for even k.*

Proof. Let G be a graph obtained by joining k copies of the hypercube Q_n where k is even. Let $V = \{u_{i1}, u_{i2}, \dots, u_{it}\}$ be the vertices of the hypercube Q_n in G; where i represents number of copies of the hypercube Q_n in G and $1 \leq t \leq 2^n$. $|V(G)| = k2^n$, $|E(G)| = kn2^{n-1} + k - 1$. To obtained a product cordial labeling of path union of k copies of the hypercube Q_n defined a vertex labelling of hypercube Q_n in G, $f : V(Q_n) \to \{0, 1\}$ as below
Case-I: $k \equiv 0(\bmod 2)$

$$f(u_{ij}) = \begin{cases} 1 & ; 1 \leq i \leq \frac{k}{2}, 1 \leq j \leq 2^n \\ 0 & ; \frac{k}{2} < i \leq k, 1 \leq j \leq 2^n. \end{cases}$$

In view of above defining labeling pattern $v_f(0) = v_f(1) = k2^{n-1}$ and $e_f(0) + 1 = e_f(1) = \frac{k}{2}\left(n2^{n-1} + 1\right)$. Thus, the graph G is a product cordial graph when k is even.
Case-II: $k \equiv 1(\bmod 2)$.
In graph G, the total number of vertices are even. Thus, the Condition $\left|v_f(0) - v_f(1)\right| \leq 1$ is satisfied if and only if we assign labels 0 and 1 to the equal number of vertices. Also, to fulfil the condition $\left|e_f(0) - e_f(1)\right| \leq 1$, we need to minimize the number of 0 labelled edge and therefore we need to assign a label 0 to adjacent vertices in G. For the graph G we assign a label 0 to all the vertices of the first 1 to $\left(\frac{k-1}{2}\right)^{th}$ copies of the hypercube Q_n and label 1 to all the vertices of the last $\left(\frac{k+3}{2}\right)^{th}$ to k^{th} copies of hypercube Q_n. Now for $\left(\frac{k+1}{2}\right)^{th}$ copy of the hypercube Q_n in G, if we assign a label 0 to first 2^{n-1} adjacent vertices and label 1 to remaining 2^{n-1} vertices, then the condition $\left|v_f(0) - v_f(1)\right| \leq 1$ satisfies but $\left|e_f(0) - e_f(1)\right| \geq 2$. For all other patterns of vertex labeling, we can easily verify that the condition $\left|e_f(0) - e_f(1)\right| \leq 1$ is not satisfied. In fact, the difference between $e_f(0)$ and $e_f(1)$ will increase for another pattern of vertex labeling. Therefore, the graph G under consideration is not a product cordial graph when k is odd. □

Example 2.3. *Product cordial labeling of path union of four copies of hypercube graph Q_3 is shown in the figure 1.*

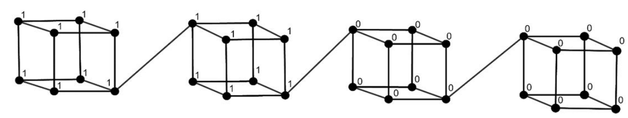

Figure 1. Product cordial labeling of path union of four copies of Q_3

Theorem 2.4. *Open star of graphs $S(t, Q_n)$ is product cordial except for odd t.*

Proof. Let G be a graph obtained by replacing each vertex of $K_{1,t}$ except the apex vertex, by the hypercube graph Q_n.

Let u_0 be the apex vertex of $K_{1,t}$ i.e. u_0 is the central vertex of graph G. Let $V = \{u_{i1}, u_{i2}, \dots, u_{ij}\}$ be the vertex set of the hypercube Q_n in G where i represents the number of branches of $K_{1,t}$ in G; and $1 \leq j \leq 2^n; n \in N$.

Assign a label 1 to the centre of graph G. Define vertex labeling $f : V(G) \to \{0, 1\}$ as below.

