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On star coloring of tensor product of graphs

V. Kowsalya[1]*

Abstract

The star chromatic number of tensor products of path and complete graphs have been investigated in this article.

Keywords

Star coloring, path, cycle, complete graph, tensor product.

AMS Subject Classification

05C15, 05C75, 05C76.

[1] *Department of Mathematics, Sri Ramakrishna College of Arts and Science, Coimbatore-641006, Tamil Nadu, India.*

***Corresponding author**: [1] vkowsalya09@gmail.com;

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1. Introduction

A star coloring of a graph G [1–3]is a proper vertex coloring in which every path on four vertices is colored such that it is not bicolored. More concisely, in a star coloring, the induced subgraphs formed by the vertices of any two colors have connected components that are star graphs. The star chromatic number $\chi_s(G)$ of G is the minimum number of colors needed to star color G. Star coloring of graphs was introduced by Branko Grünbaum in 1973 who linked star coloring to acyclic coloring by showing that any planar graph has an acyclic chromatic number less than or equal to 9 and suggested that this implies that any particular graph has star chromatic number less than or equal to $9.2^8 = 2304$.

There exist number of results for star colorings of graphs formed by certain graph operations. Guillaume Fertin et al.[3] gave the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs, and 2-dimensional grids. They also investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids ($d \geq 3$), d-dimensional tori ($d \geq 2$), graphs with bounded treewidth, and cubic graphs. In 2015, K. Venkatesan et al.[4] investigated star chromatic number of corona product of graphs and provided some bounds on chromatic number on corona product of graphs.

Albertson et al.[1] given that it is NP-complete to investigate whether $\chi_s(G) \leq 3$. The problem of finding star colorings is NP-hard and remain so even for bipartite graphs [5, 6].

2. Preliminaries

The tensor product of graphs was introduced by Bertrand [7] in 1912. In 1998, Imrich has given a polynomial time algorithm for recognizing tensor product graphs and finding a factorization of any such graph. If either G or H is bipartite, then so is their product. $G \otimes H$ is connected if and only if both factors are connected and at least one factor is nonbipartite.

The *tensor product* of two graphs G and H has the vertex set $V(G \otimes H) = V(G)V(H)$, edge set
$E(G \otimes H) = \{(a,b)(c,d) | ac \in E(G) \text{ and } bd \in E(H)\}$.

3. Main Results

Theorem 3.1. *For any positive integers $m,n \geq 3$, $\chi_s(K_m \otimes K_n) = mn - \max\{m,n\}$.*

Proof. Let $V(K_m) = \{u_i : 1 \leq i \leq m\}$ and
$V(K_n) = \{v_j : 1 \leq j \leq n\}$. By the definition of tensor product the vertex set and edge set of the graph $K_m \otimes K_n$ is given by,

$$V(K_m \otimes K_n) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \text{ and}$$
$$E(K_m \otimes K_n) = \bigcup_{i,k=1}^{m-1} \{u_i v_j, u_k v_l : 1 \leq j,l \leq n, i \neq k, j \neq l\}$$

Let $e_{(i)(j),(k)(l)}$ be the edge of $K_m \otimes K_n$ connecting the vertices $u_i v_j$ and $u_k v_l$ of $K_m \otimes K_n$. Therefore, $e_{(i)(j),(k)(l)} \in E(K_m \otimes K_n)$

if and only if $|k-i| = m-1$ and $|l-j| = n-1$, $K_m \otimes K_n$ being isomorphic to $K_n \otimes K_m$.

Let $G = K_m \otimes K_n$ and f be a funtion defined by $f : S \to C$, where $S = V(G)$ and C is the set of colors. Now use the color partition of $K_m \otimes K_n$ as follows:

Case 1 When $n \geq m$
The function f takes the values,
$f(u_i v_j) = n(i-1) + j, 1 \leq i \leq m-1, 1 \leq j \leq n$ and
$f(u_m v_j) = j, 1 \leq j \leq n$.

Case 2 When $n < m$
$f(u_i v_j) = m(j-1) + i, 1 \leq j \leq n-1, 1 \leq i \leq m$ and
$f(u_i v_n) = i, 1 \leq i \leq m$.

Clearly, in the above cases, the partition gives no bicolored path P_4 or cycle C_4. Assume that $\chi_s(K_m \otimes K_n) = mn - \max\{m,n\} - 1$, then there exists any one bicolored path P_4 or cycle C_4. A contradiction to proper star coloring. Thus, no coloring that uses $mn - \max(m,n) - 1$ colors can be a star coloring.

