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Some coupled fixed point results for rational type contraction mappings in S-metric spaces

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Abstract
We prove the existence of some coupled fixed point for rational type contraction mappings in $S-$metric space. Our results generalized, extend and enrich recently fixed point results in the literature. From the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type as an application.

Keywords
Fixed point, S-metric space, point of coincidence, continuous.

AMS Subject Classification
54H25, 47H10.

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Contents

1. Introduction and Preliminaries

Fixed point theory plays a major role in many applications, including variational and linear inequalities, optimization and applications in the field of approximation theory and minimum norm problem. In 1922, S. Banach proved the famous and well known Banach contraction principle concerning the fixed of contraction mappings defined on a complete metric space. In recent years, Gahler [10, 11] introduced the notion of 2-metric spaces, while Dhage [8] introduced the concept of D-metric spaces. Later on, Mustafa and Sims [12] introduced a new notion of generalized metric space called G-metric spaces. After then many authors studied fixed and common fixed points in generalized metric spaces see [1–6, 12, 16, 23]. Next, Sedghi et al [22] introduced a D^*-metric space, In [21], Sedghi, Shobe and Aliouche have introduced the notion of an S-metric space. Dung [19] proved coupled fixed point in partially ordered S-metric space with the help of mixed weakly monotone maps. Lakshmikantham and Ćirić [15] introduced the concept of mixed $g-$ monotone mapping and proved coincidence fixed point theorems in partially ordered metric space.

Definition 1.1. *[20] Let X be a nonempty set. A S-metric on X is a function $S : X\times X\times X \to [0,\infty)$ that satisfies the following conditions*
(S1) $S(x,y,z) \geq 0$,
(S2) $S(x,y,z) = 0$ *if and only if* $x = y = z$,
(S3) $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$,
for all $x,y,z,a \in X$. *The pair* (X,S) *is called an* $S-$*metric space.*

Some examples of such S-metric spaces are:

1. Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then
$$S(x,y,z) = \|y+z-2x\| + \|y-z\|$$
is an $S-$metric on X.

2. Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then
$$S(x,y,z) = \|x-z\| + \|y-z\|$$
is an $S-$metric on X.

3. Let X be a nonempty set, d is ordinary metric space on X, then $S(x,y,z) = d(x,y)+d(y,z)$ is an $S-$metric on X.

Lemma 1.2. *[9, 20] Let (X,S) be an $S-$metric space. Then*

$$S(x,x,y) \leq 2S(x,x,y) + S(y,y,z)$$

and

$$S(x,x,z) \leq 2S(x,x,y) + S(z,z,y)$$

for all $x,y,z \in X$.

Definition 1.3. *[20] Let (X,S) be an $S-$metric space. For $r>0$ and $x \in X$ we define the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows respectively*

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\},$$

$$B_S[x,r] = \{y \in X : S(y,y,x) \leq r\}.$$

Example 1.4. *[20] Let $X = \mathbb{R}$. Denote*

$$S(x,y,z) = |y+z-2x| + |y-z|$$

for all $x,y,z \in \mathbb{R}$. Thus

$$B_S(1,2) = \{y \in X : S(y,y,1) < 2\} = (0,2).$$

Definition 1.5. *[20] Let (X,S) be an $S-$metric space and $A \subset X$.*
(1) If for every $x \in A$ there exists $r>0$ such that $B_S(x,r) \subset A$ then the subset A is called open subset of X.
(2) Subset A of X is said to be $S-$bounded if there exists $r>0$ such that $S(x,x,y) < r$ for all $x,y \in A$.
(3) A sequence $\{x_n\}$ in X converges to x if and only if

$$S(x_n,x_n,x) \to 0 \text{ as } n \to \infty.$$

That is or each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$S(x_n,x_n,x) < \varepsilon \text{ whenever } n \geq n_0$$

and we denote this $\lim\limits_{n\to\infty} x_n = x$.
(4) A sequence $\{x_n\}$ is called Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ whenever $n,m \geq n_0$. The S-metric (X,S) is said to be complete if every Cauchy sequence is convergent.
(5) Let τ be the set of all $A \subset X$. with $x \in A$ if and only if there exists $r>0$ such that $B_S(x,r) \subset A$. Then τ is a topology on X (induced by the $S-$metric space).

Example 1.6. *Any open ball $B_S(x,r)$, $x \in X$, $r>0$ is an open set. Indeed, Let $y \in B_S(x,r)$, we prove that $B_S(y,\rho) \subset B_S(x,r)$, let z be an element of $B_S(y,\rho)$ then by definition, we have*

$$S(z,z,y) < \rho = \frac{r - S(y,y,x)}{2}.$$

Then, it follows from Lemma 1.2,

$$S(z,z,x) \leq 2S(z,z,y) + S(y,y,x) < r.$$

So, we have $S(z,z,y) < r$, and $z \in B_S(x,r)$.

Theorem 1.7. *The $S-$metric space is a T_2 space.*

Proof. Let x_0, $y_0 \in X$ and $x_0 \neq y_0$, considering the sets

$$B_1 = \{x \in X : S(x,x,x_0) < S(x,x,y_0)\},$$

$$B_2 = \{x \in X : S(x,x,x_0) > S(x,x,y_0)\}.$$

Then B_1 and B_2 are two open disjoint sets. Let $z \in B_1$, then

$$S(z,z,x_0) < S(z,z,y_0)$$

this implies

$$S(z,z,y_0) - S(z,z,x_0) > 0.$$

Setting

$$\rho = \frac{S(z,z,y_0) - S(z,z,x_0)}{4}.$$

We show that $B_S(z,\rho) \subset B_1$. Let $a \in B_S(z,\rho)$ then

$$S(a,a,z) < \rho = \frac{S(z,z,y_0) - S(z,z,x_0)}{4},$$

therefore

$$2S(a,a,z) + S(z,z,x_0) < S(z,z,y_0) - 2S(a,a,z),$$

now by Lemma 1.2 we have

$$\begin{aligned} S(a,a,x_0) &\leq 2S(a,a,z) + S(z,z,x_0) \\ &< S(z,z,y_0) - 2S(a,a,z) \\ &\leq S(a,a,y_0), \end{aligned}$$