Case-I: $t \equiv 0(\bmod 2)$

$$f(u_{ij}) = \begin{cases} 1 & ; 1 \leq i \leq \frac{t}{2}; 1 \leq j \leq 2^n \\ 0 & ; \frac{t}{2} < i \leq t; 1 \leq j \leq 2^n \end{cases}$$

Here f satisfied vertex and edge label condition of product cordial. Thus, the graph G is product cordial when t is even.
Case-II: $t \equiv 1(\bmod 2)$ In graph G, total number of vertices are even. Thus, the Condition $\left|v_f(0) - v_f(1)\right| \leq 1$ is satisfied if and only if we assign labels 0 and 1 to the equal number of vertices. Also, to fulfil the condition $\left|e_f(0) - e_f(1)\right| \leq 1$, we need to minimize the number of 0 labelled edge and thus we need to assign label 0 to the adjacent vertices in hypercube Q_n. In graph G, first assign a label 0 to all vertices to the hypercubes Q_n which are connected with the branches 1 to $\frac{t-1}{2}$ of $K_{1,t}$ in G and assign a label 0 to all vertices to the hypercube which are connected with the branches $\frac{t+3}{2}$ to t of $K_{1,t}$ in G. Now for labeling of hypercube which is connected with $\left(\frac{t+1}{2}\right)^{th}$ branch of $K_{1,t}$ in G, if we assign a label 0 to first 2^{n-1} adjacent vertices and label 1 to remaining 2^{n-1} vertices then condition $\left|v_f(0) - v_f(1)\right| \leq 1$ is satisfied but

$\left|e_f(0)-e_f(1)\right| \geq 2$.
For all other patterns of vertex labeling, we can easily verify that condition $\left|e_f(0)-e_f(1)\right| \leq 1$ is not satisfied. In fact, the difference between $e_f(0)$ and $e_f(1)$ will increase for another vertex labeling pattern. Therefore, the graph G cannot be a product cordial when t is odd. □

Example 2.5. *Product cordial labeling of $S(4, Q_3)$ as shown in the following figure 2.*

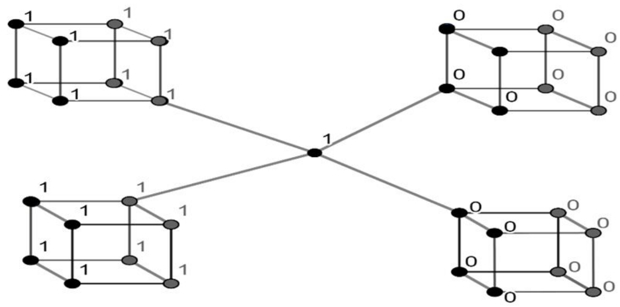

Figure 2. Product cordial labeling of $S(4,Q_3)$

Note 2.6. *In theorem 2.6 we consider the notation of hypercube as Q_k to avoid confusion.*

Theorem 2.7. *$P_n^t(t_n \cdot Q_k)$ is product cordial graph except for odd t and n.*

Proof. Let G be a graph obtained by replacing each edge of $K_{1,t}$ by path union of the hypercube graph Q_k with total length of $n-1$ of each path union and total number of n copies of Q_k Let u_0 be the apex vertex of $K_{1,t}$ i.e. u_0 be the central vertex of graph G. Let $V=\left\{u_{i,j}; i\in[1,t], j\in[1,n]\right\}$ be the vertices of the hypercube Q_k in graph G; where i represents the number of branch of $K_{1,t}$ in G and j represents the number of copies of hypercube in path union in G.

To generate a product cordial graph, assign label 1 to central vertex of graph G. i.e. $u_0=1$. Define vertex labeling $f: V(G)\to\{0,1\}$ in graph G as below
Case-I: $i\equiv 0(\bmod 2), j\in[1,n]$

$$f(u_{i,j})=\begin{cases}1 & ;1\leq i\leq \frac{t}{2}\\ 0 & ;\frac{t}{2}<i\leq t\end{cases}$$

Case-II
Subcase-II(a): $i\equiv 1(\bmod 2); j\equiv 0(\bmod 2)$

$$f(u_{i,j})=\begin{cases}1 & ;1\leq i\leq \frac{t-1}{2}\\ 0 & ;\frac{t+3}{2}\leq i\leq t\end{cases}$$

For $i=\left(\frac{t+1}{2}\right)^{th}$ branch of $K_{1,t}$ in G.

$$f(u_{i,j})=\begin{cases}1 & ;1\leq j\leq \frac{n}{2}\\ 0 & ;\frac{n}{2}<j\leq n\end{cases}$$

In case-I and subcase-II (a), f satisfies the condition of product cordiality. Thus, the graph G under consideration is product cordial when t or n or both are even.