Hence, $\chi_s(K_m \otimes K_n) = mn - \max\{m,n\}$. □

Theorem 3.2. *For any positive integers $m \geq 2$, $n \geq 3$, the star chromatic number of $P_m \otimes K_n$ is,*

$$\chi_s(P_m \otimes K_n) = \begin{cases} n & if\ m = 2,3 \\ n+1 & if\ m \geq 4 \end{cases} .$$

Proof. Let $V(P_m) = \{u_i : 1 \leq i \leq m\}$ and $V(K_n) = \{v_j : 1 \leq j \leq n\}$. By the definition of tensor product the vertex set and edge set of the graph $P_m \otimes K_n$ is given by,

$$\begin{aligned} V(P_m \otimes K_n) &= \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \quad \text{and} \\ E(P_m \otimes K_n) &= \bigcup_{i,k=1}^{m-1} \{u_i v_j, u_{k+1} v_l : 1 \leq j,l \leq n, j \neq l\} \\ &\quad \cup \bigcup_{i,k=1}^{m-1} \{u_i v_j, u_{k+1} v_{l-1} : 1 \leq j,l \leq n, i \neq k\} \end{aligned}$$

Let $e_{(i)(j),(k)(l)}$ be the edge of $P_m \otimes K_n$ connecting the vertices $u_i v_j$ and $u_i v_l$ of $P_m \otimes K_n$. Therefore, $e_{(i)(j),(k)(l)} \in E(P_m \otimes K_n)$ if and only if $|k-i| = 1$ and $|l-j| = n-1$, $P_m \otimes K_n$ being isomorphic to $K_n \otimes P_m$.

Let $G = P_m \otimes K_n$ and f be a funtion defined by $f : S \to C$, where $S = V(G)$ and C is the set of colors. Now use the color partition of $P_m \otimes K_n$ as follows:

Case 1 When $m = 2,3$
The function f takes the values,
$f(u_i v_j) = j, 1 \leq i \leq 3, 1 \leq j \leq n$. Clearly, above color partition gives a proper star coloring without bicolored paths. Suppose $\chi_s(P_m \otimes K_n) = n-1$, then there exists any one bicolored path P_4 or cycle C_4. Thus, no coloring that uses $n-1$ colors can be a star coloring and $\chi_s(P_m \otimes K_n) \geq n$. Hence, $\chi_s(P_m \otimes K_n) = n$.

Case 2 When $m \geq 4$
For $1 \leq j \leq n$ assign the values for f as,

- $f(u_{i-3} v_j) = j, 1 \leq i \leq \frac{m-3}{4}$
- $f(u_{4i-3} v_j) = j, 1 \leq i \leq \frac{m-2}{4}$
- $f(u_{4i-1} v_j) = j, 1 \leq i \leq \frac{m-1}{4}$
- $f(u_{4i} v_j) = n+1, 1 \leq i \leq \frac{m}{4}$

Clearly, above color partition gives a proper star coloring without bicolored paths. Suppose $\chi_s(P_m \otimes K_n) = n$, then there exists any one bicolored path P_4 or cycle C_4. Thus, no coloring that uses n colors can be a star coloring and $\chi_s(P_m \otimes K_n) \geq n+1$. Hence, $\chi_s(P_m \otimes K_n) = n+1$. □

Theorem 3.3. *For any positive integers m and n, the star chromatic number of the tensor product $P_m \otimes P_n$ is given by*

$$\chi_s(P_m \otimes P_n) = \begin{cases} 3 & if\ m \leq 3 \\ 4 & if\ 4 \leq m \leq 7 \\ 5 & if\ m \geq 8 \end{cases} .$$

Proof. Let $V(P_m) = \{u_i : 1 \leq i \leq m\}$ and $V(P_n) = \{v_j : 1 \leq j \leq n\}$. By the definition of tensor product the vertex set and edge set of the graph $P_m \otimes P_n$ is given by,

$$\begin{aligned} V(P_m \otimes P_n) &= \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \quad \text{and} \\ E(P_m \otimes P_n) &= \bigcup_{i=1}^{m-1} \{u_i v_j, u_{i+1} v_{j+1} : 1 \leq j \leq n-1\} \\ &\quad \cup \bigcup_{i=1}^{m-1} \{u_i v_j, u_{i+1} v_{j-1} : 2 \leq j \leq n\}. \end{aligned}$$

Let $e_{(i)(j),(k)(l)}$ be the edge of $P_m \otimes P_n$ connecting the vertices $u_i v_j$ and $u_k v_l$ of $P_m \otimes P_n$. Therefore, $e_{(i)(j),(k)(l)} \in E(P_m \otimes P_n)$ if and only if $|k-i| = |l-j| = 1$, $P_m \otimes P_n$ being isomorphic to $P_n \otimes P_m$.