this means that $S(a,a,x_0) < S(a,a,y_0)$, the desired result follows. With the same way, we prove that B_2 is also an open set. Now, we prove that $B_1 \cap B_2 = \emptyset$. Assume that

$$B_1 \cap B_2 \neq \emptyset,$$

then there exists $y \in B_1 \cap B_2$, so, we have

$$S = (y,y,x_0) < S(y,y,y_0) \text{ and } S(y,y,x_0) > S(y,y,y_0)$$

which implies that

$$S(y,y,y_0) < S(y,y,y_0)$$

which is a contradiction. □

Definition 1.8. *[15]. Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has a mixed $g-$monotone property if F is monotone $g-$non-decreasing in its first argument, and is monotone $g-$non-increasing in its second argument, that is, for any $x, x_1, x_2, y, y_1, y_2 \in X$*

$$g(x_1) \leq g(x_2) \text{ implies } F(x_1,y) \leq F(x_2,y),$$
$$g(y_1) \leq g(y_2) \text{ implies } F(x,y_1) \geq F(x,y_2).$$

Theorem 1.9. *[15] Let $(X,\leq)$ be a partially ordered set and $F: X\times X \to X$ and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume that there is a function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t$ and*

$$\lim_{r\to t^+}\varphi(r)<t \text{ for each } t>0.$$

Also suppose $F: X\times X\to X$ and $g: X\to X$ such that F has the mixed $g-$monotone property satisfying:

$$d(F(x,y),F(u,v))\leq \varphi\left(\frac{d(g(x),g(u)+d(g(y),g(v))}{2}\right)$$

for all $x,y,u,v\in X$, for which $g(x)\leq g(u)$ and $g(y)\geq g(v)$. Consider $F(X\times X)\subseteq g(X)$, g is continuous and commutes with F and also suppose either F is continuous or X has the following property
a) if $\{x_n\}$ a non-decreasing sequence with $x_n\to x$, then $x_n\leq x$ for all $n\in\mathbb{N}$,
b) if $\{y_n\}$ a non-increasing sequence with $y_n\to y$, then $y\leq y_n$ for all $n\in\mathbb{N}$.
If there exist $x_0,y_0\in X$ such that

$$g(x_0)\leq F(x_0,y_0) \text{ and } g(y_0)\geq F(y_0,x_0),$$

then there exist $x,y\in X$ such that

$$g(x)=F(x,y) \text{ and } g(y)=F(y,x).$$

In 2012, Gordji et al [13] introduced the concept of the mixed weakly increasing property of mappings and proved a coupled fixed point result. This result is generalized by Nguyen [17] to S-metric spaces

In 2015, Gupta et al [14], proved the common coupled fixed points theorems by using altering distance function and mixed weakly monotone maps in partially ordered $S-$ metric space.

Theorem 1.10. *[14] Let $(X,\leq,S)$ be a partially ordered complete $S-$metric space and the mappings $f,g: X\times X\to X$ satisfied the mixed weakly monotone property on X; $x_0\leq f(x_0,y_0)$, $f(y_0,x_0)\leq y_0$ or $x_0\leq g(x_0,y_0)$, $g(y_0,x_0)\leq y_0$ for some $x_0,y_0\in X$. Consider a function $\varphi:[0.\infty)\to[0.\infty)$ with $\varphi(t)<t$ and $\lim_{r\to t^+}\varphi(r)<t$ for each $t>0$ such that*

$$S(f(x,y),f(x,y),g(u,v))\leq\varphi\left(\frac{1}{2}\left\{(S(x,x,u)+S(y,y,v))\right\}\right),$$

for all $x,y,u,v\in X$ with $x\leq u$ and $y\leq v$. Also, assume that either f or g is continuous or X has the following property:
(1) if $\{x_n\}$ is an increasing sequence with $x_n\to x$ then $x_n\leq x$ for all $n\in\mathbb{N}$,
(2) if $\{y_n\}$ is a decreasing sequence with $y_n\to y$ then $y_n\leq y$ for all $n\in\mathbb{N}$.
Then f and g have a coupled common fixed point in X.

2. Main Results

Theorem 2.1. *Let $(X,\leq,S)$ be a partially ordered complete $S-$metric space, the mappings $f,g: X\times X\to X$ satisfied the mixed weakly monotone property on X and*

$$\begin{cases} x_0\leq f(x_0,y_0)\\ f(y_0,x_0)\leq y_0\end{cases} \text{ or } \begin{cases} x_0\leq g(x_0,y_0)\\ g(y_0,x_0)\leq y_0,\end{cases}$$

for some $x_0,y_0\in X$. Let a function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t$ and $\lim_{r\to t^+}\varphi(r)<t$ for each $t>0$ such that

$$S(f(x,y),f(x,y),g(u,v))\leq \tag{2.1}$$

$$\varphi\left(\frac{1}{2}\left\{\begin{array}{c}\alpha\left(S(x,x,u)+S(y,y,v)\right)\\ +\beta\frac{S(x,x,f(x,y))S(x,x,g(u,v))}{1+S(x,x,u)+S(y,y,v)}\\ +\gamma\frac{S(x,x,f(x,y))S(u,u,g(u,v))}{1+S(x,x,u)}\\ +\lambda\frac{S(x,x,f(x,y))S(u,u,g(u,v))}{1+S(x,x,f(x,y))}\\ +\mu\frac{S(x,x,f(x,y))S(u,u,g(u,v))}{1+S(u,u,g(u,v))}\end{array}\right\}\right), \tag{2.2}$$

for all $x,y,u,v\in X$, $\alpha,\beta,\gamma,\lambda,\mu\in\mathbb{R}^+$ such that $x\leq u$, $y\leq v$ and $2\alpha+3\beta+\gamma+\lambda+\mu<2$. Also, assume that either f or g is continuous or X has the following property:
(1) if $\{x_n\}$ is an increasing sequence with $x_n\to x$ then $x_n\leq x$ for all $n\in\mathbb{N}$,
(2) if $\{y_n\}$ is a decreasing sequence with $y_n\to y$ then $y_n\leq y$ for all $n\in\mathbb{N}$.
Then f and g have a coupled common fixed point in X.