Subcase-II(b): $i\equiv 1(\bmod 2); j\equiv 1(\bmod 2)$ To generate a product cordial graph G with both t and n odd, assign label 0 to the all vertices of the hypercube which lie on the path union on 1^{st} to $\left(\frac{t-1}{2}\right)^{th}$ branches in graph G and assign label 1 to the all vertices of the hypercube which lie on the path union on $\left(\frac{t+3}{2}\right)^{th}$ to t^{th} branches in graph G. Now to satisfy the vertex and edge condition we assign label 0 to the vertices of first $\frac{n-1}{2}$ copies of hypercube graph Q_k and assign label 1 to the vertices of last $\frac{n-1}{2}$ copies of hypercube graph Q_k of the path union which lie on $\left(\frac{t+1}{2}\right)^{th}$ branch of the graph G. For the remaining $\left(\frac{n+1}{2}\right)^{th}$ copy of the hypercube which lie on the path union of $\left(\frac{t+1}{2}\right)^{th}$ branch of G. If we assign label 0 to the first 2^{n-1} adjacent vertices and assign label 1 to the last 2^{n-1} adjacent vertices then the vertex condition $\left|v_f(0)-v_f(1)\right|\leq 1$ is satisfied but $\left|e_f(0)-e_f(1)\right|>1$.
Also, the above vertex arrangement gives minimum number of 0 labelled edges, and therefore, for any other arrangement we can easily verify the difference between $e_f(0)$ and $e_f(1)$ will increase. Therefore, the graph G cannot be product cordial when t and n both are odd. □

Example 2.8. *Product cordial labeling of $P_4^3(3_4, Q_3)$ is shown in the figure 3.*

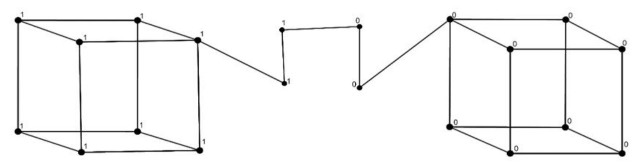

Figure 3. Joining two copies of Q_3 by P_6

Theorem 2.9. *The graph obtained by joining two copies of the hypercube Q_n by path of arbitrary length admits product cordial labeling.*

Proof. Let G be the graph obtained by joining two copies of hypercube graph Q_n by path P_k of the length $k-1$. Let $V=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_k\}$ be the vertex set of graph G where u_i and v_i; $1\leq i\leq n$ denote the consecutive vertices of first copy and second copy of the hypercube respectively and w_j; $1\leq j\leq k$ represent the vertices of the path P_k with $u_n=w_1$ and $w_k=v_1$. First, assign a label 1 to all vertices of first copy of Q_n and assign a label 0 to all vertices of second copy of Q_n.
It can be asserted that the vertex conditions and the edge conditions of product cordial labeling are satisfied. Now we need to label the vertices of path P_k for which we define a labeling function $f: V(G)\to\{0,1\}$ as follows.
Case-I: $k\equiv 0(\bmod 2)$

$$f(w_i)=\begin{cases}0 & ;1\leq i\leq \frac{k}{2}\\ 1 & ;\frac{k}{2}+1\leq i\leq k.\end{cases}$$

It can be asserted that the vertex condition and edge condition of product cordial labeling are satisfied in this case.

Case-II: $k \equiv 1(\bmod 2)$

$$f(w_i) = \begin{cases} 1 & ; 1 \leq i \leq \frac{k+1}{2} \\ 0 & ; \frac{k+3}{2} \leq i \leq k. \end{cases}$$

It can again be asserted that the vertex condition and edge condition of product cordial labeling are satisfies in this case too.

Hence, in each case, we noted that the graph G under consideration satisfies the condition of product cordial labeling i.e. $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence G is a product cordial graph. □

Example 2.10. *Graph obtained by joining two copies of hypercube graph Q_3 by P_6 is shown in the figure 4.*

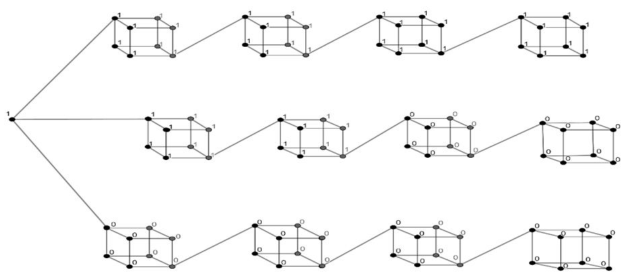

Figure 4. Product cordial labeling of $P_4^3(3_4, Q_3)$

3. Conclusion

In this article we discussed Product cordial labeling of n-dimensional Cube (hypercube) and its related graph. We also investigate the product cordial labeling on Path union of the hypercube graphs, Open star of the hypercube graphs and One point for path of graphs.We prove that graph obtained by joining two copies of hypercube with arbitrary length of path admits product cordial graph.

4. Further Scope of Research

Total product cordial labeling and edge product cordial labeling on hypercube and its related graphs are further scope of the research in this area.

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