Let $G = P_m \otimes P_n$ and f be a funtion defined by $f : S \to C$, where $S = V(G)$ and C is the set of colors. Now use the color partition of $P_m \otimes P_n$ as follows:

Case 1 When $m \leq 3$
For $1 \leq j \leq n$, the function f takes the values, $f(u_i v_j) = \begin{cases} j \bmod 3 & \text{if } j \not\equiv 0 \bmod 3 \\ 3 & \text{if } j \equiv 0 \bmod 3 \end{cases}$, where $1 \leq i \leq 3$. The function f is a proper star coloring with no bicolored path P_4. It is clear that $\chi_s(P_n) = 3$, thus, $\chi_s(P_m \otimes P_n) \geq 3$. Hence, $\chi_s(P_m \otimes P_n) = 3$.

Case 2 When $4 \leq m \leq 7$
For $1 \leq j \leq n$, the function f takes the values,

- $f(u_{4i-3} v_j) = \begin{cases} j \bmod 3 & \text{if } j \not\equiv 0 \bmod 3 \\ 3 & \text{if } j \equiv 0 \bmod 3 \end{cases}$, where $1 \leq i \leq \frac{m+3}{4}$
- $f(u_{4i-2} v_j) = 4$, $1 \leq i \leq \frac{m+2}{4}$

- $f(u_{4i-1}v_j) = \begin{cases} j+2 \bmod 3 & \text{if } j+2 \not\equiv 0 \bmod 3 \\ 3 & \text{if } j+2 \equiv 0 \bmod 3 \end{cases}$, where $1 \le i \le \frac{m-1}{4}$
- $f(u_{4i}v_j) = \begin{cases} j+2 \bmod 3 & \text{if } j+2 \not\equiv 0 \bmod 3 \\ 3 & \text{if } j+2 \equiv 0 \bmod 3 \end{cases}$, where $1 \le i \le \frac{m}{4}$

Clearly, f is a proper star coloring. Suppose, that $\chi_s(P_m \otimes P_n) = 3$, then there exists bicolored path P_4 or cycle C_4. Thus, no coloring can be used with 3 colors. Hence, $\chi_s(P_m \otimes P_n) = 4$.

Case 3 When $m \ge 8$
The function f takes the values,

- $f(u_{4i-3}v_j) = \begin{cases} j \bmod 3 & \text{if } j \not\equiv 0 \bmod 3 \\ 3 & \text{if } j \equiv 0 \bmod 3 \end{cases}$, where $1 \le i \le \frac{m+3}{4}$ and $1 \le j \le n$
- $f(u_{4i-2}v_{4j-3}) = 4$, where$1 \le i \le \frac{m+2}{4}$ and $1 \le j \le \frac{n+3}{4}$
- $f(u_{4i-2}v_{4j-2}) = 4$, $1 \le i \le \frac{m+2}{4}$ and $1 \le j \le \frac{n+2}{4}$
- $f(u_{4i-2}v_{4j-1}) = 5$, $1 \le i \le \frac{m+2}{4}$ and $1 \le j \le \frac{n+1}{4}$
- $f(u_{4i-2}v_{4j}) = 5$, $1 \le i \le \frac{m+2}{4}$ and $1 \le j \le \frac{n}{4}$
- $f(u_{4i-1}v_j) = \begin{cases} j+2 \bmod 3 & \text{if } j+2 \not\equiv 0 \bmod 3 \\ 3 & \text{if } j+2 \equiv 0 \bmod 3 \end{cases}$, $1 \le i \le \frac{m-1}{4}$ and $1 \le j \le n$.
- $f(u_{4i}v_j) = \begin{cases} j+2 \bmod 3 & \text{if } j+2 \not\equiv 0 \bmod 3 \\ 3 & \text{if } j+2 \equiv 0 \bmod 3 \end{cases}$, $1 \le i \le \frac{m}{4}$ and $1 \le j \le n$.

Clearly, f is a proper star coloring. Suppose, that $\chi_s(P_m \otimes P_n) = 4$, then there exists bicolored path P_4 or cycle C_4. Thus, no coloring can be used with 4 colors. Hence, $\chi_s(P_m \otimes P_n) = 5$.

□

4. Conclusion

In this paper, some results regarding star chromatic number of tensor product of complete graphs and paths have been discussed. In future this can be extended to tensor product of some more special graphs.

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