Proof. Consider $x_0\leq f(x_0,y_0)$ and $f(y_0,x_0)\leq y_0$ and let

$$f(x_0,y_0)=x_1 \text{ and } f(y_0,x_0)=y_1.$$

Now

$$x_1=f(x_0,y_0)\leq g(f(x_0,y_0)),f(y_0,x_0)=g(x_1,y_1)=x_2$$

and

$$y_1=f(y_0,x_0)\leq g\left(f(y_0,x_0),f(x_0,y_0)\right)=g\left(y_1,x_1\right)=y_2.$$

Continuing in this way, we have

$$x_{2n+1}=f(x_{2n},y_{2n}),y_{2n+1}=f(y_{2n},x_{2n})$$

and

$$x_{2n+2}=g(x_{2n+1},y_{2n+1}),y_{2n+2}=g(y_{2n+1},x_{2n+1}).$$

Thus, we conclude that $\{x_n\}$ is an increasing sequence and $\{y_n\}$ is a decreasing sequence. Similarly, from the condition

$$x_0\leq g(x_0,y_0) \text{ and } y_0\geq g(y_0,x_0),$$

we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ are increasing or decreasing. Now from (2.2), we obtain

$$S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))\leq$$

$$\varphi\left(\frac{1}{2}\left\{\begin{array}{c}\alpha\left(S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})\right)\\ +\beta\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n}))S(x_{2n},x_{2n},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}\\ +\gamma\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n}))S(x_{2n+1},x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n},x_{2n},x_{2n+1})}\\ +\lambda\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n}))\cdot S(x_{2n+1},x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n},x_{2n},x_{2n+1})}\\ +\mu\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n}))\cdot S(x_{2n+1},x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n+1},x_{2n+1},x_{2n+2})}\end{array}\right\}\right)$$

which implies

$$S(x_{2n+1},x_{2n+1},x_{2n+2}) \leq \varphi\left(\frac{1}{2}\left\{\begin{array}{c} \alpha\left(S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})\right) \\ +\beta\frac{S(x_{2n},x_{2n},,x_{2n+1})\cdot S(x_{2n},,x_{2n},,x_{2n+2})}{1+S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n}y_{2n},y_{2n+1})} \\ \gamma\frac{S(x_{2n},x_{2n},x_{2n+1})\cdot S(x_{2n+1},x_{2n+1},x_{2n+2})}{1+S(x_{2n},x_{2n},x_{2n+1})} \\ +\lambda\frac{S(x_{2n},x_{2n},x_{2n+1})\cdot S(x_{2n+1},x_{2n+1}x_{2n+2})}{1+S(x_{2n},x_{2n},x_{2n+1})} \\ +\mu\frac{S(x_{2n},x_{2n}f(x_{2n},y_{2n}))\cdot S(x_{2n+1},x_{2n+1},x_{2n+2})}{1+S(x_{2n+1},x_{2n+1},x_{2n+2})} \end{array}\right\}\right).$$

Since $\varphi(t)<t$, we have

$$\begin{aligned} & 2S(x_{2n+1},x_{2n+1},x_{2n+2}) \leq \alpha\left(S(x_{2n},x_{2n},x_{2n+1})\right. \\ + & \left.S(y_{2n},y_{2n},y_{2n+1})\right)+\beta S(x_{2n},x_{2n},x_{2n+2}) \\ + & \gamma S(x_{2n+1},x_{2n+1},x_{2n+2})+\lambda S(x_{2n+1},x_{2n+1},x_{2n+2}) \\ + & \mu S(x_{2n},x_{2n},x_{2n+1}). \end{aligned}$$

With the property of the metric S

$$\begin{aligned} S(x_{2n},,x_{2n},x_{2n+2}) & \leq 2S(x_{2n+1},x_{2n+1},x_{2n+2}) \\ & + S(x_{2n},x_{2n},x_{2n+1}) \end{aligned}$$

we get

$$\begin{aligned} & 2S(x_{2n+1},x_{2n+1},x_{2n+2}) \leq \alpha\left(S(x_{2n},x_{2n},x_{2n+1})\right. \\ + & \left.S(y_{2n},y_{2n},y_{2n+1})\right)+2\beta S(x_{2n+1},x_{2n+1},x_{2n+2}) \\ + & \beta S(x_{2n},x_{2n},x_{2n+1})+\gamma S(x_{2n+1},x_{2n+1},x_{2n+2}) \\ + & \lambda S(x_{2n+1},x_{2n+1},x_{2n+2})+\mu S(x_{2n},x_{2n},x_{2n+1}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} 2S(x_{2n+1},x_{2n+1},x_{2n+2}) & \leq \alpha S(y_{2n},y_{2n},y_{2n+1}) \\ & + (2\beta+\gamma+\lambda) \\ & \quad S(x_{2n+1},x_{2n+1},x_{2n+2}) \\ & + (\beta+\mu+\alpha)S(x_{2n},x_{2n},x_{2n+1}). \end{aligned} \tag{2.3}$$

By using the same step, we get

$$\begin{aligned} 2S(y_{2n+1},y_{2n+1},y_{2n+2}) & \leq \alpha S(x_{2n},x_{2n},x_{2n+1}) \\ & + 2\left(\beta+\frac{\gamma}{2}+\frac{\lambda}{2}\right) \\ & \quad S(y_{2n+1},y_{2n+1},y_{2n+2}) \\ & + (\beta+\mu+\alpha)S(y_{2n},y_{2n},y_{2n+1}). \end{aligned} \tag{2.4}$$

Adding (2.4) and (2.5), we get

$$\begin{aligned} & 2\left[S(x_{2n+1},x_{2n+1},x_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2})\right] \leq \\ & (2\beta+\gamma+\lambda)\left[S(x_{2n+1},x_{2n+1},x_{2n+2})\right. \\ + & \left.S(y_{2n+1},y_{2n+1},y_{2n+2}\right] \\ + & (2\alpha+\beta+\mu)\left[(S(y_{2n},y_{2n},y_{2n+1})+(S(x_{2n},x_{2n},x_{2n+1})\right]. \end{aligned}$$

Thus

$$\begin{aligned} & [2-(2\beta+\gamma+\lambda)]\left[S(x_{2n+1},x_{2n+1},x_{2n+2}).\right. \\ + & \left.S(y_{2n+1},y_{2n+1},y_{2n+2})\right] \leq \\ & (2\alpha+\beta+\mu)[S(y_{2n},y_{2n},y_{2n+1}) \\ + & S(x_{2n},x_{2n},x_{2n+1})]. \end{aligned}$$

Hence

$$\begin{aligned} & S(x_{2n+1},x_{2n+1},x_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2}) \leq \\ & \frac{2\alpha+\beta+\mu}{2-(2\beta+\gamma+\lambda)}\left[S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})\right]. \end{aligned}$$

Setting

$$t_{2n+2}=S(x_{2n+1},x_{2n+1},x_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2}),$$

we obtain

$$t_{2n+2} \leq \frac{2\alpha+\beta+\mu}{[2-(2\beta+\gamma+\lambda)]}t_{2n+1}.$$

Interchanging the role of mappings f and g, using (2.2) and proceeding as above one can get

$$t_{2n+1} \leq \frac{2\alpha+\beta+\mu}{[2-(2\beta+\gamma+\lambda)]}t_{2n}.$$

Therefore we conclude that

$$t_{n} \leq \frac{2\alpha+\beta+\mu}{[2-(2\beta+\gamma+\lambda)]}t_{n-1}.$$

So,

$$t_{n} \leq \left(\frac{2\alpha+\beta+\mu}{[2-(2\beta+\gamma+\lambda)]}\right)^{n}t_{0}.$$

Since $\frac{2\alpha+\beta+\mu}{[2-(2\beta+\gamma+\lambda)]}<1$, by going to the limit as $n\to\infty$, we have

$$\lim_{n\to\infty}\left(\frac{2\alpha+\beta+\mu}{[2-(2\beta+\gamma+\lambda)]}\right)^{n}=0.$$

So,

$$\lim_{n\to\infty}t_{n}=\lim_{n\to\infty}\left[S(x_{n},x_{n},x_{n+1})+S(y_{n},y_{n},y_{n+1})\right]=0. \tag{2.5}$$

Using the property of $S-$metric space, for $n\leq m$, we have

$$\begin{aligned} S(x_{n},x_{n},x_{m})+S(y_{n},y_{n},y_{m}) & \leq 2S(x_{n},x_{n},x_{n+1}) \\ & + S(x_{n+1},x_{n+1},x_{m}) \\ & + 2S(y_{n},y_{n},y_{n+1}) \\ & + S(y_{n+1},y_{n+1},y_{m}). \end{aligned}$$

On taking limit as $n,m\to\infty$ and using (2.5), we obtain

$$\begin{aligned} & \lim_{n,m\to\infty}\left[S(x_{n},x_{n},x_{m})+S(y_{n},y_{n},y_{m})\right] \leq \\ & \lim_{n,m\to\infty}\left[S(x_{n+1},x_{n+1},x_{m})+S(y_{n+1},y_{n+1},y_{m}\right] \leq \\ & \lim_{n,m\to\infty}\left[\begin{array}{c} 2S(x_{n+1},x_{n+1},x_{n+2})+S(x_{n+2},x_{n+2},x_{m}) \\ +2S(y_{n+1},y_{n+1},y_{n+2})+S(y_{n+2},y_{n+2},y_{m}) \end{array}\right] \\ & =\lim_{n,m\to\infty}\left[S(x_{n+2},x_{n+2},x_{m})+S(y_{n+2},y_{n+2},y_{m}\right]. \end{aligned}$$

By repeatedly use of property of $S-$metric space, we get

$$\lim_{n,m\to\infty}[S(x_n,x_n,x_m)+S(y_n,y_n,y_m)] \leq \lim_{n,m\to\infty}\left[\begin{array}{c} 2S(x_{m-1},x_{m-1},x_m)+S(x_m,x_{n+2},x_{m+1}) \\ +2S(y_{m-1},y_{m-1},y_m)+S(y_m,y_m,y_{m+1}) \end{array}\right]$$

which gives

$$\lim_{n,m\to\infty}[S(x_n,x_n,x_m)+S(y_n,y_n,y_m)]=0.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X. Since X is complete, there exist $x,y\in X$ such that

$$x_n\to x \text{ and } y_n\to y \text{ as } n\to\infty.$$

Since f is continuous, we obtain

$$x=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}f(x_{2n},y_{2n})=f(x,y),$$

$$y=\lim_{n\to\infty}y_{2n+1}=\lim_{n\to\infty}f(y_{2n},x_{2n})=f(y,x).$$

From (2.2) we have

$$S(f(x,y),f(x,y),g(x,y)) \leq \varphi\left(\frac{1}{2}\left\{\begin{array}{c} \alpha[S(x,x,x)+S(y,y,y)] \\ +\beta\frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,x)+S(y,y,y)} \\ +\gamma\frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,x)} \\ +\lambda\frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,x)} \\ +\mu\frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,g(x,y))} \end{array}\right\}\right),$$

therefore

$$S(f(x,y),f(x,y),g(x,y)) \leq \varphi\left(\frac{1}{2}\left\{\begin{array}{c} \beta S(x,x,x)\cdot S(y,y,g(x,y)) \\ +\gamma S(x,x,x)\cdot S(x,x,g(x,y)) \\ +\lambda\frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,x)} \\ +\mu\frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,g(x,y))} \end{array}\right\}\right) \leq \varphi(0)=0$$

which implies that

$$S(x,x,g(x,y))+S(y,y,g(y,x))=0.$$

Thus we have $g(x,y)=x$ and $g(y,x)=y$.
Hence (x,y) is coupled common fixed point of f and g. Similarly, the result follows, when g is assumed to be continuous. Now, consider the other assumption that for an increasing sequence $\{x_n\}$ with $x_n\to x$ we have $x_n\leq x$ and for decreasing sequence y_n with $y_n\to y$ we have $y_n\leq y$ for all $n\in\mathbb{N}$,

$$\begin{aligned} S(x,x,f(x,y)) &\leq 2S(x,x,x_n)+S(f(x,y),f(x,y),x_n) \\ &= 2S(x,x,x_n)+ \\ &\quad S(f(x,y),f(x,y),g(x_{n-1},y_{n-1})). \end{aligned}$$

Therefore by using (2.2), one has

$$S(x,x,f(x,y)) \leq 2S(x,x,x_n)+ \varphi\left(\frac{1}{2}\left\{\begin{array}{c} \alpha(S(x,x,x_{n-1})+S(y,y,y_{n-1})) \\ +\beta\frac{S(x,x,f(x,y))\cdot S(x,x,x_n)}{1+S(x,x,x_{n-1})+S(y,y,y_{n-1})} \\ +\gamma\frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n))}{1+S(x,x,x_{n-1}))} \\ +\lambda\frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x,x,f(x,y))} \\ +\mu\frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x_{n-1},x_{n-1},x_n)} \end{array}\right\}\right)$$

then we have

$$S(x,x,f(x,y)) \leq 2S(x,x,x_n)+\frac{1}{2}\left\{\begin{array}{c} \alpha(S(x,x,x_{n-1})+S(y,y,y_{n-1})) \\ +\beta\frac{S(x,x,f(x,y))\cdot S(x,x,x_n)}{1+S(x,x,x_{n-1})+S(y,y,y_{n-1})} \\ +\gamma\frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n))}{1+S(x,x,x_{n-1}))} \\ +\lambda\frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x,x,f(x,y))} \\ +\mu\frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x_{n-1},x_{n-1},x_n)} \end{array}\right\}.$$

Since x_n converges to x and y_n converges to y then

$$\begin{aligned} \lim_{n\to\infty}(S(x,x,x_{n-1}) &= \lim_{n\to\infty}S(y,y,y_{n-1}) \\ &= \lim_{n\to\infty}S(x_{n-1},x_{n-1},x_n))=0. \end{aligned}$$

So by taking the limit as $n\to\infty$ in the above inequality, one can get $S(x,x,f(x,y))\leq 0$, Hence $f(x,y)=x$. Similarly, by the same way we obtain $S(y,y,f(y,x))\leq 0$ and $f(y,x)=y$. By interchanging the role of function f and g, we get the same result for g. Thus (x,y) is the common coupled fixed point for f and g. □

Theorem 2.2. *Let $(X,\leq,S)$ be a partially ordered complete $S-$metric space and the mappings $f,g:X\times X\to X$ satisfied the mixed weakly monotone property on X, $x_0\leq f(x_0,y_0)$, $f(y_0,x_0)\leq y_0$ or $x_0\leq g(x_0,y_0)$, $g(y_0,x_0)\leq y_0$ for some $x_0,y_0\in X$ and*

$$S(f(x,y),f(x,y),g(u,v)) \leq \frac{1}{2}\left\{\begin{array}{c} \alpha(S(x,x,u)+S(y,y,v)) \\ +\beta\frac{S(x,x,f(x,y))\cdot S(x,x,g(u,v))}{1+S(x,x,u)+S(y,y,v)} \\ +\gamma\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(x,x,u)} \\ +\lambda\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(x,x,f(x,y))} \\ +\mu\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(u,u,g(u,v))} \end{array}\right\} \tag{2.6}$$

for all $x,y,u,v\in X$ with $x\leq u$ and $y\leq v$, $\alpha,\beta,\gamma,\lambda,\mu\in\mathbb{R}^+$ such that $2\alpha+3\beta+\gamma+\lambda+\mu<2$. Also, assume that either f or g is continuous or X has the following property:
(1) if $\{x_n\}$ is an increasing sequence with $x_n\to x$ then $x_n\leq x$ for all $n\in\mathbb{N}$,
(2) if $\{y_n\}$ is a decreasing sequence with $y_n\to y$ then $y_n\leq y$ for all $n\in\mathbb{N}$.
Then f and g have a coupled common fixed point in X.

Theorem 2.3. *Let $(X,\leq,S)$ be a partially ordered complete $S-$metric space and the mappings $f,g:X\times X\to X$ satisfied the mixed weakly monotone property on X,*

$$x_0\leq f(x_0,y_0),f(y_0,x_0)\leq y_0$$

or

$$x_0\leq g(x_0,y_0),g(y_0,x_0)\leq y_0$$

for some $x_0,y_0\in X$ and

$$S(f(x,y),f(x,y),g(u,v))\leq\frac{1}{2}\left\{\begin{array}{c}\varphi\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right)\\+\beta\frac{S(x,x,f(x,y))\cdot S(x,x,g(u,v))}{1+S(x,x,u)+S(y,y,v)}\\+\gamma\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(x,x,u)}\\+\lambda\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(x,x,f(x,y))}\\+\mu\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(u,u,g(u,v))}\end{array}\right\}\tag{2.7}$$

for all $x,y,u,v\in X$ with $x\leq u$ and $y\leq v$, $\beta,\gamma,\lambda,\mu\in\mathbb{R}^+;3\beta+3\lambda+\gamma+\mu<1$; Also, assume that either f or g is continuous or X has the following property:
(1) if $\{x_n\}$ is an increasing sequence with $x_n\to x$ then $x_n\leq x$ for all $n\in\mathbb{N}$,
(2) if $\{y_n\}$ is a decreasing sequence with $y_n\to y$ then $y_n\leq y$ for all $n\in\mathbb{N}$.
Then f and g have a coupled common fixed point in X.

Proof. Consider

$$x_0\leq f(x_0,y_0)\text{ and }f(y_0,x_0)\leq y_0$$

and let

$$f(x_0,y_0)=x_1\text{ and }f(y_0,x_0)=y_1.$$

Now

$$x_1=f(x_0,y_0)\leq g(f(x_0,y_0),f(y_0,x_0)=g(x_1,y_1)=x_2$$

and

$$y_1=f(y_0,x_0)\leq g\left(f(y_0,x_0),f(x_0,y_0)\right)=g\left(y_1,x_1\right)=y_2.$$

Continuing in this way, we have

$$x_{2n+1}=f(x_{2n},y_{2n}),y_{2n+1}=f(y_{2n},x_{2n})$$

and

$$x_{2n+2}=g(x_{2n+1},y_{2n+1}),y_{2n+2}=g(y_{2n+1},x_{2n+1}).$$

Thus, we conclude that $\{x_n\}$ is an increasing sequence and $\{y_n\}$ is a decreasing sequence. Similarly, from the condition $x_0\leq g(x_0,y_0)$ and $y_0\geq g(y_0,x_0)$, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ are increasing or decreasing. Now from (2.2), we obtain

$$S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))\leq$$
$$\frac{1}{2}\left\{\begin{array}{c}\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\+\beta\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n})).S(x_{2n},x_{2n},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}\\+\gamma\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n})).S(x_{2n+1},x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n},x_{2n},x_{2n+1})}\\+\lambda\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n}))\cdot S(x_{2n+1},x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n},x_{2n},x_{2n+1})}\\+\mu\frac{S(x_{2n},x_{2n},f(x_{2n},y_{2n}))\cdot S(x_{2n+1},x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+S(x_{2n+1},x_{2n+1},x_{2n+2})}\end{array}\right\}$$

which implies

$$\begin{array}{c}S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))\leq\\\frac{1}{2}\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\+\frac{1}{2}\left[\beta S(x_{2n},x_{2n},x_{2n+2})+\gamma S(x_{2n+1},x_{2n+1},x_{2n+2})\right]\\+\frac{1}{2}\left[\lambda S(x_{2n+1},x_{2n+1},x_{2n+2})+\mu S(x_{2n},x_{2n},x_{2n+1})\right]\end{array}$$

With the property of the metric S

$$S(x_{2n},x_{2n},x_{2n+2})\leq 2S(x_{2n+1},x_{2n+1},x_{2n+2})+S(x_{2n},x_{2n},x_{2n+1})$$

we get

$$S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))\leq$$
$$\frac{1}{2}\left\{\begin{array}{c}\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\+2\beta S(x_{2n+1},x_{2n+1},x_{2n+2})+\beta S(x_{2n},x_{2n},x_{2n+1})\\+\gamma S(x_{2n+1},x_{2n+1},x_{2n+2})\\+\lambda S(x_{2n+1},x_{2n+1},x_{2n+2})+\mu S(x_{2n},x_{2n},x_{2n+1})\end{array}\right\}.$$

Therefore, we have

$$\begin{array}{c}S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))\leq\\+\frac{1}{2}\left\{\begin{array}{c}\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\(2\beta+\gamma+\lambda)S(x_{2n+1},x_{2n+1},x_{2n+2})\\(\beta+\mu)S(x_{2n},x_{2n},x_{2n+1})\end{array}\right\}.\end{array}\tag{2.8}$$

By using the same step, we get

$$\begin{array}{c}S(y_{2n+1},y_{2n+1},y_{2n+2})\leq\\+\frac{1}{2}\left\{\begin{array}{c}\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\(2\beta+\gamma+\lambda)S(y_{2n+1},y_{2n+1},y_{2n+2})\\(\beta+\mu+\lambda)S(y_{2n},y_{2n},y_{2n+1})\end{array}\right\}.\end{array}\tag{2.9}$$

Adding (2.8) and (2.9), we get

$$\begin{array}{rl}&S(x_{2n+1},x_{2n+1},x_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2})\leq\\&\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\+&\frac{1}{2}(2\beta+\gamma+\lambda)[S(x_{2n+1},x_{2n+1},x_{2n+2})\\+&S(y_{2n+1},y_{2n+1},y_{2n+2}]\\+&\frac{1}{2}(\beta+\mu)[(S(y_{2n},y_{2n},y_{2n+1})+(S(x_{2n},x_{2n},x_{2n+1})].\end{array}$$

Thus

$$\begin{array}{rl}&[2-(2\beta+\gamma+\lambda)][S(x_{2n+1},x_{2n+1},x_{2n+2})\\+&S(y_{2n+1},y_{2n+1},y_{2n+2})]\\\leq&2\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\+&(2\beta+\gamma+\lambda)[S(y_{2n},y_{2n},y_{2n+1})+S(x_{2n},x_{2n},x_{2n+1})]\\+&(\beta+\mu)[S(y_{2n},y_{2n},y_{2n+1})+S(x_{2n},x_{2n},x_{2n+1})],\end{array}$$

hence we have

$$\begin{array}{c}S(x_{2n+1},x_{2n+1},x_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2})\leq\\+\frac{2}{2-(2\beta+\gamma+\lambda)}\varphi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)\\+\frac{\beta+\mu}{2-(2\beta+\gamma+\lambda)}\left[S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})\right].\end{array}$$

Setting

$$t_{2n+2} = S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})$$

we obtain

$$\begin{aligned} t_{2n+2} &\leq \frac{\beta+\mu}{[2-(2\beta+\gamma+\lambda)]} t_{2n+1} \\ &+ \frac{2}{2-(2\beta+\gamma+\lambda)} \varphi\left(\frac{t_{2n+1}}{2}\right). \end{aligned}$$

Interchanging the role of mappings f and g, using (2.2) and proceeding as above one can get

$$\begin{aligned} t_{2n+1} &\leq \frac{\beta+\mu}{[2-(2\beta+\gamma+\lambda)]} t_{2n} \\ &+ \frac{2}{2-(2\beta+\gamma+\lambda)} \varphi\left(\frac{t_{2n}}{2}\right). \end{aligned}$$

Therefore, since $\varphi(t) < t$, we conclude that

$$t_{n+1} \leq \frac{\beta+\mu+1}{[2-(2\beta+\gamma+\lambda)]} t_n$$

which implies that

$$t_n \leq \left(\frac{\beta+\mu+1}{[2-(2\beta+\gamma+\lambda)]}\right)^n t_0.$$

So

$$0 = \lim_{n\to\infty} t_n = \lim_{n\to\infty} \left[S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1})\right]. \quad (2.10)$$

Using the property of $S-$metric space, for $n \leq m$, we have

$$\begin{aligned} &S(x_n, x_n, x_m) + S(y_n, y_n, y_m) \leq \\ &2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &+2S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_m). \end{aligned}$$

On taking limit as $n, m \to \infty$ and using (2.10), we obtain

$$\begin{aligned} &\lim_{n,m\to\infty} [S(x_n, x_n, x_m) + S(y_n, y_n, y_m)] \leq \\ &\lim_{n,m\to\infty} [S(x_{n+1}, x_{n+1}, x_m) + S(y_{n+1}, y_{n+1}, y_m] \leq \\ &\lim_{n,m\to\infty} \begin{bmatrix} 2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(x_{n+2}, x_{n+2}, x_m) \\ +2S(y_{n+1}, y_{n+1}, y_{n+2}) + S(y_{n+2}, y_{n+2}, y_m) \end{bmatrix} \\ &= \lim_{n,m\to\infty} [S(x_{n+2}, x_{n+2}, x_m) + S(y_{n+2}, y_{n+2}, y_m]. \end{aligned}$$

By repeatedly use of property of $S-$metric space, we get

$$\begin{aligned} &\lim_{n,m\to\infty} [S(x_n, x_n, x_m) + S(y_n, y_n, y_m)] \leq \\ &\lim_{n,m\to\infty} \begin{bmatrix} 2S(x_{m-1}, x_{m-1}, x_m) + S(x_m, x_{n+2}, x_{m+1}) \\ +2S(y_{m-1}, y_{m-1}, y_m) + S(y_m, y_m, y_{m+1}) \end{bmatrix} \end{aligned}$$

which gives

$$\lim_{n,m\to\infty} [S(x_n, x_n, x_m) + S(y_n, y_n, y_m)] = 0.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, there exist $x, y \in X$ such that

$$x_n \to x \text{ and } y_n \to y \text{ as } n \to \infty.$$

Now, as in Theorem 2.1 we conclude that

$$x = f(x,y), y = f(y,x).$$

Using condition (2.2) we obtain

$$S(f(x,y), f(x,y), g(x,y)) \leq \frac{1}{2} \left\{ \begin{array}{c} \varphi\left(\frac{S(x,x,x)+S(y,y,y)}{2}\right) \\ +\beta \frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,x)+S(y,y,y)} \\ +\gamma \frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,x)} \\ +\lambda \frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,x)} \\ +\mu \frac{S(x,x,x)\cdot S(x,x,g(x,y))}{1+S(x,x,g(x,y))} \end{array} \right\},$$

so,

$$S(f(x,y), f(x,y), g(x,y)) \leq$$

$$\frac{1}{2} \left\{ \begin{array}{c} \beta S(x,x,x) + S(y,y,g(x,y)) \\ +\gamma S(x,x,x) + S(x,x,g(x,y)) \\ +\lambda \frac{S(x,x,x)+S(x,x,g(x,y))}{1+S(x,x,x)} \\ +\mu \frac{S(x,x,x)+S(x,x,g(x,y))}{1+S(x,x,g(x,y))} \end{array} \right\} = 0,$$

this implies that

$$S(x,x,g(x,y)) + S(y,y,g(y,x)) = 0.$$

Thus we have $g(x,y) = x$ and $g(y,x) = y$.
Hence (x,y) is coupled common fixed point of f and g. Similarly, the result follows, when g is assumed to be continuous.
Now, consider the other assumption that for an increasing sequence $\{x_n\}$ with $x_n \to x$ we have $x_n \leq x$ and for decreasing sequencey_n with $y_n \to y$ we have $y_n \leq y$ for all $n \in \mathbb{N}$.
Consider

$$\begin{aligned} S(x,x,f(x,y)) &\leq 2S(x,x,x_n) + S(f(x,y), f(x,y), x_n) \\ &= 2S(x,x,x_n) + \\ &\quad S(f(x,y), f(x,y), g(x_{n-1}, y_{n-1})). \end{aligned}$$

Therefore by using (2.2), we obtain

$$S(x,x,f(x,y)) \leq 2S(x,x,x_n)$$
$$+ \quad \frac{1}{2} \left\{ \begin{array}{c} \varphi\left(\frac{S(x,x,x_{n-1})+S(y,y,y_{n-1})}{2}\right) \\ +\beta \frac{S(x,x,f(x,y))\cdot S(x,x,x_n)}{1+S(x,x,x_{n-1})+S(y,y,y_{n-1})} \\ +\gamma \frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n))}{1+S(x,x,x_{n-1}))} \\ +\lambda \frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x,x,f(x,y))} \\ +\mu \frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x_{n-1},x_{n-1},x_n)} \end{array} \right\}$$

then we have

$$S(x,x,f(x,y)) \leq 2S(x,x,x_n)$$
$$+ \quad \frac{1}{2} \left\{ \begin{array}{c} \varphi\left(\frac{S(x,x,x_{n-1})+S(y,y,y_{n-1})}{2}\right) \\ +\beta \frac{S(x,x,f(x,y))\cdot S(x,x,x_n)}{1+S(x,x,x_{n-1})+S(y,y,y_{n-1})} \\ +\gamma \frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n))}{1+S(x,x,x_{n-1}))} \\ +\lambda \frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x,x,f(x,y))} \\ +\mu \frac{S(x,x,f(x,y))\cdot S(x_{n-1},x_{n-1},x_n)}{1+S(x_{n-1},x_{n-1},x_n)} \end{array} \right\}.$$

Since x_n converges to x and y_n converges to y then

$$\begin{aligned}\lim_{n\to\infty}(S(x,x,x_{n-1}) &= \lim_{n\to\infty} S(y,y,y_{n-1}) \\ &= \lim_{n\to\infty} S(x_{n-1},x_{n-1},x_n)) = 0.\end{aligned}$$

So by taking the limit as $n\to\infty$ in the above inequality , one can get $S(x,x,f(x,y)) \le 0$, Hence $f(x,y)=x$. Similarly , by the same way we obtain $S(y,y,f(y,x)) \le 0$ and $f(y,x)=y$. By interchanging the role of function f and g, we get the same result for g. Thus (x,y) is the common coupled fixed point for f and g. □

3. Applications

Later, from the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type as an application of Theorem 2.1 above. For this purpose, let

$$Y = \left\{ \begin{array}{c} \chi:\mathbb{R}^+\to\mathbb{R}^+, \text{ satisfies that } \chi \text{ is a Lebesgue} \\ \text{integrable, summable on each compact} \\ \text{subset of } \mathbb{R}^+ \text{and } \int_0^\varepsilon \chi(t)dt > 0 \text{ for each } \varepsilon > 0 \\ \text{and subadditive,that is} \\ \int\limits_0^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p} \chi(t)dt \le \sum\limits_{i=1}^{p}\int\limits_0^{\varepsilon_i}\chi(t)dt \\ \text{for each } \varepsilon_i>0, i=1,\ldots,p \end{array} \right\}.$$

Example 3.1. *We consider $\chi(x)=\frac{1}{1+x}, \chi:\mathbb{R}^+\to\mathbb{R}^+, \chi$ is Lebesgue integrable which is nonnegative, satisfies*

$$\begin{aligned}\int\limits_0^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p}\chi(t)dt &= \ln(1+\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p) \\ &\le \ln[(1+\varepsilon_1)(1+\varepsilon_2)\cdots(1+\varepsilon_p)] \\ &= \ln(1+\varepsilon_1)+\cdots+\ln(1+\varepsilon_p) \\ &= \sum_{i=1}^{p}\int\limits_0^{\varepsilon_i}\chi(t)dt.\end{aligned}$$

This shows that χ is an example of subadditive, nonnegative, Lebesgue integrable function.

Theorem 3.2. *Let $(X,\le,S)$ be a partially ordered complete $S-$metric space and the mappings $f,g: X\times X\to X$ satisfied the mixed weakly monotone property on X;$x_0 \le f(x_0,y_0)$, $f(y_0,x_0)\le y_0$ or $x_0\le g(x_0,y_0)$, $g(y_0,x_0)\le y_0$ for some $x_0,y_0\in X$.*

$$\int\limits_0^{S(f(x,y),f(x,y),g(u,v))}\chi(t)dt \le \int\limits_0^{M_{f,g}(x,y,u,v)}\chi(t)dt$$

where

$$M^{\varphi}_{f,g}(x,y,u,v) = \frac{1}{2}\left\{ \begin{array}{c} \varphi\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right) \\ +\beta\frac{S(x,x,f(x,y))\cdot S(x,x,g(u,v))}{1+S(x,x,u)+S(y,y,v)} \\ +\gamma\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(x,x,u)} \\ +\lambda\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(x,x,f(x,y))} \\ +\mu\frac{S(x,x,f(x,y))\cdot S(u,u,g(u,v))}{1+S(u,u,g(u,v))} \end{array}\right\} \quad (3.1)$$

for all $x,y,u,v\in X$ with $x\le u$ and $y\le v$. $\beta,\gamma,\lambda,\mu\in\mathbb{R}^+;3\beta+\lambda+\gamma+\mu<1$. Also, assume that either f or g is continuous or X has the following property:
(1) if $\{x_n\}$ is an increasing sequence with $x_n\to x$ then $x_n\le x$ for all $n\in\mathbb{N}$,
(2) if $\{y_n\}$ is a decreasing sequence with $y_n\to y$ then $y_n\le y$ for all $n\in\mathbb{N}$.
Then f and g have a coupled common fixed point in X.

Proof. For $\chi\in Y$, we defined the function $\Lambda:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\Lambda(x)=\int_0^x\chi(t)dt,$$

note that $\Lambda\in\Psi$. Thus the inequality (3.1) becomes

$$\Lambda(S(f(x,y),f(x,y),g(u,v))) \le \Lambda\left(M_{f,g}(x,y,u,v)\right). \quad (3.2)$$

Setting in (3.2), $\Lambda\circ M^{\varphi}_{f,g}=M^1_{f,g}$ and $\Lambda\circ S=S_1$ we obtain

$$\begin{aligned}S_1(f(x,y),f(x,y),g(u,v)) &\le \\ M^1_{f,g}(x,y,u,v)\end{aligned}$$

therefore from Theorem 2.1 the desired result follows. □

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